The aim of logic is to alert us to fallacious forms of reasoning that can lead to incorrect conclusions. We have shown how to test whether an argument is valid using truth tables. But when the argument is made up of many individual propositions, we may not be able to use truth tables because of the number of computations required. Truth tables are not how we ordinarily evaluate an argument’s validity.

A more natural way of determining validity is to identify the conclusion and then show how it can be derived from the premises provided, using only accepted valid inference rules. The valid rules of inference make it possible to construct and assess arguments by natural deduction. If each step in deriving the conclusion from the premises is justified, then we will have proven that the conclusion in question is a valid consequence of the premises being used. Thus, if the premises are true, then its conclusion must be true.

A proof of C is constructed by identifying: the conclusion (C); the premises (Pn) that C is supposed to be inferred from; and the inference rules (Rn) used to derive C from Pn. Beginning
with the premises, we infer each subsequent proposition by applying a specific valid inference rule to specific prior lines in the proof. As long as each step in the derivation of C from Pn is justified by a recognized rule of inference, the conclusion C is justified.  

As we saw in classical logic, immediate inferences require only one premise. Thus, if ‘no rats are dogs’ is true, we can infer that ‘no dogs are rats’ is true. And if ‘Some dogs are brown’ is true, we can infer immediately that ‘No dogs are brown’ is false. But if ‘all dogs are canines’ is true, we cannot infer that ‘all canines are dogs’ is true: some immediate inferences are valid and some are invalid.

Following are valid immediate inference rules that have been identified for the propositional calculus. The reader is encouraged to carefully assess each, in order to be convinced that the rule allows only valid inferences. First, we will consider valid inferences that have only one proposition as its premise. A conclusion, C, is derived from a premise, P1, if there is a rule of inference that justifies inferring C from P1.

---

3 Parenthetically, these are the basic assumptions of a deterministic universe. For if we could identify all the true facts about reality, and all the valid rules of inference, we could then we construct all possible truths about reality, past and future.
Immediate Inferences using only a single proposition  \((p)\)

1. Double negation (DN)

\[ p \quad /\quad \sim \sim p \]

Ms Flotmos will take J for a treat. // It is false that Ms F will not take J for a treat.

\[ \sim \sim p \quad /\quad p \]

It is false that J will not clean her room. // J will clean her room.

2. Tautology (Taut)

\[ p \quad /\quad p \lor p \]

Ms Flotmos is Jamie’s mother // Ms Flotmos is Jamie’s mother or Ms Flotmos is Jamie’s mother

\[ p \quad /\quad p \cdot p \]

Ms Flotmos is Jamie’s mother // Ms Flotmos is Jamie’s mother and Ms Flotmos is Jamie’s mother

Immediate Inferences using two propositions  \((p, q)\)

3. Addition  (Add)

\[ p \quad /\quad p \lor q \]
J cleaned J’s room. / J cleaned J’s room or ms F cleaned J’s room.

4. Simplification (Simp)

\[(p \cdot q) \equiv p\]

J cleaned J’s room and J cleaned F’s room. // J cleaned J’s room.

5. Commutation (Com)

\[(p \cdot q) \equiv (q \cdot p)\]

J cleaned J’s room and J cleaned F’s room. // J cleaned F’s room and J cleaned J’s room.

\[(p \lor q) \equiv (q \lor p)\]

J cleaned J’s room or J cleaned F’s room. // J cleaned F’s room or J cleaned J’s room.

6. De Morgan’s Rule (DM)

\[\neg (p \cdot q) \equiv (\neg p \lor \neg q)\]

It is not the case that J cleaned J’s room and J cleaned F’s room. // J did not clean J’s room or J did not clean F’s room.

\[\neg (p \lor q) \equiv (\neg p \cdot \neg q)\]
It is not the case that J cleaned J’s room or J cleaned F’s room. // J did not clean J’s room and J did not clean F’s room.

7. Transposition (Trans)

\((p \supset q) // (\sim q \supset \sim p)\)

If J cleans J’s room then Ms F will give J a treat. // if Ms F did not give J a treat then it must be because J did not clean J’s room

8. Material Implication (MI)

\((p \supset q) // (\sim p \lor q)\)

If J cleans J’s room then Ms Flotmos will give J a treat. // J does not clean J’s room or Ms F will give J a treat

9. Material Equivalence (ME)

\((p \equiv q) // [(p \supset q) \cdot (q \supset p)]\)

…..

\((p \equiv q) // [(p \cdot q) \lor (\sim p \cdot \sim q)]\)

…..
Immediate Inferences using three propositions \((p, q, r)\)

10. Association (Assoc)

\[
[p \lor (q \lor r)] \equiv [(p \lor q) \lor r]
\]
\[
[p \cdot (q \cdot r)] \equiv [(p \cdot q) \cdot r]
\]

11. Distribution (Dist)

\[
[p \cdot (q \lor r)] \equiv [(p \cdot q) \lor (p \cdot r)]
\]

I’m going home and either read or sleep. // I’m going home and I will read or I’m going home and I will sleep.

\[
[p \lor (q \cdot r)] \equiv [(p \lor q) \cdot (p \lor r)]
\]

I’m going home or I will read and sleep. // I’m going home or I will read and I’m going home and I will sleep.

12. Exportation (Exp)

\[
[(p \cdot q) \supset r] \equiv [p \supset (q \supset r)]
\]

....

\[
[p \supset (q \supset r)] \equiv [(p \cdot q) \supset r]
\]

....
Each of the above is a valid immediate inference rule that requires only one premise to infer its conclusion. The following are syllogistic inference rules, and each requires the use of two premises to infer a conclusion. A conclusion, C, is derived from premises, P1 and P2, if there is a rule of inference that justifies inferring C from P1 and P2.

The following four syllogistic rules of inference are widely used in ordinary and professional discourse: Modus Ponens, Modus Tollens, Hypothetical Syllogism, and Disjunctive Syllogism.

13. Modus Ponens (MP) \( p \supset q / p \rightarrow q \)

\( p \supset q \)
\( p \)
\( q \)

If Jamie cleans her room then Ms Flotmos will take Jamie for a treat.
Jamie cleans her room.
Ms Flotmos will take Jamie for a treat.

14. Modus Tollens (MT) \( p \supset q / \neg q \rightarrow \neg p \)

\( p \supset q \)
\( \neg q \)
\( \neg p \)

If Jamie cleans her room then Ms Flotmos will take her for a treat.
Ms Flotmos will not take Jamie for a treat.
Jamie did not clean her room
If it is true that Ms Flotmos will take Jamie for a treat if Jamie cleans her room, and it is also true that Ms Flotmos will not take Jamie for a treat, then we are justified in inferring that Jamie did not clean her room.

15. Hypothetical Syllogism (HS)  
\[ p \supset q / q \supset r // p \supset r \]

- If Jamie cleans her room then Ms Flotmos will take J for a treat.
- If Ms Fotmos takes Jamie for a treat then Jamie will be happy.
- If Jamie cleans her room then Jamie will be happy.

16. Disjunctive Syllogism (DS)  
\[ p \lor q / \neg p // q \]

- Jamie is three years old or four years old.
- Jamie is not three years old.
- Jamie is four years old.

Ms Flotmos is Jamie’s mother or Ms Flotmos has deceived Jamie.

- Ms Flotmos is not Jamie’s mother.
- Ms Flotmos has deceived Jamie.
17. Conjunction: \( p \land q \equiv p \cdot q \)

\[\begin{align*}
& p \\
& q \\
& p \cdot q
\end{align*}\]

Ms Flotmos is Jamie’s mother.
Ms Flotmos has deceived Jamie.
Ms Flotmos is Jamie’s mother and Ms Flotmos has deceived Jamie.

18. Constructive Dilemma (CD)

\[(p \supset q) \cdot (r \supset s) \equiv (p \lor r) \equiv (q \lor s)\]

……

In syllogisms, a conclusion, \( C \), is derived from premises, \( P1 \) and \( P2 \), if there is a rule of inference that justifies inferring \( C \) from \( P1 \) and \( P2 \).

