12-2010

Slider-Pinning Rigidity: a Maxwell-Laman-Type Theorem

Ileana Streinu
Smith College, streinu@cs.smith.edu

Louis Theran
Temple University

Slider-pinning Rigidity: a Maxwell-Laman-type Theorem

Ileana Streinu · **Louis Theran**

Received: date / Accepted: date

Abstract We define and study slider-pinning rigidity, giving a complete combinatorial characterization. This is done via direction-slider networks, which are a generalization of Whiteley's direction networks.

1 Introduction

A planar *bar-and-joint framework* is a planar structure made of fixed-length bars connected by universal joints with full rotational degrees of freedom. The allowed continuous motions preserve the lengths and connectivity of the bars. Formally, a bar-and-joint framework is modeled as a pair $(G, \boldsymbol{\ell})$, where $G = (V, E)$ is a simple graph with n vertices and m edges, and $\boldsymbol{\ell}$ is a vector of positive numbers that are interpreted as squared edge lengths.

A realization $G(\mathbf{p})$ of a bar-and-joint framework is a mapping of the vertices of G onto a point set $\mathbf{p} \in (\mathbb{R}^2)^n$ such that $||\mathbf{p}_i - \mathbf{p}_j||^2 = \boldsymbol{\ell}_{ij}$ for every edge $ij \in E$. The realized framework $G(\mathbf{p})$ is rigid if the only motions are trivial rigid motions; equivalently, $\mathbf{p}$ is an isolated (real) solution to the equations giving the edge lengths, modulo rigid motions. A framework $G(\mathbf{p})$ is *minimally rigid* if it is rigid, but ceases to be so if any bar is removed.

The Slider-pinning Problem. In this paper, we introduce an elaboration of planar bar-joint rigidity to include *sliders*, which constrain some of the vertices of a framework to move on given lines. We define the combinatorial model for a bar-slider framework to be a graph $G = (V, E)$ that has edges (to represent the bars) and also self-loops (that represent the sliders).

A realization of a bar-slider framework $G(\mathbf{p})$ is a mapping of the vertices of G onto a point set that is compatible with the given edge lengths, with the additional requirement that if a vertex is on a slider, then it is mapped to a point on the slider's line. A bar-slider

Ileana Streinu
Computer Science Department, Smith College, Northampton, MA
E-mail: istreinu@smith.edu,streinu@cs.smith.edu

Louis Theran
Mathematics Department, Temple University, Philadelphia, PA
E-mail: theran@temple.edu

framework $G(\mathbf{p})$ is *slider-pinning rigid* (shortly *pinned*) if it is completely immobilized. It is minimally pinned if it is pinned and ceases to be so when any bar or slider is removed. (Full definitions are given in Section 7).

Historical note on pinning frameworks. The topic of immobilizing bar-joint frameworks has been considered before. Lovász [13] and, more recently, Fekete [3] studied the related problem of pinning a bar-joint frameworks by a minimum number of *thumbtacks*, which completely immobilize a vertex. Thumbtack-pinning induces a different (and non-matroidal) graph-theoretic structure than slider-pinning. In terms of slider-pinning, the minimum thumbtack-pinning problem asks for a slider-pinning with sliders on the minimum number of distinct vertices. Recski [16] also previously considered the specific case of vertical sliders, which he called tracks.

We give, for the first time, a *complete combinatorial characterization* of planar slider-pinning in the most general setting. Previous work on the problem is concerned either with thumbtacks (Fekete [3]) or only with the algebraic setting (Lovász [13], Recski [16]).

On the algorithmic side, we [10] have previously developed algorithms for generic rigidity-theoretic questions on bar-slider frameworks. The theory developed in this paper provides the theoretical foundation for their correctness.

Generic combinatorial rigidity. The purely geometric question of deciding rigidity of a framework seems to be computationally intractable, even for small, fixed dimension d. The best-known algorithms rely on exponential time Gröbner basis techniques, and specific cases are known to be NP-complete [17]. However, for *generic* frameworks in the plane, the following landmark theorem due to Maxwell and Laman states that rigidity has a combinatorial characterization, for which several efficient algorithms are known (see [8] for a discussion of the algorithmic aspects of rigidity). The Laman graphs and looped-Laman graphs appearing in the statements of results are combinatorial (not geometric) graphs with special sparsity properties. The technical definitions are given in Section 2.

