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A New Lower Bound on Guard Placement for Wireless Localization

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A New Lower Bound on Guard Placement for Wireless Localization

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Abstract

The problem of *wireless localization* asks to place and orient stations in the plane, each of which broadcasts a unique key within a fixed angular range, so that each point in the plane can determine whether it is inside or outside a given polygonal region. The primary goal is to minimize the number of stations. In this paper we establish a lower bound of $\lfloor 2n/3 \rfloor - 1$ stations for polygons in general position, for the case in which the placement of stations is restricted to polygon vertices, improving upon the existing $\lceil n/2 \rceil$ lower bound.

The problem of *wireless localization* introduced in [EGS07] asks to place a set of fixed localizers (*guards*) in the plane so as to enable mobile communication devices to prove that they are inside or outside a secure region, defined by the interior of a polygon P. The guards are equipped with directional transmitters that can broadcast a key within a fixed angular range. The polygon P is *virtual* in the sense that it does not block broadcasts. A mobile device (henceforth, a point in the plane) determines whether it is inside or outside P from a monotone Boolean formula composed from the broadcasts using AND($\cdot$) and OR($+$) operations only. The primary goal is to minimize the number of guards. Solutions for convex and orthogonal polygons were established [EGS07], but for general polygons, a considerable gap between a lower bound of $\lceil n/2 \rceil$ and an upper bound of $n-2$ guards remains to be closed. See also [O'R07].

In this paper we establish a lower bound of $\lfloor 2n/3 \rfloor - 1$ guards for polygons in general position, for the case in which the placement of guards is restricted to polygon vertices (*vertex* guards). In [EGS07], the authors use vertex guards only, and leave open the question of whether general guards (i.e, guards placed at arbitrary points) are more efficient. In this paper we answer their question positively by establishing a solution with $n/2$ general guards for a polygon that requires no fewer than $2n/3 - 1$ vertex guards for localization.

A vertex guard that broadcasts over the full internal or external angle at that vertex is called *natural*. Natural guards alone do not suffice to localize a region [EGS07], so non-natural guards must be employed as well.

Theorem 1 *There exist n-vertex simple polygons that require at least $\lfloor 2n/3 \rfloor - 1$ guards placed at polygon vertices for localization.*

Proof: The proof is by construction. Let $n = 3m$. Let P be a polygon consisting of m narrow spikes, as illustrated in Figure 1. P is parameterized in terms of w, h, and δ, where $\delta < h < w$. The first $m-1$ spikes each consists of three vertices l_i, t_i, and r_i, for $1 \le i < m$. Edge $t_i r_i$ is vertical and of height $h/2$; edge $r_i l_{i+1}$ is horizontal. The vertical distance separating l_i and r_i is h; the horizontal distance between l_i and r_i is δ. The horizontal distance between r_i and r_{i+1} is w.

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To close the polygon, the mth spike deviates from this pattern slightly; its vertical edge $t_m r_m$ has height $1.5h$ and the edge $r_m l_1$ closes the polygon.[1] We now show that P cannot be localized with fewer than $2n/3 - 1$ guards placed at vertices.

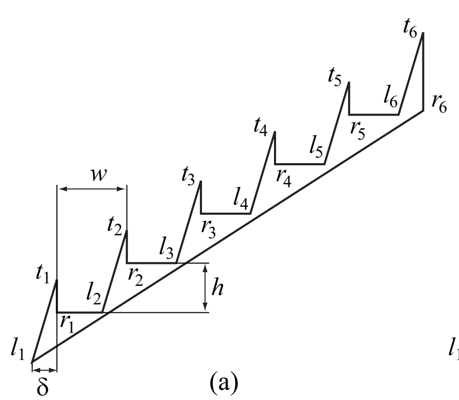

Figure 1: Polygon construction.

For any i, call a guard stationed at a vertex t_i a *tip* guard, and a guard stationed at a vertex ℓ_i or r_i a *base* guard. One critical observation is that each polygon edge e must align with the broadcast boundary line of a guard G [EGS07]; we say that G *covers* e. Since the only vertices in P collinear with a spike edge (polygon edges incident to t_i, for some i) are the vertices incident to the edge, a guard covering a spike edge must be stationed at a vertex of that edge. Counting spike edges and ignoring horizontal edges for the moment, we get a total number of $2n/3$ spike edges that need coverage. Next we analyze the employment of natural tip guards in an optimal localization solution for P.