Suppose we are given the following syllogism.

\[\begin{align*}
(p \cdot q \cdot \neg r \cdot \neg m \cdot s \cdot t) \supset (j \lor k \lor \neg m \lor \neg r \lor s \lor t) \\
(p \cdot q \cdot \neg r \cdot \neg m \cdot s \cdot t) \\
(j \lor k \lor \neg m \lor \neg r \lor s \lor t)
\end{align*}\]

This argument has 8 propositions \((p,q,r,m,s,t,j,k)\), so a truth table would have 256 lines. But we can see that the argument has the general form of MP:
We can therefore conclude that it is a valid argument, without having recourse to calculations.

Now, suppose we are given the following argument:

\[(p \cdot q \cdot \sim r \cdot \sim m \cdot s \cdot t) \supset (j \vee k \vee \sim m \vee \sim r \vee s \vee t)\]

\[\sim r \cdot \sim m \cdot s \cdot t \cdot p \cdot q\]

\[(k \vee j \vee \sim m \vee \sim r \vee s \vee t)\]

\[(p \cdot q \cdot \sim r \cdot \sim m \cdot s \cdot t)\] is not identical to \[\sim r \cdot \sim m \cdot s \cdot t \cdot p \cdot q\]. But by using the rule of association, \[(p \cdot q \cdot \sim r \cdot \sim m \cdot s \cdot t)\] can be derived from \[\sim r \cdot \sim m \cdot s \cdot t \cdot p \cdot q\] as follows:

\[\frac{1}{(p \cdot q \cdot \sim r \cdot \sim m \cdot s \cdot t) \supset (j \vee k \vee \sim m \vee \sim r \vee s \vee t)}{P1}\]

\[\frac{2}{\sim r \cdot \sim m \cdot s \cdot t \cdot p \cdot q}{P2}\]

\[\frac{3}{[(\sim r \cdot \sim m \cdot s \cdot t) \cdot (p \cdot q)]} {2, \text{Assoc}}\]

\[\frac{4}{[(p \cdot q) \cdot (\sim r \cdot \sim m \cdot s \cdot t)]} {3, \text{Assoc}}\]

\[\frac{5}{(p \cdot q \cdot \sim r \cdot \sim m \cdot s \cdot t)} {4, \text{Assoc}}\]

\[\frac{6}{(j \vee k \vee \sim m \vee \sim r \vee s \vee t)} {1, 5 \text{ MP}}\]

Consider the following argument:

\[(p \cdot q \cdot \sim r \cdot \sim m \cdot s \cdot t) \supset (j \vee k \vee \sim m \vee \sim r \vee s \vee t)\]

\[\sim (j \vee k \vee \sim m \vee \sim r \vee s \vee t)\]

\[\sim (p \cdot q \cdot \sim r \cdot \sim m \cdot s \cdot t)\]
To prove that this argument is valid using truth-tables would require a truth-table of $2^8 = 256$ lines. However, we can see that the conditional first premise and the negation of the consequent clause of the first premise allows us to derive the negation of the antecedent clause of the first premise by the inference rule MT.

Consider the following argument:

$$(p \land q \land \neg r \land \neg m \land s \land t) \supset (j \lor k \lor \neg m \lor r \lor s \lor t)$$

$$(j \lor k \lor \neg m \lor r \lor s \lor t) \supset (p \land q \land \neg r \land \neg m \land s \land t)$$

$$(p \land q \land \neg r \land \neg m \land s \land t) \supset (p \land q \land \neg r \land \neg m \land s \land t)$$

This argument would also require a truth table of 256 lines to prove its validity. Yet we are convinced that it is a valid argument because it has the HS syllogistic form. And any argument with a HS form is a valid argument.

Likewise we are convinced that the following argument is valid, not because of truth tables, but because we recognize that it has the form of a disjunctive syllogism, and any argument with a DS form is a valid argument:

$$(p \land q \land r \land d \land e \land f) \lor (j \lor k \lor \neg m \lor r \lor s \lor t)$$

$$\neg (p \land q \land r \land d \land e \land f)$$

$$(j \lor k \lor \neg m \lor r \lor s \lor t)$$
Examples:  (need example using HS)

1  p ⊃ (q > r)
2  r v p
3  ~r  //q ⊃ r
4  p     2, 3 DS
5  q ⊃ r     1, 4 MP

1  ~(p ⊃ q) ⊃ (~r > s)
2  (p ⊃ q) ⊃ r
3  ~r   // s
4  ~(p ⊃ q)   2, 3 MT
5  ~r ⊃ s   1, 4 MP
6  s     5, 3 MP

1  p
2  p ⊃ ~q
3  ~q ⊃ ~t
4  t v s   // s
5  ~q     2, 1MP
6  ~t     3, 5 MP
7  s     4, 6 DS
When we make an inference, only certain information out of all the information available to us is relevant. From the possible premises given below, select those that can be used to infer certain of the possible conclusions:

Possible Premises:

1. \( p \supset (q \supset r) \)
2. \( r \lor p \)
3. \( s \lor \neg p \)
4. \( (q \supset r) \supset (p \supset q) \)
5. \( \neg(p \supset q) \supset \neg(r \supset s) \)
6. \( (p \supset q) \supset r \)
7. \( \neg p \supset (q \supset \neg r) \)
8. \( \neg s \supset (\neg r \supset p) \)
9. \( \neg s \)
10. \( \neg r \)

Possible Conclusions:

C1 \( p \)

C2 \( \neg p \)

C3 \( q \)

C4 \( \neg q \)

C5 \( r \)

C6 \( \neg r \supset p \)

C7 \( s \)

C8 \( \neg r \supset p \)
Suppose we are given the following premises:

P1: \((P \supset Q) \cdot (Q \supset \neg P)\)

P2: \(P \lor Q\)

P3: \(\neg Q\)

P4: \((D \supset Q) \cdot (R \supset D)\)

P5: \((Q \supset D) \cdot (E \supset R)\)

P6: \(Q \lor E\)

P7: \(\neg P \supset Q\)

P8: \(R \supset \neg D\)

P9: \(\neg P \lor R\)

P10:
Which of the above ten propositions can be used to derive which of the following conclusions:

C1: E . F
C2: D v Q
C3: Q v ~D
C4: D
C5: E
C6: 
C7

**Natural Deduction: Predicate Logic**

Following are valid inference rules involving quantifiers:

**Quantifier Elimination:**

Suppose we have a finite universe, U1, consisting of individuals a, b, and c. Then in universe U1 the following relationships hold:

(x)Mx // (Ma . Mb . Mc)

(∃x)Mx // (Ma v Mb v Mc)

(Ma . Mb . Mc) // (x)Mx

(Ma v Mb v Mc) // (∃x)Mx

In this way, quantified statements in any finite universe can always be replaced by statements with no quantifier.
Quantifier Negation:

The following inferences hold for quantifiers that are negated:

\[ \neg(x) \ Ax \ // \ (\exists x) \ \neg Ax \]

\[ \neg(\exists x) \ Ax \ // \ (x) \ \neg Ax \]

There are four possible inference rules using the operations of generalization and instantiation.

Two of them are valid and two are invalid.

Universal generalization (UG): \( Fa \ // \ (x) \ Fx \) invalid

This is generally an invalid argument. For there are many cases where what is true of a need not be true of b and c. (There are exceptions to this in certain proof strategies used in natural deduction.)

Existential generalization (EG): \( Fa \ // \ (\exists x) \ Fx \) valid

It is always valid to infer that something has the property F if we already know that a has that property.

Universal instantiation (UI): \( (x)Fx \ // \ Fa \) valid

If it is true that all members of U1 have property F, then if a is a member of U1 then a must have the property F.

Existential instantiation (EI): \( (\exists x) \ Fx \ // \ Fa \) invalid

If we know that some individual in U1 has the property F, we cannot conclude that a must be that individual.

(Exercises needed)
The 18 rules of propositional logic and the additional valid rules of predicate logic make it is possible to do natural deductions in both propositional and predicate logic. These formal languages make it possible to express ordinary and technical inferences in a form that can be used by modern information technology.

(Exercises needed)

4.1. Logic and Computers

4.1.1 People normally process verbal information in much the same way it is processed in truth-functional logic, and inferences such as “P ⊃ Q / P // Q” are intuitively made and recognized as correct. We will now indicate how devices such as calculators and computers are constructed so that they process information in this manner. This is accomplished by first representing propositions in terms of circuit diagrams.