Theorem A (Maxwell-Laman Theorem: Generic bar-joint rigidity [7, 14]). *Let $(G,\boldsymbol{\ell})$ be a generic abstract bar-joint framework. Then $(G,\boldsymbol{\ell})$ is minimally rigid if and only if G is a Laman graph.*

Our main rigidity result is a Maxwell-Laman-type theorem for slider-pinning rigidity.

Theorem B (Generic bar-slider rigidty). *Let $(G,\ell,\mathbf{n},\mathbf{s})$ be a generic bar-slider framework. Then $(G,\ell,\mathbf{n},\mathbf{s})$ is minimally rigid if and only if G is looped-Laman.*

Our proof relies on a new technique and proceeds via direction networks, defined next.

Direction networks. A *direction network* $(G,\mathbf{d})$ is a graph G together with an assignment of a direction vector $\mathbf{d}_{ij} \in \mathbb{R}^2$ to each edge. A realization $G(\mathbf{p})$ of a direction network is an embedding of G onto a point set $\mathbf{p}$ such that $\mathbf{p}_i - \mathbf{p}_j$ is in the direction $\mathbf{d}_{ij}$; if the endpoints of every edge are distinct, the realization is *faithful*.

The *direction network realizability problem* is to find a realization $G(\mathbf{p})$ of a direction network $(G,\mathbf{d})$.

Direction-slider networks. We define a *direction-slider network* $(G,\mathbf{d},\mathbf{n},\mathbf{s})$ to be an extension of the direction network model to include sliders. As in slider-pinning rigidity, the combinatorial model for a slider is defined to be a self-loop in the graph G. A realization $G(\mathbf{p})$ of a direction-slider network respects the given direction for each edge, and puts $\mathbf{p}_i$ on the line specified for each slider. A realization is *faithful* is the endpoints of every edge are distinct.

Generic direction network realizability. Both the direction network realization problem and the direction-slider network realization problem give rise to a *linear* system of equations, in contrast to the quadratic systems arising in rigidity, greatly simplifying the analysis of the solution space.

The following theorem was proven by Whiteley.

We give a new proof, using different geometric and combinatorial techniques, and we give an explicit description of the set of generic directions.

Theorem C (Generic direction network realization (Whiteley [20–22])). *Let $(G, \mathbf{d})$ be a generic direction network, and let G have n vertices and $2n-3$ edges. Then $(G, \mathbf{d})$ has a (unique, up to translation and rescaling) faithful realization if and only if G is a Laman graph.*

For direction-slider networks we have a similar result to Theorem C.

Theorem D (Generic direction-slider network realization). *Let $(G, \mathbf{d}, \mathbf{n}, \mathbf{s})$ be a generic direction-slider network. Then $(G, \mathbf{d}, \mathbf{n}, \mathbf{s})$ has a (unique) faithful realization if and only if G is a looped-Laman graph.*

From generic realizability to generic rigidity. Let us briefly sketch how the rigidity theorems A and B follow from the direction network realization theorems C and D (full details are given in Section 7). For brevity, we sketch only how Theorem C implies Theorem A, an implication that can be traced back to Whiteley in [21]. The proof that Theorem D implies Theorem B will follow a similar proof plan.

All known proofs of the Maxwell-Laman theorem proceed via *infinitesimal rigidity*, which is a linearization of the rigidity problem obtained by taking the differential of the system of equations specifying the edge lengths and sliders to obtain the *rigidity matrix* $\mathbf{M}_{2,3}(G)$ of the abstract framework (see Figure 9(a)).

One then proves the following two statements about bar-joint frameworks $(G, \boldsymbol{\ell})$ with n vertices and $m = 2n-3$ edges:

- In realizations where the rigidity matrix achieves rank $2n-3$ the framework is rigid.
- The rigidity matrix achieves rank $2n-3$ for almost all realizations (these are called *generic*) if and only if the graph G is Laman.