Let $\mathcal{S}$ be the set of guards in an optimal localization solution for P, and let n_0 be the number of natural tip guards in $\mathcal{S}$. The natural tip guards cover precisely $2n_0$ spike edges, leaving $2n/3 - 2n_0$ spike edges to be covered by other guards. Note however that any other (base or non-natural tip) guard can cover at most one spike edge (since no two spike edges are collinear). This implies that at least $(2n/3 - 2n_0) + n_0$ guards are necessary to cover all spike edges and therefore $|\mathcal{S}| \geq 2n/3 - n_0$. Thus, if $n_0 = 0$, then $|S| \geq 2n/3$ and the proof is finished.

Consider now the case $n_0 > 0$ and let G be an arbitrary natural tip guard at t_i, for $i < m$. Let A and B be the ε-neighborhoods along the outside of G's broadcast cone between horizontal lines through ℓ_i and r_i, with B restricted to the interior of P. See Fig. 1b. Observe that G is only able to delineate its cone-shaped broadcast region, leaving A and B with ambiguous inside/outside status. This ambiguity can be easily resolved by a guard positioned at t_i, l_i, or r_i. We show now that this is the only way to resolve the ambiguity. Specifically, we will show that, if A and B are separated by combinations of guards other than at t_i, l_i, r_i, then the bound of 2n/3 is exceeded for sufficiently small δ and sufficiently large w.

First observe that any horizontal line segment ab, with $a \in A$ and $b \in B$, must be crossed by at least one cone edge besides G's broadcast cone edges; if this were not the case, then a and b would be covered by a same set of cones and $\mathcal{S}$ would not localize P. For any cone ray α not belonging to G, we therefore say that its contribution to separating A and B is the difference in the y-coordinates of the intersection points between α and the cone ray boundaries for G. Since

[1] This polygon can be seen as a variation on the "comb" polygon that establishes a lower bound on the original art gallery problem [O'R87, p. 2] [O'R98, p. 6].

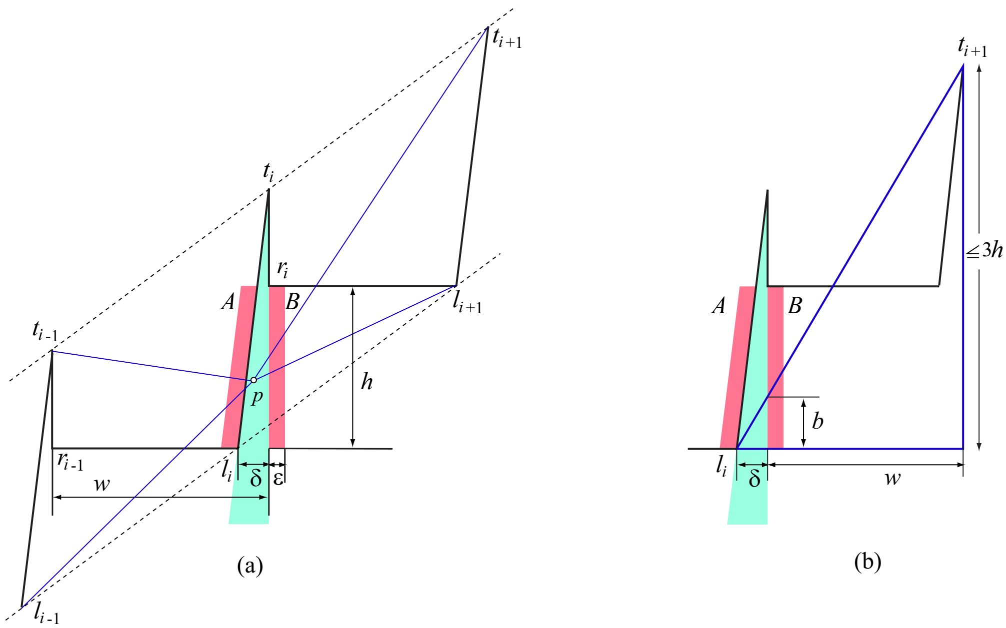

Figure 2: An A-B region requires many guards for localization.

the height of A and of B is at least $h - \delta$ (the δ term arising because the edge $l_1 r_m$ cuts off a bit of B), the sum of the contributions must be at least $h - \delta$.