1. Let the propositional form P be represented by the following circuit:

   ![Circuit Diagram]

   Electricity can flow from beginning to end continuously if and only if gate P is down:

   \[
   \begin{array}{cc}
   P & \text{is down} \\
   \text{T} & \text{T} \\
   \text{F} & \text{F}
   \end{array}
   \]

   Path a-b is continuous
2. Let the propositional form \( \neg P \) be represented by the following \textit{inverted circuit}:

\[ \begin{array}{c}
\neg P \\
\end{array} \]

Here, the path is continuous if and only if gate P is not down:

\[
\begin{array}{c|c}
\text{P is down} & \text{Path a-b is continuous} \\
T & F \\
F & T \\
\end{array}
\]

3. Let the propositional form \( P \cdot Q \) be represented by the following \textit{series circuit}:

\[ P \cdot Q \]

Here, the path from a to b is continuous if and only if gate P is down and gate Q is down:

\[
\begin{array}{c|c|c|c}
\text{P (is down)} & \text{and} & \text{Q (is down)} & \text{Path a-b is continuous} \\
T & T & T & T \\
T & F & F & F \\
F & F & T & F \\
F & F & F & F \\
\end{array}
\]
4. Let the propositional form \((P \lor Q)\) be represented by the following parallel circuit:

\[
\begin{array}{c}
P \lor Q \\
P \quad Q
\end{array}
\]

In this diagram, there is a continuous path from a to b in all cases except that in which it is false that P is down and it is false that Q is down:

<table>
<thead>
<tr>
<th>P is down</th>
<th>or</th>
<th>Q is down</th>
<th>Path is continuous</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

When we envision the path from beginning to end as the circuit followed by an electric current, the above diagrams are called switching circuits: diagram 2 is called an inverted circuit; diagram 3 is called a series circuit; and diagram 4 is called a parallel circuit. By giving circuit analogues of the connectives ~, \(\lor\), and \(\land\), it is possible to construct circuits that process electrical signals in exactly the same manner that compound propositions process truth values. The output of a circuit is determined by the combined operation of the switches in the same way as the truth value of a compound proposition is determined by the operation of the truth functional connectives on the atomic propositions.
Some examples of circuit diagrams for compound truth functional propositions are as follows:

<table>
<thead>
<tr>
<th>Propositional Form</th>
<th>Circuit Diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P \cdot \sim Q$</td>
<td><img src="image1" alt="Diagram" /></td>
</tr>
<tr>
<td>$\sim P \cdot Q$</td>
<td><img src="image2" alt="Diagram" /></td>
</tr>
<tr>
<td>$\sim P \cdot \sim Q$</td>
<td><img src="image3" alt="Diagram" /></td>
</tr>
<tr>
<td>$P \lor \sim Q$</td>
<td><img src="image4" alt="Diagram" /></td>
</tr>
</tbody>
</table>
4.1.1. **Exercises:** Draw the circuit diagram for each of the following propositional forms:

1. \( \sim p \lor (\sim q \lor p) \)
2. \( p \land (q \lor \sim p) \)
3. \( (p \land \sim q) \lor (\sim p \lor q) \)
4. \( (p \land q) \lor (\sim p \land \sim q) \)
5. \( (p \lor q) \land (\sim p \lor \sim q) \)

4.1.2  **Disjunctive Normal Forms**

Many propositional forms, such as \( \sim (P \lor Q) \) and \( (P \supset Q) \), can not be represented directly by a circuit diagram. But by deriving propositional forms that are the equivalent of such propositions, it is possible to construct circuits that give equivalent outputs. The propositional form \( (P \supset Q) \) is constructed from the simple propositions \( P \) and \( Q \) and has the following truth table:

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( P \supset Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>
To derive its disjunctive normal form, follow the steps below:

Step 1. On each row where $P \supset Q$ is true, write down the simple propositions that are true and write down the negation of the simple propositions that are false.

Step 2. On each row where $P \supset Q$ is true, form the conjunction of the simple propositions and the negations of the simple propositions as determined in Step 1.

Step 3. Form the disjunction of the conjoined statements determined in Step 2.

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>$P \supset Q$</th>
<th>Step 1</th>
<th>Step 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>$P \land Q$</td>
<td>$P \land Q$</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>$\neg P \land Q$</td>
<td>$\neg P \land Q$</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>$\neg P \land \neg Q$</td>
<td>$\neg P \land \neg Q$</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>$\neg P \land \neg Q$</td>
<td>$\neg P \land \neg Q$</td>
</tr>
</tbody>
</table>

Step 3 is then $(P \land Q) \lor (\neg P \land Q) \lor (\neg P \land \neg Q)$

Because $P \supset Q$ has the same truth table as $(P \land Q) \lor (\neg P \land Q) \lor (\neg P \land \neg Q)$, they are truth-functionally equivalent. $P \supset Q$ also has the same truth table as $\neg P \lor Q$. Thus

$$P \supset Q \equiv (P \land Q) \lor (\neg P \land Q) \lor (\neg P \land \neg Q) \equiv (\neg P \lor Q).$$

The DNF circuit diagram for $P \supset Q$ and all other statements that are truth functionally equivalent to it is thus:
Any compound of two propositions can be represented by a parallel circuit consisting of at most four series circuits by constructing its DNF.

4.1.2. Exercises:

1. Derive the Disjunctive Normal Form of each of the following propositions:
   a. \( P \supset \neg Q \)
   b. \( \neg (P \supset Q) \)
   c. \( P \equiv Q \)
   d. \( \neg P \equiv \neg Q \)
   e. \( \neg P \supset \neg Q \)

2. Draw the circuit diagram of each of the above DNFs.

3. For each of the following propositional forms, draw a circuit diagram that gives equivalent outputs:
   1. \( (P \supset Q) \cdot \neg Q \)
   2. \( (P \cdot Q) \supset (\neg Q \lor \neg P) \)
   3. \( \neg P \equiv (\neg Q \cdot P) \)
4. \( P \supset (Q \supset P) \)
5. \( \neg P \lor (\neg Q = P) \)

I.4 Because there are so many ways of engineering the on-off action of a switch (relays, electro-magnetics, vacuum tubes, transistors, etc.), we ignore details as to how a switch is constructed and concern ourselves only with their inputs and outputs. In computer language, a switch is a “black box” and only its input and output is considered relevant. The logical connectives are conventionally represented as “black boxes” in logic diagrams:

<table>
<thead>
<tr>
<th>Propositional Form</th>
<th>Logic Connective</th>
<th>Computer Gate</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P \cdot Q )</td>
<td>( P \cdot Q )</td>
<td>( P \cdot Q )</td>
</tr>
<tr>
<td>( P \lor Q )</td>
<td>( P \lor Q )</td>
<td>( P \lor Q )</td>
</tr>
<tr>
<td>( \neg P )</td>
<td>( \neg P )</td>
<td>( \neg P )</td>
</tr>
</tbody>
</table>

We have seen that one way of resolving the ambiguity involved in a statement like \( (P \cdot Q \lor R) \) is to represent it in terms of a logic diagram. Logic diagrams indicate the order in which simple propositions are compounded in order to form successive levels of compound propositions. The propositional form \( (P \cdot Q \lor R) \) is ambiguous because it does not tell us
whether P should be grouped with Q and their compound grouped with R, or whether Q should be grouped with R, and then their compound grouped with P. It is ambiguous between the following two propositional forms:

a. \((P \cdot Q) \lor R\)

b. \(P \cdot (Q \lor R)\)

Their respective logic diagrams are as follows:

a. 

\[
\begin{array}{c}
\text{P} \\
\cdot \\
\text{Q} \\
\lor \\
\text{R} \\
\end{array}
\]

b. 

\[
\begin{array}{c}
\text{P} \\
\cdot \\
\text{Q} \\
\lor \\
\text{R} \\
\end{array}
\]

These diagrams make clear the order in which logical operators are to be applied in order to produce an unambiguous propositional form.

4.1.5. **Exercises:** Draw the logic gate diagrams for H2 and H3.