The second step, where the rank of the rigidity matrix is established from only a combinatorial assumption, is the (more difficult) "Laman direction". The plan is in two steps:

- We begin with a matrix $\mathbf{M}_{2,2}(G)$, arising from the direction network realization problem, that has non-zero entries in the same positions as the rigidity matrix, but a simpler pattern: $\mathbf{d}_{ij} = (a_{ij}, b_{ij})$ instead of $\mathbf{p}_i - \mathbf{p}_j$ (see Figure 7). The rank of the simplified matrices is established in Section 3 via a matroid argument.
- We then apply the direction network realization Theorem C to a Laman graph. For generic (defined in detail in Section 4) edge directions $\mathbf{d}$ there exists a point set $\mathbf{p}$ such that $\mathbf{p}_i - \mathbf{p}_j$ is in the direction $\mathbf{d}_{ij}$, with $\mathbf{p}_i \neq \mathbf{p}_j$ when ij is an edge. Substituting the $\mathbf{p}_i$ into $\mathbf{M}_{2,2}(G)$ recovers the rigidity matrix while preserving rank, which completes the proof.

Genericity. In this paper, the term *generic* is used in the standard sense of algebraic geometry: a property is generic if it holds on the (open, dense) complement of an algebraic set defined by a finite number of polynomials. In contrast, the rigidity literature employs a number of definitions that are not as amenable to combinatorial or computational descriptions. Some authors [12, p. 92] define a *generic framework* as being one where the points $\mathbf{p}$ are algebraically independent. Other frequent definitions used in rigidity theory require that generic properties hold *for most of* the point sets (measure-theoretical) [23, p. 1331] or focus on properties which, if they hold for a point set $\mathbf{p}$ (called generic for the property), then they hold for any point in some open neighborhood (topological) [4].

For the specific case of Laman bar-joint rigidity we identify two types of conditions on the defining polynomials: some arising from the genericity of directions in the direction network with the same graph as the framework being analyzed; and a second type arising from the constraint the the directions be realizable as the difference set of a planar point set. To the best of our knowledge, these observations are new.

Organization. The rest of this paper is organized as follows. Section 2 defines Laman and looped-Laman graphs and gives the combinatorial tools from the theory of (k,ℓ)-sparse and $(k,\boldsymbol{\ell})$-graded sparse graphs that we use to analyze direction networks and direction-slider networks. Section 3 introduces the needed results about (k,ℓ)-sparsity-matroids, and we prove two matroid representability results for the specific cases appearing in this paper. Section 4 defines direction networks, the realization problem for them, and proves Theorem C. Section 5 defines slider-direction networks and proves the analogous Theorem D. In Section 6 we extend Theorem D to the specialized situation where all the sliders are axis-parallel.

In Section 7 we move to the setting of frameworks, defining bar-slider rigidity and proving the rigidity Theorems A and B from our results on direction networks. In addition, we discuss the relationship between our work and previous proofs of the Maxwell-Laman theorem.

Notations. Throughout this paper we will use the notation $\mathbf{p} \in (\mathbb{R}^2)^n$ for a set of n points in the plane. By identification of $(\mathbb{R}^2)^n$ with $\mathbb{R}^{2n}$, we can think of $\mathbf{p}$ either as a vector of point $\mathbf{p}_i = (a_i, b_i)$ or as a flattened vector $\mathbf{p} = (a_1, b_1, a_2, b_2, \dots, a_n, b_n)$. When points are used as unknown variables, we denote them as $\mathbf{p}_i = (x_i, y_i)$.

Analogously, we use the notation $\mathbf{d} \in (\mathbb{R}^2)^m$ for a set of m directions in $\mathbb{R}^2$. Since directions will be assigned to edges of a graph, we index the entries of $\mathbf{d}$ as $\mathbf{d}_{ij} = (a_{ij}, b_{ij})$ for the direction of the edge ij.