For any point p interior to or on the boundary of the broadcast cone for G, let R_p denote the double-cone region bounded by the four rays originating at p and passing through t_{i-1}, ℓ_{i-1}, t_{i+1} and either ℓ_{i+1} or r_m, depending on whether p lies above or below line$(l_{i+1}r_m)$. See Fig. 2a. Any other ray originating at p and passing through a vertex of P lies inside R_p. Thus the contribution of R_p to separating A and B, defined as the maximal contribution among all such rays, is achieved by one of the four rays bounding R_p. Furthermore, the contribution of R_p to separating A and B is maximized for $p = \ell_i$ and is achieved by $t_{i+1}\ell_i$. This contribution value is $2.5h\delta/(\delta + w)$, which decreases with decreasing δ and increasing w.

Now consider k contributing rays working together to separate A and B. It follows from the previous observations that $k > (h - \delta)(\delta + w)/2.5h\delta$. If we choose, for instance, $\delta = h/2$ and $w = 5nh/3$, then we get $k > 2n/3$. Thus, more than $2n/3$ guards are required to separate A and B if they are placed at vertices other than t_i, l_i, r_i.

So it must be that for each natural tip guard placed at a vertex other than t_m, $\mathcal{S}$ includes an additional guard either at the base or at the tip of the spike in order to separate regions A and B. Note however that such an additional guard cannot cover any spike edges other than the ones already covered by G. So the total number of guards necessary to localize P is at least $n_0 + (n_0 - 1) + (2n/3 - 2n_0)$: the first term counts the natural tip guards; the second term counts the additional guards required to separate A and B for each natural tip guard (with the exception of a natural guard placed at t_m); and the third term counts the guards necessary to cover the spike edges left uncovered by the natural tip guards. Thus at least $\lfloor 2n/3 \rfloor - 1$ guards are necessary to localize P. This is also true for polygons in general position, since the arguments here hold even when P's vertices are perturbed within an ε-neighborhood, for small $\varepsilon > 0$. $\square$

We now show that it is possible to localize the polygon P constructed in Theorem 1 with $n/2$ guards, if we eliminate the restriction that they be placed at polygon vertices, and allow them to sit at arbitrary points. The placement of guards is illustrated in Fig. 3. Three guards are used for

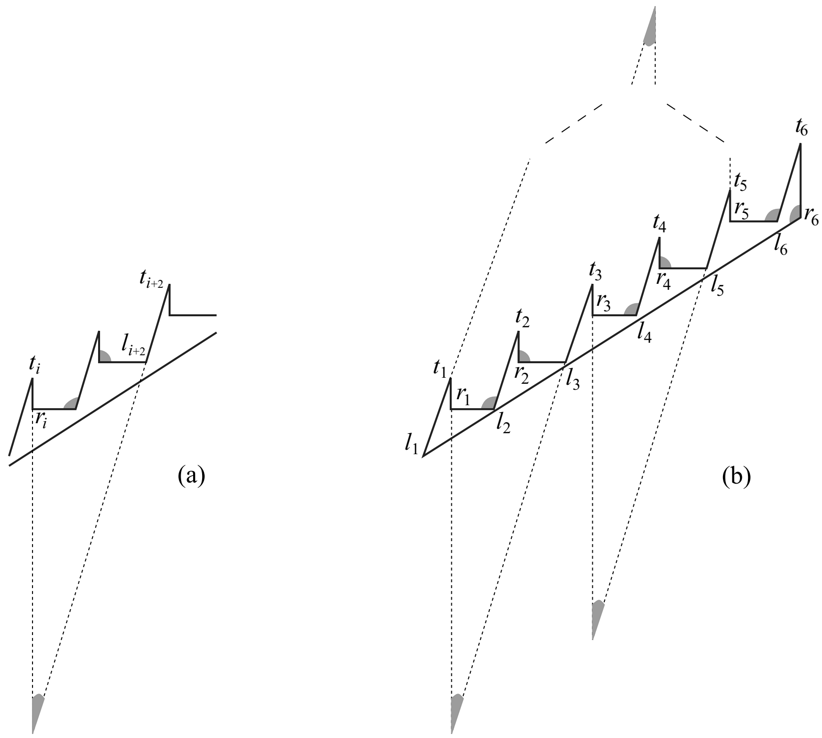

Figure 3: $n/2$ general guards localize P.

every six edges (see Fig. 3a), which implies $n/2$ guards for n edges (see Fig. 3b). In general, if n is not a multiple of 6, then P can be localized with $\lceil n/2 \rceil + 1$.

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