For each row of a truth table, we can take the truth-values of the atomic propositions as inputs and the truth-value of the compound as output. The truth tables can then be represented as follows:

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 2 3 4</td>
<td>T F F F</td>
</tr>
<tr>
<td>T T F F</td>
<td>T F F F</td>
</tr>
<tr>
<td>T F T F</td>
<td>T F F F</td>
</tr>
</tbody>
</table>

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If we adopt the convention of representing T by 1 and F by 0, then we can present the input-output function of series, parallel, and inverted circuits as follows:

By replacing the decimal representation of numbers with the binary representation of numbers, we are able to use the circuit analogues of the logical connectives to add, subtract, and multiply. Thus, inputs to the ‘or’ gate give outputs that are equivalent to the mathematical operation of
addition (when a carry register is incorporated). And inputs to the ‘and’ gate give outputs that are equivalent to the mathematical operation of multiplication.

By replacing the decimal representation of numbers with the binary representation of numbers, we are able to use the circuit analogues of the logical connectives to add, subtract, and multiply. Thus, inputs to the ‘or’ gate give outputs that are equivalent to the mathematical operation of addition (when a carry register is incorporated). And inputs to the ‘and’ gate give outputs that are equivalent to the mathematical operation of multiplication.

Diagram Summaries:

Circuit Diagrams:

1. P:

2. ~P:

3. ~P ⋅ ~Q:

4. P ∨ Q:

5. P ⋅ (~P ∨ ~Q)
i) $\sim P$: Logic Diagram:

Gate Diagram:

ii) $P \land Q$: Logic Diagram:

Gate Diagram:

iii) $P \lor Q$: Logic Diagram:

Gate Diagram:
Write out the output for $P \cdot (\sim P \lor \sim Q)$:

Logic Diagram:

Gate Diagram:

Computer chips combine binary number inputs and produce binary number outputs in accordance with the logical operations of negation, disjunction, and conjunction.
4.1.4. Exercises: For each of the following propositional forms, (a) draw its logic diagram; (b) construct its DNF; (c) draw the circuit diagram for the DNF; (d) draw the gate diagram for the DNF.

1. \( \sim(p \cdot q) \)
2. \( (p \lor q) \sim p \)
3. \( (p \cdot q) \lor \sim q \)
4. \( p \supset q) \lor (p \cdot q) \)
5. \( (p \cdot q) \cdot \sim p \)
6. \( (p \supset q) \lor (q \supset p) \)

**Logic and Probability**

While this text has focused primarily on deductive arguments, it is important to recognize that not all arguments are deductive. Some arguments are inductive. But, unlike deductive arguments, an acceptable inductive argument is not one where the truth of the premises guarantees the truth of the conclusion. Rather, an acceptable inductive argument is one where the truth of the premises makes the truth of the conclusion more or less probable.

Propositional and predicate logic shows how, from the truth values assigned to constituent simple statements, the truth-value of compound statements formed from them can be calculated. Probabilistic logic supplements truth-functional logic by showing how statements that are only probably true can be combined into compounds whose subsequent probabilities can be calculated. It provides us with a way of defining probabilities, a way of combining probability statements, and a way of calculating the probability of compound statements. The modern theory of probability provides the foundation of inductive logic.
A. PROBABILISTIC INFERENCE

I. The Modern Theory of Probability

From time immemorial, human beings have attempted to benefit themselves by taking risks, hoping to succeed where others may have failed. Businessmen, traders, gamblers, and lovers are known for taking chances on uncertain outcomes. But while lovers typically do not act rationally, businessmen and serious gamblers do. And being rational means making choices that have the highest expectation of success and avoiding choices that have the lowest expectation of success.

Gamblers are particularly shrewd observers of games of chance where wealth is wagered, won, and lost. Much of the modern analysis of chance and probability comes from the observations and conjectures of gamblers, many of whom appealed to contemporary mathematicians for help in deciding what bets to make and what bets to avoid. In the 13\textsuperscript{th} century, King Alfonso of Castile (1221-1284) produced seven treatises on dice and other games of chance. In the 16\textsuperscript{th} century, Girolamo Cardano (1501-1576) wrote Liber de Ludo Aleae (The Book on Games of Chances), one of the first written outlines of modern probability.\footnote{Cardano was one of the most famous physicians of his age, but he was also a gambling addict, as were many of the aristocracy. He confessed to “immoderate devotion to table games and dice….During many years I have played not off and on but, as I am ashamed to say, every day.” Bernstein, Peter L. Against the Gods: The Remarkable Story of Risk. New York: John Wiley & Sons, 1996. Print. p. 45} Cardano’s aim was to show how to calculate the probability of outcomes in games of chance so that bettors truly had equal chances of winning. Cardano’s cryptic remarks prefigure what became the mathematical theory of probability where the probability of $E$, $\text{Pr}(E)$, is defined as follows:
Let \( n(G) \) = the total number of outcomes in the game;

\[ n(E) = \text{the total number of outcomes in the game that are } E. \]

Then, if each outcome is equally likely,

\[ \Pr(E) = \frac{n(E)}{n(G)}. \]

In the mathematical theory of probability, the probability of an event \( E \) is the number of outcomes describable as \( E \), divided by the total number of possible outcomes. In other words, the probability that \( E \) is true, \( \Pr(E) \), is based on the proportion of events in our universe of discourse that are \( E \) events. \( \Pr(E) \) can range from a value of zero, which means there is total certainty that \( E \) will be false; to a value of one, which means that there is total certainty that \( E \) will be true. This is expressed as \( 0 \leq \Pr(E) \leq 1 \).

It may seem paradoxical that by using probabilities we can know what to expect, even when the outcome is a matter of chance. But this paradox is resolved when we frame the question in terms of individual events and kinds of events. Each of the individual events of a game has an equal probability of being realized. And if the different kinds each have the same number of elements, then each kind has an equal chance of being picked.

Flipping a coin is one model of choice by chance. There are 2 possible outcomes, H or T, and any flip produces either H up or T up, but not both. Thus,

\[ \Pr(H) = \Pr(T) = \frac{1}{2} \]

Suppose A bets B that the coin will land H up, and B accepts the bet. If the coin is randomly flipped, and lands H up, then A wins. (If the coin is randomly flipped, and lands T up, then B wins.) The bet was a fair bet because each participant had an equal chance of winning. If A was a wealthy merchant and B a poor farmer, we may be appalled at the justice of a wealthy person.
gaining at the expense of the poor person who loses, but the bet was fair. (reference to the ethics of gambling)

Throwing a die is another model of choice by chance. There are 6 possible outcomes, and any throw is either 1 or 2 or 3 or 4 or 5 or 6 (but not more than one of these):

\[ Pr(1) = Pr(2) = Pr(3) = Pr(4) = Pr(5) = Pr(6) = \frac{1}{6} \]

Each number on a die is either even or odd: E(2,4,6) or O(1,3,5). Thus, in throwing a single die, there are two kinds of outcome: even or odd number up, but not both:

\[ Pr(E) = Pr(O) = \frac{3}{6} = \frac{1}{2} \]

drawing a card from a standard deck that has been shuffled is another model of choice by chance. A standard deck consists of 52 distinct cards divided into 4 kinds (suites). Therefore the probability of drawing any particular card is 1/52:

\[ Pr(9H) = Pr(6C) = Pr(2D) = Pr(7S) = \frac{1}{52} \]

There are 4 kinds of outcome: Heart (H), Diamond (D), Spade (S), or Club (C). Since each kind has 13 members,

\[ Pr(H) = Pr(D) = Pr(S) = Pr(C) = \frac{13}{52} = \frac{1}{4} \]

In each of the above examples, each kind of event has an equal chance of occurring. But there are many cases where the different kinds of events in the game may have different chances of occurring. An example would be a roulette wheel (urn of marbles), on which there are 10 (G)reen, 20 (W)hite, 30 (R)ed, and 40 (B)lack slots (marbles). This gives a total of 100 slots (marbles). If we turn (shake) the wheel (urn) and randomly choose a slot (marble) on the wheel (from the urn), the probability of picking a slot (marble) with a certain kind of color would be:

\[ Pr(G) = \frac{10}{100} = \frac{1}{10} \]

\[ Pr(W) = \frac{20}{100} = \frac{2}{10} \]

\[ Pr(R) = \frac{30}{100} = \frac{3}{10} \]

\[ Pr(B) = \frac{40}{100} = \frac{4}{10} \]
In the example with cards, there are four kinds of cards, and each kind has thirteen instances. Thus, Pr(H) = Pr(D) = Pr(S) = Pr(C) = 1/4. But in the roulette wheel example, while there are four kinds of events, each kind has a different number of instances. Thus, Pr(R) ≠ Pr(W) ≠ Pr(B) ≠ Pr(G).

