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Characterizing Sparse Graphs by Map Decompositions

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Abstract

A **map** is a graph that admits an orientation of its edges so that each vertex has out-degree exactly 1. We characterize graphs which admit a decomposition into k edge-disjoint maps after: (1) the addition of *any* ℓ edges; (2) the addition of *some* ℓ edges. These graphs are identified with classes of *sparse* graphs; the results are also given in matroidal terms.

1 Introduction and related work

Let $G = (V, E)$ be a graph with n vertices and m edges. In this paper, graphs are multigraphs, possibly containing loops. For a subset $V' \subset V$, we use the notation $E(V')$ to denote the edges spanned by V'; similarly, $V(E')$ denotes the vertex set spanned by E'.

A graph $G = (V, E)$ is (k, ℓ)-**sparse**, or simply sparse,[1] if no subset V' of n' vertices spans more than $kn' - \ell$ edges; when $m = kn - \ell$, we call the graph **tight**.

Our interest in this problem stems from our prior work on pebble game algorithms [7, 8]. The (k, ℓ)-pebble game takes as its input a graph, and

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[1]For brevity, we omit the parameters k and ℓ when the context is clear.

outputs **tight**, **sparse** or **failure** and an orientation of a sparse subgraph of the input. We had previously considered the problem in terms of tree decompositions, suggesting the natural range of $k \leq \ell \leq 2k - 1$. In, fact, the pebble game generalizes to the range $0 \leq \ell \leq 2k - 1$. In this paper we examine the graphs that the general pebble game characterizes.

A **map** is a graph that admits an orientation of its edges so that each vertex has out-degree exactly 1. This terminology and definition is due to Lovász [9]. This class of graphs is also known as the bases of the **bicycle matroid** [12] or **spanning pseudoforests** [2], where the equivalent definition of having at most one cycle per connected component is used.

Our choice of the former definition is motivated by the pebble game algorithms. In the $(k, 0)$-pebble game, the output orientation of a tight graph has out-degree exactly k for every vertex. The motivation for studying the pebble game was to have a good algorithm for recognizing sparse and tight graphs. These compute an orientation of a sparse graph that obeys a specific set of restrictions on the out degree of each vertex.

The focus of this paper is the class of graphs that decompose into k edge-disjoint **maps** after the addition of ℓ edges; we call such a graph a k-**map**. Our goal is to extend the results on adding $\ell - k$ edges to obtain k edge-disjoint spanning trees [3] to the range $0 \leq \ell \leq k - 1$. A theorem of [7] identifies the graphs recognized by the (k, ℓ)-pebble game as (k, ℓ)-sparse graphs.

The complete graph K_4 in Figure 1(a) is $(2, 2)$-tight; i.e., adding any two edges to K_4 we obtain a 2-map. The graphs in Figure 1(b) and Figure 1(c) are obtained by adding two edges to K_4; the edges are dashed and oriented to show a decomposition into two maps.
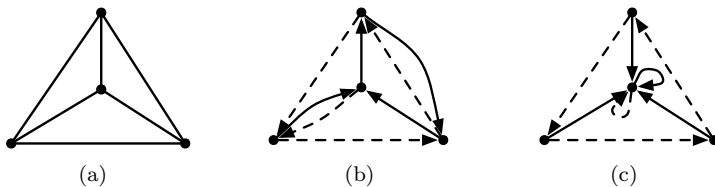

(a) (b) (c)

Figure 1: Adding any two edges to K_4 results in two maps.

White and Whiteley [20] observe the matroidal properties of sparse graphs for $0 \leq \ell \leq 2k - 1$ in the context of bar-and-joint rigidity for frameworks embedded on surfaces [19]. In [15], Szegő characterized exactly when tight graphs exist.

We also state our results in the context of matroid truncations. If $\mathcal{M} = (E, \mathcal{I})$ is a matroid given by its independent sets, then the **truncation** of $\mathcal{M}$ is the matroid $(E, \{E' \in \mathcal{I} : |E'| \leq k\})$, for some nonnegative integer k.

See, e.g., [12] for a more complete treatment of the topic of truncations.

The connection between sparse graphs and decompositions into edge-disjoint spanning trees has been extensively studied. The classical results of Tutte [18] and Nash-Williams [11] show the equivalence of (k, k)-tight graphs and graphs that can be decomposed into k edge-disjoint spanning trees; such a graph is called a k-**arborescence**. A theorem of Tay [16, 17] relates such graphs to generic rigidity of bar-and-body structures in arbitrary dimension.

The particular case in which $k = 2$ and $\ell = 3$ has an important application in rigidity theory: the minimally $(2, 3)$-sparse graphs, known as Laman graphs, correspond to minimally generically rigid bar-and-joint frameworks in the plane [6]. Crapo [1] showed the equivalence of Laman graphs and those graphs that have a decomposition into 3 edge-disjoint trees such that each vertex is incident to exactly 2 of the trees; such a decomposition is called a $3T2$ **decomposition**.