The graphs appearing in this paper have edges and also self-loops (shortly, loops). Both multiple edges and multiple self loops will appear, but the multiplicity will never be more than two copies. We will use n for the number of vertices, m for the number of edges, and c for the numbers of self-loops. Thus for a graph $G = (V, E)$ we have $|V| = n$ and $|E| = m + c$. Edges are written as $(ij)_k$ for the kth copy of the edge ij, $(k = 1, 2)$. As we will not usually need to distinguish between copies, we abuse notation and simply write ij, with the understanding that multiple edges are considered separately in "for all" statements. The jth loop on vertex i is denoted i_j $(j = 1, 2)$.

For subgraphs G' of a graph G, we will typically use n' for the number of vertices, m' for the number of edge and c' for the number of loops.

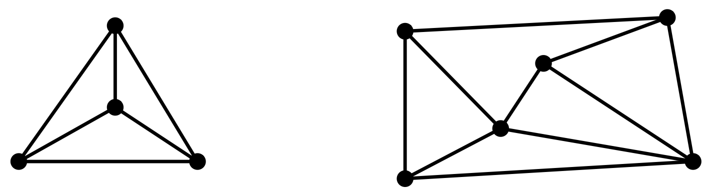
A contraction of a graph G over the edge ij (see Section 2 for a complete definition) is denoted G/ij.

We use the notation $[n]$ for the set $\{1, 2, \dots, n\}$. If $\mathbf{A}$ is an $m \times n$ matrix, then $\mathbf{A}[M, N]$ is the sub-matrix induced by the rows $M \subset [m]$ and $N \subset [n]$.

2 Sparse and graded-sparse graphs

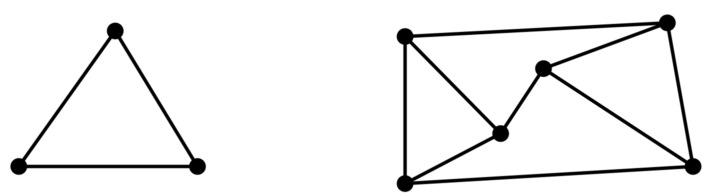
Let G be a graph on n vertices, possibly with multiple edges and loops. G is (k,ℓ)-*sparse* if for all subgraphs G' of G on n' vertices, the numbers of induced edges and loops $m'+c' \leq kn'-\ell$. If, in addition, G has $m+c = kn-\ell$ edges and loops, then G is (k,ℓ)-*tight*. An induced subgraph of a (k,ℓ)-sparse graph G that is (k,ℓ)-tight is called a *block* in G; a maximal block is called a *component* of G.

Throughout this paper, we will be interested in two particular cases of sparse graphs: $(2,2)$-tight graphs and $(2,3)$-tight graphs. For brevity of notation we call these $(2,2)$-*graphs* and *Laman graphs* respectively. We observe that the sparsity parameters of both $(2,2)$-graphs and Laman graphs do not have self-loops. Additionally, Laman graphs are simple, but $(2,2)$-graphs may have two parallel edges (any more would violate the sparsity condition). See Figure 1 and Figure 2 for examples.



(a) (b)

Fig. 1 Examples of $(2,2)$-graphs: (a) K_4; (b) a larger example on 6 vertices.



(a) (b)

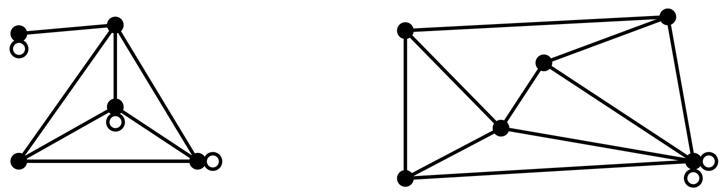
Fig. 2 Examples of Laman graphs.

Graded sparsity. We also make use of a specialization of the $(k,\boldsymbol{\ell})$-graded-sparse graph concept from our paper [9]. Here, $\boldsymbol{\ell}$ is a vector of integers, rather than just a single integer value. To avoid introducing overly general notation that is immediately specialized, we define it only for the specific parameters we use in this paper.

Let G be a graph on n vertices with edges and also self-loops. G is $(2,0,2)$-graded-sparse if:

- All subgraphs of G with only edges (and no self-loops) are $(2,2)$-sparse.
- All subgraphs of G with edges and self-loops are $(2,0)$-sparse.