When multiple choices are made from the same deck, we must distinguish choice with replacement of the card chosen from choice without replacement of the card chosen. Thus, with a deck of 52 cards, Pr(8D) = 1/52 and Pr(D) = 13/52 = 1/4. If 8D is chosen, and replaced, then Pr(8D) remains 1/52 and Pr(D) = 13/52 = 1/4. But if 8D is not replaced, then on the next choice, Pr(8D) = 0 and Pr(D) = 12/51.

5.A.1. Probability Exercises for Games of Chance:
A. Deck without wild card with wild card
   1. Pr(5H) =
   2. Pr(H) =
   3. Pr(4D) =
   4. Pr(4) =
   5. Pr(KC) =

B. Assume we have 6 bananas, 7 oranges, 8 apples, and 9 peaches, each in a bag that is indistinguishable from the other bags. What is the probability that the bag you choose will have in it:

   1. an orange?
   2. an apple?
   3. a peach?
   4. a banana?
C. Assume we have 6 bananas, 7 oranges, 8 apples, and 9 peaches, each in a bag that is indistinguishable from the other bags.

1. remove 2 banana bags only; \( \Pr(b) = \)
2. remove 3 apple bags only; \( \Pr(a) = \)
3. remove 4 orange bags only; \( \Pr(o) = \)
4. remove 5 peach bags only; \( \Pr(p) = \)
5. remove all bags with bananas. \( \Pr(a) = \)
6. remove all bags with peaches. \( \Pr(a) = \)

D. Remove 3H, 8H, 3C, 8C from a regular deck. What is:

1. \( \Pr(8D) = \)
2. \( \Pr(8H) = \)
3. \( \Pr(H) = \)
4. \( \Pr(6H) = \)
5. \( \Pr(D) = \)
6. \( \Pr(6D) = \)
7. \( \Pr(C) = \)
8. \( \Pr(2C) = \)
9. \( \Pr(S) = \)

(slot machines, lotteries, blackjack, )
II. The Relative Frequency Theory of Probability

The definition of probability we have developed using games can be extended to events that are not parts of a game. The probability of many kinds of events in everyday life, business, and science can be estimated using the frequency of past events to estimate the probability of similar events in the present and future. Suppose there were 10,000 auto accidents in City A over the last five years, and 9,000 of them involved drivers 20-30 yrs. old, 800 between 30-40, 100 between 40-50 yrs. old, 50 between 50-60 yrs. old, 30 between 60-70 yrs. old, and 20 were > 70 yrs. old. We calculate the probabilities of Auto Accidents in A for each age group as follows:

a. \[ \text{Pr}(\text{AA20-30}) = \frac{9,000}{10,000} = \frac{900}{1000} \]
b. \[ \text{Pr}(\text{AA30-40}) = \frac{800}{10,000} = \frac{80}{1000} \]
c. \[ \text{Pr}(\text{AA40-50}) = \frac{100}{10,000} = \frac{10}{1000} \]
d. \[ \text{Pr}(\text{AA50-60}) = \frac{50}{10,000} = \frac{5}{1000} \]
e. \[ \text{Pr}(\text{AA60-70}) = \frac{30}{10,000} = \frac{3}{1000} \]
f. \[ \text{Pr}(\text{AA>70}) = \frac{20}{10,000} = \frac{2}{1000} \]

If city A in the next year is pretty much the same as city A in the previous years, and an automobile accident takes place in City A, it is highly probable that a 20-30 yr. old was involved, and highly unlikely that someone >70 was involved. If insurance company C1 has a high percentage of 20 yr. olds, then it should expect to pay more auto insurance claims than a company, C2, that has a higher percentage of 50 yr. olds. Accordingly, premiums or deductibles might have to be higher at firm C1 than at firm C2. Or, C1 may keep lower rates but introduce tougher requirements.

Fire insurance rates are determined in similar fashion. Suppose that for the last 10,000 accidental house fires, 8000 were in houses >70 yrs. old, 1000 were in houses 60-70 yrs. old, 800
were in houses 50-60 yrs. old, 100 were in houses 40-50 yrs. old, 50 in houses 30-40 yrs. old, 30
in houses 20-30 yrs. old, and 20 in houses <20 yrs. old.

Yrs. old       # fires       Probability of accidental house fire

a.  < 20       20           20/10,000
b.  20-30      30           30/10,000
c.  30-40      50           50/10,000
d.  40-50      100          100/10,000
e.  50-60      800          800/10,000
f.  60-70      1000         1000/10,000
g.  > 70       8000         8000/10,000

If history has shown that older houses have a higher probability of catching fire, then premiums
or deductibles on older houses are likely to be higher than those on more recently built houses.
The insurance company wagers that the total premiums they collect will exceed their total
payouts for claims. The insurance company is not making a fair bet. The insurance company is
making a bet where the odds are in it’s favor, in order to produce a profit.

The above scenarios use a relative frequency theory of probability, where the
probability of an event E is determined by the frequency with which E has occurred in the past,
relative to the total number of all A cases:

\[ n(A) = \text{number of all past outcomes of A}; \]

\[ n(E) = \text{number of past outcomes of A’s that are E}, \]

\[ \text{Pr (E) = } n(E)/n(A) \]
In estimating the relative frequency of A’s that are E, \( n(E) \), it is easy to be misled by the availability and vividness of the events we take notice of. As a result, there is a tendency to overestimate the frequency of events that are described in personal terms, and underestimate the frequency of events described by impersonal statistical data. Despite the data that Yototas have the highest repair rate, Phyllis may believe Yototas are good cars because her cousin has one, and likes it a lot.

(earthquake insurance & fracking, flood insurance & global warming, horse racing, sports betting, )

5.A.2. Probability Exercises for Relative Frequencies:

1) Let each sample consists of 10 consecutive flips. Record the relative frequencies of H and T in each sample.

2) Place a piece of clear tape on the H side. Record the relative frequencies in 10 throws.

3) In the last year (365 days) the precipitation was 70 days of (ra)in, 61 days of (sn)ow, 40 days of (sl)eeet, and 10 days of (ha)il. For that year, what was:

   a) \( \Pr(Ra) \)

   b) \( \Pr(Sn) \)

   c) \( \Pr(Sl) \)

   d) \( \Pr(Ha) \)

The modern theory of probability derives from using games to provide objective measures of what we should expect, based on the particular parameters of the game used (coins, dice, cards, roulette, ..). But in calculating relative frequencies, we are often called upon to give probability estimates of events that are not as well defined as the events within a game. To illustrate, determining whether a fire was accident or arson is not like determining whether two
dice fall showing snake-eyes\(^5\) or whether five cards are a flush. Part of the problem in deciding what will be counted as an accidental fire is that the criteria for whether something is accidental or not has changed (witchcraft, sorcery, divine curse, lightning, electrical short circuit, spontaneous combustion). We are also sometimes called upon to give probability estimates of events that have not occurred in the past. Thus, how probable is it that a man will become pregnant and give birth to a child within the next 5 yrs., within the next 20 yrs., within the next 100 yrs.? Such estimates involve factors that often may differ from person to person.

\(^5\) (1,1)
III. Rules for Calculating Probabilities

Once the probabilities of our simple propositions have been determined, we can calculate the probabilities of compound propositions formed from them using the truth-functional connectives for negation, conjunction, and disjunction.