Of particular relevance to our work are results of Recski [13, 14] and Lovasz and Yemini [10], which identify Laman graphs as those that decompose into two spanning trees after doubling any edge. In [4, 5] Hendrickson characterized Laman graphs in terms of the existence of certain bipartite matchings. Stated in the terminology of this paper, the results of [4] show the Laman graphs are precisely those that decompose into 2 edge-disjoint maps after any edge is quadrupled.

The most general results linking sparse graphs to tree decompositions are found in Haas [3], who shows the equivalence of sparsity, adding $\ell - k$ edges to obtain a k-arborescence, and ℓTk decompositions for the case where $k \leq \ell \leq 2k-1$. Our results provide an analog of the first equivalences in terms of graphs which decompose into k edge-disjoint maps.

Another decomposition theorem involving sparse graphs is due to Whiteley, who proved in [19] that for the range $0 \leq \ell \leq k - 1$, the tight graphs are those that can be decomposed into ℓ edge-disjoint spanning trees and $k - \ell$ edge-disjoint maps.

2 Our Results

Our results characterize the graphs which admit a decomposition into k edge-disjoint maps after adding ℓ edges. Since the focus of this paper is on the families of matroidal sparse graphs, we assume that $0 \leq \ell \leq 2k - 1$ unless otherwise stated.

First we consider the case in which we may add *any* ℓ edges, including multiple edges and loops, to G. Let $K_n^{k,2k}$ be the complete graph on n vertices with k loops on each vertex and edge multiplicity $2k$. It is easily seen that any sparse graph is a subgraph of $K_n^{k,2k}$, and we assume this in

the following discussion.

Theorem 1. *Let $G = (V, E)$ be a graph on n vertices and $kn - \ell$ edges. The following statements are equivalent:*

1. *G is (k, ℓ)-sparse (and therefore tight).*

2. *Adding any ℓ edges from $K_n^{k,2k} - G$ to G results in a k-map.*

Theorem 1 directly generalizes the characterization of Laman graphs in [4]. It also generalizes the results of Haas [3] to the range $0 \le \ell \le k - 1$.

As an application of Theorem 1 we obtain the following decomposition result.

Corollary 2. *Let $0 \le \ell \le k$. Let G be a graph with n vertices and $kn - \ell$ edges. The following statements are equivalent:*

1. *G is the union of ℓ edge-disjoint spanning trees and $k - \ell$ edge-disjoint maps.*

2. *Adding any ℓ edges to G results in a k-map.*

We also characterize the graphs for which there are *some* ℓ edges that can be added to create a k-map.

Theorem 3. *Let $G = (V, E)$ be a graph on n vertices and $kn - \ell$ edges. The following statements are equivalent:*

1. *G is $(k, 0)$-sparse.*

2. *There is some set of ℓ edges, which when added to G results in a k-map.*

Stating Theorem 3 in matroid terms, we obtain the following.

Corollary 4. *Let $\mathcal{N}_{k,\ell}$ be the family of graphs G such that $m = kn - \ell$ and G is $(k, 0)$-sparse. Then $\mathcal{N}_{k,\ell}$ is the class of bases of a matroid that is a truncation of the k-fold union of the bicycle matroid.*

Generalizing Theorem 1 and Theorem 3 we have the following theorem.

Theorem 5. *Let $G = (V, E)$ be a graph on n vertices and $kn - \ell - p$ edges and let $0 \le \ell + p \le 2k - 1$. The following statements are equivalent:*

1. *G is (k, ℓ)-sparse.*

2. *There is some set P of p edges which when added to G results in a graph $G' = (V, E \cup P)$, such that adding any ℓ edges to G' (but no more than k loops per vertex) results in a k-map.*

In the next section, we provide the proofs.

4

3 Proofs

The proof of Theorem 1 relies on the following lemma.

Lemma 6. *A graph G is a k-map if and only if G is $(k,0)$-tight.*

Proof. Let $B_k(G) = (V_k, E, F)$ be the bipartite graph with one vertex class indexed by E and the other by k copies of V. The edges of $B_k(G)$ capture the incidence structure of G. That is, we define $F = \{v_i e : e = vw, e \in E, i = 1, 2, \ldots, k\}$; i.e., each edge vertex in B is connected to the k copies of its endpoints in $B_k(G)$. Figure 2 shows K_3 and $B_1(K_3)$.

Figure 2: $B_1(K_3)$ is shown on the right with the one copy of V at the top. The style of line of the edges on the left matches the style of line of the vertex in the bipartite graph corresponding to that edge.
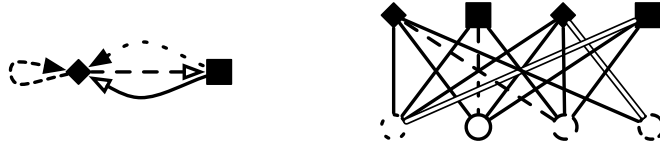

Figure 3: $B_2(G)$ for the graph G on the left is shown on the right with the two copies of V at the top. G is a 2-map; one possible decomposition is indicated by the orientation of the edges and the style of arrow heads. The matching corresponding to this decomposition is indicated in the bipartite graph by dashed and doubled edges.

Observe that for $E' \subset E$, $N_{B_k(G)}(E')$, the neighbors of E' in $B_k(G)$ of E', are exactly the k copies of the vertices of the subgraph spanned by E' in G. It follows that

$$\left| N_{B_k(G)}(E') \right| = k \left| V_G(E') \right| \geq |E'| \tag{1}$$

holds for all $E' \subset E$ if and only if G is $(k,0)$-sparse. Applying Hall's theorem shows that G is $(k,0)$-tight if and only if $B_k(G)$ contains a perfect matching.

The edges matched to the ith copy of V correspond to the ith map in the k-map, as shown for a 2-map in Figure 3. Orient each edge away from

5

the vertex to which it is matched. It follows that each vertex has out degree one in the spanning subgraph matched to each copy of V as desired. $\qquad\square$

Proof of Theorem 1. Suppose that G is tight, and let G' be the graph obtained by adding any ℓ edges to G from $K_n^{k,2k} - G$. Then G' has kn edges; moreover G' is $(k,0)$-sparse since at most ℓ edges were added to the span of any subset V' of V of size at least 2. Moreover, since the added edges came from $K_n^{k,2k}$, they do not violate sparsity on single-vertex subsets. It follows from Lemma 6 that G' can be decomposed into k edge-disjoint maps.

For the converse, suppose that G is not tight. Since G has $kn - \ell$ edges, G is not sparse. It follows that G contains a subgraph $H = (V', E')$ such that $|E'| \geq k|V'| - \ell + 1$. Add ℓ edges to the span of V' to form G'. By construction G' is not $(k,0)$-sparse; V' spans at least $k|V'| + 1$ edges in G'. Applying Lemma 6 shows that G' is not a k-map. $\qquad\square$

Proof of Corollary 2. The equivalence of tight graphs for $0 \leq \ell \leq k$ and the existence of a decomposition into ℓ edge-disjoint spanning trees and $(k - \ell)$ edge-disjoint maps is shown in [19]. By Theorem 1, the tight graphs are exactly those that decompose into k edge-disjoint maps after adding any ℓ edges. $\qquad\square$

Proof of Theorem 3. By hypothesis, G is $(k,0)$-sparse but not tight. By a structure theorem of [7], G contains a single maximal subgraph H that is $(k,0)$-tight. It follows that any edge with at least one end in $V - V(H)$ may be added to G without violating sparsity. Adding ℓ edges inductively produces a tight graph G' as desired. Apply Lemma 6 to complete the proof. $\qquad\square$

Proof of Corollary 4. Let $\mathcal{M}_k$ be the k-fold union of the bicycle matroid. The bases of $\mathcal{M}_k$ are exactly the k-maps. Combining this with Theorem 3 shows that $G \in \mathcal{N}_{k,\ell}$ if and only if G is independent in $\mathcal{M}_k$ and $|E(G)| = kn - \ell$ as desired. $\qquad\square$

Proof of Theorem 5. Suppose that G is sparse. Since G has $kn - \ell - p$ edges, G does not contain a spanning (k,ℓ)-tight subgraph. Hence there exist vertices u and v not both in the same (k,ℓ)-tight subgraph. Add the edge uv. Inductively add p edges this way. The resulting graph G' is (k,ℓ)-tight. By Theorem 1, adding any ℓ edges to G' results in a k-map.

Now suppose that G is not sparse. As in Theorem 1, there is no set of edges that can be added to G to create a (k,ℓ)-tight G', which proves the converse. $\qquad\square$

4   Conclusions and open problems

We characterize the graphs for which adding ℓ edges results in a k-map. These results are an analog to those of Haas [3] using k-maps as the primary object of study. In this setting, we obtain a uniform characterization of the tight graphs for all the matroidal values of ℓ. Figure 4 compares our results to other characterizations of sparse graphs. In this paper we extend the results of [3] to a larger range of ℓ. While we do not have an analog of ℓTk decompositions for the new $0 \leq \ell \leq k-1$ range, we do show the equivalence of adding ℓ edges and the existence of a decomposition into maps and trees.
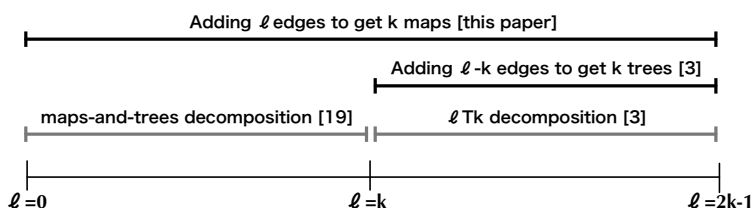

Figure 4: Equivalent characterizations of sparse graphs in terms of decompositions and adding edges.

In [3], there are two additional types of results: inductive sequences for the sparse graphs and the ℓTk decompositions. Describing an analog of ℓTk decompositions for the maps-and-trees range of ℓ is an open problem.

Lee and Streinu describe inductive sequences based on the pebble game for all the sparse graphs in [7], but these do not give the explicit decomposition shown to exist in Corollary 2. Providing this decomposition explicitly with an inductive sequence, as opposed to algorithmically as in [2], is another open problem. The theorem of [19] used in the proof of Corollary 2 is formulated in the setting of matroid rank function and does not describe the decomposition.

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