**Negation Rule:** $\Pr (\sim A) = 1 - \Pr (A)$

\[
\Pr (A) = \frac{n(A)}{n(G)}
\]

\[
\Pr (\sim A) = \frac{n(\sim A)}{n(G)}
\]

\[
1 = \frac{n(A) + n(\sim A)}{n(G)} = \frac{n(A)}{n(G)} + \frac{n(\sim A)}{n(G)} = \Pr (A) + \Pr (\sim A)
\]

Given $1 = \Pr (A) + \Pr (\sim A)$, the Negation Rule follows:

\[
1 - \Pr (A) = \Pr (\sim A).
\]

In throwing a white die, W, with six sides numbered 1 through 6, $\Pr (W2) = 1/6$. Thus,

\[
\Pr (\sim W2) = 1 - \Pr (W2) = 1 - 1/6 = 5/6.
\]

The **odds** that E will occur is the ratio of E events to $\sim$E events:

\[
O(E) = \frac{n(E)}{n(\sim E)}
\]

Since $n(G) = n(E) + n(\sim E)$, the odds tell us how often we can expect E events relative to $\sim$E events. If $n(E)$ is greater than $n(\sim E)$, then betting on E is more likely to produce wins than betting on $\sim$E. On the other hand, if $n(E)$ is less than $n(\sim E)$, then betting on $\sim$E is more likely to produce wins than bets on E. Thus, $O(W2) = 1:5$ and $O(\sim E) = 5:1$. When $n(E) \neq n(\sim E)$, payoffs have to be adjusted in order to make the bet fair.
5. A. 3. Exercises on determining simple probabilities and odds with a deck of 52 cards:

1. J proposes to K: If you are dealt a Heart then I’ll give you $1. If you are not dealt a Heart, then you give me $1. Is this a fair bet?

2. Without wild card:
   a. \( P(H) = \)
   b. \( Pr(\sim H) = \)
   c. \( O(H: \sim H) = \)

3. With 1 wild card:
   a. \( P(H) = \)
   b. \( Pr(\sim H) = \)
   c. \( O(H: \sim H) = \)

4. With 2 wild cards:
   a. \( P(H) = \)
   b. \( Pr(\sim H) = \)
   c. \( O(H: \sim H) = \)

5. Suppose Joan receives $1 each time she throws shows W2 and pays $1 each time she does not throw W2 with a fair die. Is this a fair bet?
   a. \( Pr(W2) = \)
   b. \( Pr(\sim W2) = \)
   c. \( O(W2: \sim W2) = \)

6. Drawing an ace from a standard deck of cards:
   a. \( Pr(A) = \)
   b. \( Pr(\sim A) = \)
   c. \( O(A: \sim A) = \)
7. Suppose we are given 6 bananas, 7 oranges, 8 apples, and 9 peaches, each in a box that is indistinguishable from the other boxes. Determine the following values:

a. \( \Pr(B) = \) \( \Pr(\neg B) = \) O(B: \( \neg B) = \)

b. \( \Pr(O) = \) \( \Pr(\neg O) = \) O(O: \( \neg O) = \)

c. \( \Pr(A) = \) \( \Pr(\neg A) = \) O(A: \( \neg A) = \)

d. \( \Pr(P) = \) \( \Pr(\neg P) = \) O(P: \( \neg P) = \)

**Conjunction Rule:** \( \Pr(A \cdot B) = \Pr(A) \times \Pr(B) \)

where A and B are independent events

The probability of a conjunctive compound is the product of the probabilities of each conjunct, so long as the conjuncts are independent of one another. Two outcomes, A and B, are independent of one another if the occurrence of one has no influence on the occurrence of the other. We can illustrate this with the example of throwing a white die (W) and a black die (B).


Throws of the black die, B: B1, B2, B3, B4, B5, B6.

The probability of throwing W2 is 1/6. The probability of throwing B5 is 1/6. Thus, the probability of throwing W2 and B5 is:

\( \Pr(W2 \cdot B5) = \Pr(W2) \times \Pr(B5) = 1/6 \times 1/6 = 1/36. \)

But this is only because the occurrence of W2 is independent of the occurrence of B5.

Many situations are such that what happens on one occasion can affect the probability of what happens subsequently. In a deck of cards, there are 13 hearts, 13 diamonds, 13 clubs, and 13 spades, giving a total of 52 cards. The probability of pulling a heart on the first draw is
n(H)/n(cards) = 13/52. However, the probability of pulling a heart on the second draw is 13/52 if the first card was replaced, but is 13/51 if the first card was not a heart and was not replaced; and is 12/51 if the first card was a heart and was not replaced. Given two outcomes, if the first outcome, A, alters the probability of the second outcome, B, then A and B are not independent. In cases where A and B are events that are not independent, the Conjunction Rule as given above can not be applied, and must be expressed as follows:

Expanded Conjunction Rule: Pr(A • B) = Pr(A) x Pr(B given A)

**Disjunction Rule:**

\[
Pr (A \lor B) = Pr (A) + Pr (B) \\
\text{where A and B are exclusive events}
\]

The probability of a disjunctive compound is the sum of the probabilities of each disjunct, so long as the disjuncts are exclusive. Recall that ‘\lor’ allows for inclusive disjuncts, where both disjuncts may be true. But the disjunction rule in probability is only applicable for exclusive disjuncts, where if one of the disjuncts is true, then all other disjuncts must be false.

Throwing a **single die** illustrates the application of the disjunction rule. A die (W) has six sides – W1,W2,W3,W4,W5,W6 – and they are mutually exclusive. If the die falls showing a particular number, it can show none of the other numbers on that throw. Thus, the probability that W will show a 2 or a 5 on a single throw is

\[
Pr (W2 \lor W5) = Pr (W2) + Pr(W5) = 1/6 + 1/6 = 1/3.
\]

And the probability that a single die will show a 1 or 2 or 3 or 4 or 5 or 6 on a single throw is

\[
Pr (W1 \lor W2 \lor W3 \lor W4 \lor W5 \lor W6) = \\
Pr (W1) + Pr (W2) + Pr (W3) + Pr (W4) + Pr (W5) + Pr (W6) = \\
1/6 + 1/6 + 1/6 + 1/6 + 1/6 + 1/6 = 1
\]
Now suppose we are throwing, not one, but two dice: one white and one black. Then all possible combinations are:

<table>
<thead>
<tr>
<th>W1B1</th>
<th>W2B1</th>
<th>W3B1</th>
<th>W4B1</th>
<th>W5B1</th>
<th>W6B1</th>
</tr>
</thead>
<tbody>
<tr>
<td>W1B3</td>
<td>W2B3</td>
<td>W3B3</td>
<td>W4B3</td>
<td>W5B3</td>
<td>W6B3</td>
</tr>
<tr>
<td>W1B4</td>
<td>W2B4</td>
<td>W3B4</td>
<td>W4B4</td>
<td>W5B4</td>
<td>W6B4</td>
</tr>
<tr>
<td>W1B5</td>
<td>W2B5</td>
<td>W3B5</td>
<td>W4B5</td>
<td>W5B5</td>
<td>W6B5</td>
</tr>
<tr>
<td>W1B6</td>
<td>W2B6</td>
<td>W3B6</td>
<td>W4B6</td>
<td>W5B6</td>
<td>W6B6</td>
</tr>
</tbody>
</table>

The probability of throwing a 2 with the white die or of throwing a 5 with the same die on a given throw is

$$Pr(W2 \lor W5) = Pr(W2) + Pr(W5) = \frac{1}{6} + \frac{1}{6}.$$ 

But the probability of throwing a 2 with the white die or of throwing a 5 with the black die is not

$$Pr(W2 \lor B5) = Pr(W2) + Pr(B5) = \frac{1}{6} + \frac{1}{6}.$$ 

This is because throwing W2 excludes the possibility of throwing W5 on that throw. But throwing W2 does not exclude the possibility of also throwing B5. Both W2 and B5 may occur on the same throw. W2 and W5 are exclusive. W2 and B5 are not exclusive.

In cases where A and B are not exclusive events, the disjunction rule is:

**Expanded Disjunction Rule:**  
$$Pr(A \lor B) = Pr(A) + Pr(B) - Pr(A \cdot B)$$

Thus, the probability of W2 or B5 when throwing two dice is

$$Pr(W2 \lor B5) = Pr(W2) + Pr(B5) - Pr(W2 \cdot B5) =$$

$$(\frac{6}{36} + \frac{6}{36}) - [Pr(W2) \times Pr(B5)] = \frac{12}{36} - \frac{1}{36} = \frac{11}{36}.$$ 

Each of the 36 possible combinations of throwing two dice has an equal probability of occurring, which is 1/36. But when the outcomes of rolling 2 dice are grouped in terms of their sums, the probabilities of those sums are not equal.
Let the number of black and white pairs that sum to $k$ be called $n(sum \ k)$. Then

$$n(sum \ k) = \text{the number of pairs that sum to } k.$$ 

And the 36 combinations of throwing two dice produce sums from 2 through 12 are as follows:

Sum 2 = (W1,B1)  
Sum 3 = (W1,B2), (W2,B1)  
Sum 4 = (W1,B3), (W2,B2), (W3,B1)  
Sum 5 = (W1,B4), (W2,B3), (W3,B2), (W4,B1),  
Sum 6 = (W1,B5), (W2,B4), (W3,B3), (W4,B2), (W5,B1),  
Sum 7 = (W1,B6), (W2,B5), (W3,B4), (W4,B3), (W5,B2), (W6,B1),  
Sum 8 = (W6,B2), (W5,B3), (W4,B4), (W3,B5), (W2,B6),  
Sum 9 = (W6,B3), (W5,B4), (W4,B5), (W3,B6),  
Sum 10 = (W6,B4), (W5,B5), (W4,B6),  
Sum 11 = (W6,B5), (W5,B6),  
Sum 12 = (W6,B6)

If $n(sum \ k) = \text{the number of B and W pairs that sum to } k$, then

$$n(sum \ 2) = 1 \quad n(sum \ 3) = 2 \quad n(sum \ 4) = 3 \quad n(sum \ 5) = 4,$$

$$n(sum \ 6) = 5 \quad n(sum \ 7) = 6 \quad n(sum \ 8) = 5 \quad n(sum \ 9) = 4,$$

$$n(sum \ 10) = 3 \quad n(sum \ 11) = 2 \quad n(sum \ 12) = 1.$$
Thus,

\[ \Pr(\text{sum}2) = 1/36 \quad \Pr(\text{sum}3) = 2/36 \quad \Pr(\text{sum}4) = 3/36 \]
\[ \Pr(\text{sum}5) = 4/36 \quad \Pr(\text{sum}6) = 5/36 \quad \Pr(\text{sum}7) = 6/36 \]
\[ \Pr(\text{sum}8) = 5/36 \quad \Pr(\text{sum}9) = 4/36 \quad \Pr(\text{sum}10) = 3/36 \]
\[ \Pr(\text{sum}11) = 2/36 \quad \Pr(\text{sum}12) = 1/36 \]

The probability of each individual outcome of rolling two dice is 1/36. But the probabilities of the sums of the two dice are not equal. Thus, \( \Pr(W2,B2) \) is different from \( \Pr(\text{sum}4) \) because there is only one way for the two dice to simultaneously show a 2, but there are three ways that two dice can sum to 4: \((W1,B3), (W2,B2), (W3,B1)\). This is why an even money bet on rolling a \((2,2)\) is not as good as an even money bet on rolling a sum 4.

5.A.4. Exercises on negation, conjunction, disjunction:

1. What is the probability of getting at least one tail in three tosses of a coin?

2. Is an even money bet that you will not throw a 1 on any of three successive throws of a die a fair bet? Explain.

3. Calculate the \( \Pr \) and odds when flipping 2 Coins:
   a. \( \Pr(\text{HH}) = \quad \text{O(\text{HH}: \sim\text{HH})} = \)
   b. \( \Pr(\text{TT}) = \quad \text{O(\text{TT}: \sim\text{TT})} = \)
   c. \( \Pr(\text{TH}) = \quad \text{O(\text{TH}: \sim\text{TH})} = \)
   d. \( \Pr(\text{HT}) = \quad \text{O(\text{HT}: \sim\text{HT})} = \)

4. Calculate the \( \Pr \) and odds for throwing all combinations of 2 Dice.
5. \[\text{Sum 6} = (W1.B5) \lor (W2.B4) \lor (W3.B3) \lor (W4.B2) \lor (W5.B1)\]
   a. \[\Pr(\text{Sum 6}) = \]
   b. \[\Omega(\text{Sum 6}) = \]
   c. \[\text{Sum 7} = \]
   d. \[\Pr(\text{Sum 7}) = \]
   e. \[\Omega(\text{Sum 7}) = \]

6. Let \[\text{sum } n_k = \text{sum } n \text{ on throw } k\]. Thus, \[\text{sum 6}_5 = \text{sum 6} \text{ on throw five}\].
   a. \[\Pr(\text{sum 6}_1 \text{ or sum 5}_1 \text{ or sum 4}_1 \text{ or sum 3}_1) = \]
   b. \[\Omega(\text{sum 6}_1 \text{ or sum 5}_1 \text{ or sum 4}_1 \text{ or sum 3}_1) = \]
   c. \[\Pr[(\text{sum 6}_1 \text{ and sum 6}_2) \text{ or } (\text{sum 6}_3 \text{ and sum 6}_4)] = \]
   d. \[\Omega[(\text{sum 6}_1 \text{ and sum 6}_2) \text{ or } (\text{sum 6}_3 \text{ and sum 6}_4)] = \]
   e. \[\Pr(\text{sum 6}_1 \text{ and sum 6}_2) = \]
   f. \[\Omega(\text{sum 6}_1 \text{ and sum 6}_2) = \]
   g. \[\Pr(\text{sum 6}_3 \text{ or sum 6}_4) = \]
   h. \[\Omega(\text{sum 6}_3 \text{ or sum 6}_4) = \]

7. Assume a standard deck of cards is shuffled, without wild card and with replacement:
   a. \[\Pr(2 \text{ cards drawn of the same suit}) = \]
   b. \[\Pr(3 \text{ cards drawn of the same suit}) = \]
   c. \[\Pr(\text{ace high straight: five sequential cards with ace high}) = \]
   d. \[\Pr(\text{flush: five cards of the same suit}) = \]

8. Show why an even money bet on rolling a sum 4 is not as good as an even money bet on rolling a sum 7.
Origins of The Modern Theory of Probability

Around the year 1605, the founder of modern science, Galileo Galilei, was asked by his patron, the Grand Duke Cosmo II, for help in solving the following gambling problem:

“Three dice are thrown: … long observation has made dice players consider (sum) ten to be more advantageous than (sum) nine. Why?”

Proceeding as Cardano had indicated, Galileo listed all the \(6^3 (=216)\) possible combinations of 3 dice, and then listed those combinations that produce 9 when summed, and those combinations that produce 10 when summed. He showed that there were 25 combinations which summed to 9, and 27 combinations that summed to 10.\(^6\) Thus

\[
\text{Pr (sum 9)} = \frac{25}{216} \quad \text{Pr (sum 10)} = \frac{27}{216}.
\]

In a circuit of 216 throws, 3 dice will sum 10 more often than they sum 9 (on the average), confirming the Grand Duke’s suspicion.

Another 17\(^{th}\) century gambling enthusiast, the Chevalier de Mere (1607-1684), would bet even money that he could get at least one six in every four rolls of a die. This seemed counter-intuitive because we would expect, on the average, one six in every six throws of a die. Thus, de Mere was able to entice many to bet against him. But de Mere’s conjecture proved correct, and he won a considerable amount of money on the wager.

What is the probability that 6 will turn up in four throws of a six sided die? Let \(6_1 = 6\) on throw 1 and \(6_n = 6\) on throw \(n\). Then it is tempting to represent this problem as follows:

\[
\text{Pr}(6_1 \text{ or } 6_2 \text{ or } 6_3 \text{ or } 6_4) = \text{Pr}(6_1) + \text{Pr}(6_2) + \text{Pr}(6_3) + \text{Pr}(6_4) =
\]

\[
\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}.
\]

\(^6\) For 9, (1,2,6) appears 6 times, (1,3,5) appears 6 times, (1,4,4) appears 3 times, (2,2,5) appears 3 times, (2,3,4) appears 6 times, (3,3,3) appears 1 time. So throwing a total of 9 can appear 25 times in all; For 10, (1,3,6) appears 6 times, (1,4,5) appears 6 times, (2,4,4) appears 3 times, (2,2,6) appears 3 times, (2,3,5) appears 6 times, (3,3,4) appears 3 time. So throwing a total of 10 can appear 27 times in all. Therefore, the chance of throwing a total of 9 with three fair dice was less than that of throwing a total of 10.
This would lead us to expect the following:

\[
Pr(6_1 \text{ or } 6_2 \text{ or } 6_3 \text{ or } 6_4 \text{ or } 6_5 \text{ or } 6_6) = \\
Pr(6_1) + Pr(6_2) + Pr(6_3) + Pr(6_4) + Pr(6_5) + Pr(6_6) = \\
1/6 + 1/6 + 1/6 + 1/6 + 1/6 + 1/6 = 1
\]

On this line of reasoning, we are certain to get 6 in six throws of a die, and more than certain to get it in seven throws. Using the Disjunction Rule in this way commits the fallacy of treating the constituent events as if they were mutually exclusive when they are not. Representing the problem as \( Pr [(6_1) \text{ or } (6_2) \text{ or } (6_3) \text{ or } (6_4)] \) hides the fact that each of these alternatives could be true. They are not exclusive alternatives.

The problem is correctly represented by asking for the probability of not getting a 6 in either of the four throws. Using the Negation Rule \( Pr (A) + Pr (~A) = 1 \), it follows that \( Pr (A) = 1 - Pr (~A) \). Thus, the probability of getting a 6 in four throws is one minus the probability of not getting a 6 on throw 1 or on throw 2 or on throw 3 or on throw 4:

\[
Pr (6 \text{ in } 4 \text{ throws}) = 1 - Pr (~6 \text{ in } 4 \text{ throws}) = 1 - Pr (~6_1 \cdot ~6_2 \cdot ~6_3 \cdot ~6_4).
\]

Since \((~6_1), (~6_2), (~6_3), \text{ and } (~6_4)\) are independent events, using the Conjunction Rule we get:

\[
1 - Pr (~6_1 \cdot ~6_2 \cdot ~6_3 \cdot ~6_4) = 1 - [Pr (~6_1) \times Pr (~6_2) \times Pr (~6_3) \times Pr (~6_4)] = \\
1 - \{5/6 \times 5/6 \times 5/6 \times 5/6\} = 1 - 625/1296 = 1296/1296 - 625/1296 = 671/1296
\]

It follows that in betting even money on 6 in four throws, the Chevalier de Mere could, on the average, expect to win 671 times and lose 625 times in a circuit of \(6^4 (=1296)\) throws. The odds of throwing a 6 in four throws are 671: 625, slightly in favor of the Chevalier. But because this occurs only on the average over every 1296 throws, it is unlikely to be detected by the casual observer. Yet, this slight advantage in the odds gave the Chevalier a respectable 7% profit on the wager.
Cardano began his inquiries with the ethics of gambling in order to determine when a bet was fair. In the modern theory of probability a fair bet is one in which the odds of winning are equal to the odds of losing. But the Chevalier was not looking to make a fair bet. He, like most gamblers, wanted a bet where the odds were in his favor, but not noticeably so. In such cases, most losers tend to be unaware of the true source of their losses, and instead blame their bad luck on a spiritual affliction of some sort. Many seek a remedy through charms and prayers, instead of through a rational appraisal of the odds. When the Chevalier lost money on a similar bet he sought an explanation from the mathematician Fermat. Fermat in turn involved the mathematician Blasé Pascal, and together they introduced the basis for the modern theory of probability.

5.A.5. Exercises:

1. \( \Pr(6 \text{ in 2 throws}) = \)

2. \( \Pr(6 \text{ in 3 throws}) = \)

3. \( \Pr(6 \text{ in 5 throws}) = \)

4. \( \Pr(6 \text{ in 6 throws}) = \)

5. What is the probability of getting heads each time in three throws of a coin?

6. What is the probability of rolling two dice such that they sum three in each of three consecutive throws?

7. Out of 36 possible combinations of a pair of dice, only one, W6B6, gives a sum of 12. So we would expect a pair of dice to sum twelve once in every thirty six throws, but not in every twenty-four throws. However, the Chevalier de Mere wagered that in a sequence of 24 rolls of a pair of dice, he would roll at least one twelve. Is this a good bet?

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7 One would expect that \( (6,6) \) would occur only once in every 36 throws of two dice. The Chevalier bet he could throw \( (6,6) \) in 24 throws of two dice.
8. What are the odds that 2 will show in three throws of a 6-sided die? [Cardano could expect to win (on the average) 91 times and lose 125 times for every circuit of 216 throws.]

9. There are ten balls in a bag. 7 are Green and 3 are Blue. Jane is blindfolded, the bag is thoroughly shaken, and Jane draws a ball without looking. What is the probability that the ball will be: B? G?

10. Two dice are thrown. What are the odds that: (a) the tops of the two thrown dice sum 2; or (b) the tops of the two thrown dice sum 6.

11. Jonita has flipped a fair coin three times, and each time it has come up heads. What are the odds that the coin will come up heads on the next flip?

**Risk** is the probability that a specific harm will occur from a specific choice. Risk analysis allows us to describe and quantify the possibilities of incurring losses. It helps make clear how certain kinds of choices are more likely to lead to a harm than other choices.

If \( P(E) \) = probability of \( E \) and \( V(E) \) = value of \( E \), then the risk associated with \( E \) is

\[
R(E) = Pr(E) \times V(E). \quad 8
\]

Suppose, on a scale of 0 to 100, the value of a benign accident \( V(\text{Ab}) = 0 \) while the value of a fatal accident \( V(\text{Af}) = 100 \). Then the value of a fatal accident using motorcycle \( V(\text{Afm}) = \) value of fatal accident using auto \( V(\text{Afa}) = \) value of fatal accident using public transport \( V(\text{Afp}) = 100 \). But if \( \text{Pr}(\text{Afm}) = 20/1000, \text{Pr}(\text{Afa}) = 10/1000, \) and \( \text{Pr}(\text{Afp}) = 5/1000, \) then

\[
\begin{align*}
R(\text{Afm}) &= 2; \\
R(\text{Afa}) &= 1; \\
R(\text{Afp}) &= .5
\end{align*}
\]

This shows how, based on the relative frequencies given, the risk of a fatal accident using a motorcycle is twice that using an auto and four times that using public transport.

We take risks in order to obtain benefits we otherwise might not get. For the benefit of a regular salary, we accept the increased risk of an accident traveling to work. For the benefit of a

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8 Also called Expected Value: See Against the Gods: The Remarkable Story of Risk by Peter Bernstein, p.103. see also Risk and Rationality by K.S.Shrader-Frechette, chpt.10. Calculated Risks by J.Rodricks, chpt. 10.
warm house in the winter, we risk our house catching fire. For the benefit of vegetables and fruits during the winter, we increase our risk of botulism poisoning. For the benefits of increased industrialization, we increase our risks from exposure to chemical, biological, and mechanical hazards. We may buy insurance in order to guard against the exceptional losses that certain risks entail, but it is impossible to insure against all risks.

The possibility of incurring a loss is perceived and evaluated differently by different individuals. Some people are risk-aversive, and prefer to avoid losses rather than acquire benefits. Some people are risk-takers, and are prepared to suffer a loss in exchange for the opportunity to gain something they value. Some people are calculative, and prefer to take only risks that give them the highest odds of winning. One person might decline the opportunity for a great benefit in order to avoid the slightest risk to their family. Another might take great risks in order to provide benefit for their family. There is no one way that all people describe their experiences, assign probabilities, and take risks.

The mathematical theory of probability derives from games in which there are a finite number of clearly specifiable outcomes. Thus, the probability of drawing a spade from a regular deck of playing cards is \(\frac{1}{4}\), because there are 52 cards total, and 13 spades: \(\frac{13}{52} = \frac{1}{4}\). But relative frequencies are not as clear-cut as features of games. Determining the probability that a particular house fire was accident or arson is in many cases a matter of contention. Likewise, deciding whether (publicly provided) bikes will count as public transportation may be contentious. And this will in turn affect the probabilities we assign.

The features of real life are often not as well-defined as features of a game. There is not always one right way to describe real life events that objectively determines their probability. This makes it difficult to identify all the different outcomes there are, estimate the probability of
the different outcomes identified, and rank how important those outcomes are for the decision maker. As such, there may be no one right way of estimating a real life event’s probability. But for any particular set of assumptions that we do make, the logic of probability provides us with a way to estimate the probabilities of the more complex events that derive from those assumptions. The modern theory of probability provides us with a way of exploring the consequences of different assumptions so that we can avoid bad bets, losing investments, and faulty inferences.