If, additionally, G has $m+c=2n$ edges and loops, then G is $(2,0,2)$-*tight* (shortly looped-$(2,2)$). See Figure 3 for examples of looped-$(2,2)$ graphs.



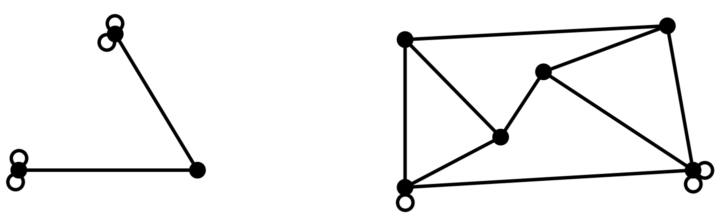
(a) (b)

Fig. 3 Examples of looped-$(2,2)$ graphs.

Let G be a graph on n vertices with edges and also self-loops. G is $(2,0,3)$-graded-sparse if:

- All subgraphs of G with only edges (and no self-loops) are $(2,3)$-sparse.
- All subgraphs of G with edges and self-loops are $(2,0)$-sparse.

If, additionally, G has $m+c=2n$ edges and loops, then G is $(2,0,3)$-*tight* (shortly looped-Laman). See Figure 4 for examples of looped-Laman graphs.



(a) (b)

Fig. 4 Examples of looped-Laman graphs.

Characterizations by contractions. We now present characterizations of Laman graphs and looped-Laman graphs in terms of graph contractions. Let G be a graph (possibly with loops and multiple edges), and let ij be an edge in G. The *contraction of G over ij*, G/ij is the graph obtained by:

- Discarding vertex j.
- Replacing each edge jk with an edge ik, for $k \neq i$.
- Replacing each loop j_k with a loop i_k.

By symmetry, we may exchange the roles of i and j in this definition without changing it. We note that this definition of contraction *retains* multiple *edges* created during the contraction, but that *loops* created by contracting are discarded. In particular, any loop in G/ij corresponds to a loop in G. Figure 5 shows an example of contraction.

(a) (b)

Fig. 5 Contracting an edge of the triangle: (a) before contraction; (b) after contraction we get a doubled edge but *not* a loop, since there wasn't one in the triangle before contracting.

The following lemma gives a characterization of Laman graphs in terms of contraction and $(2,2)$-sparsity.

Lemma 1. *Let G be a simple $(2,2)$-sparse graph with n vertices and $2n-3$ edges. Then G is a Laman graph if and only if after contracting* any *edge $ij \in E$, G/ij is a $(2,2)$-graph on $n-1$ vertices.*

Proof. If G is not a Laman graph, then some subset $V' \subset V$ of n' vertices induces a subgraph $G' = (V', E')$ with $m' \geq 2n'-2$ edges. Contracting any edge ij of G' leads to a contracted graph G'/ij with $n'-1$ vertices and at least $2n'-3 = 2(n'-1)-1$ edges, so G'/ij is not $(2,2)$-sparse. Since G'/ij is an induced subgraph of G/ij for this choice of ij, G/ij is not a $(2,2)$-graph.

For the other direction, we suppose that G is a Laman graph and fix a subgraph $G' = (V', E')$ induced by n' vertices. Since G is Laman, G' spans at most $2n'-3$ edges, and so for any edge $ij \in E'$ the contracted graph G'/ij spans at most $2n'-4 = 2(n'-1)-2$ edges, so G'/ij is $(2,2)$-sparse. Since this G'/ij is an induced subgraph of G/ij, and this argument holds for *any* $V' \subset V$ and edge ij, G/ij is $(2,2)$-sparse for any edge ij. Since G/ij has $2n-2$ edges, it must be a $(2,2)$-graph. $\square$

For looped-Laman graphs, we prove a similar characterization.

Lemma 2. *Let G be a looped-$(2,2)$ graph. Then G is looped-Laman if and only if for* any *edge $ij \in E$ there is a loop v_w (depending on ij) such that $G/ij - v_w$ is a looped-$(2,2)$ graph.*

Proof. Let G have n vertices, m edges, and c loops. Since G is looped-$(2,2)$, $2n = m + c$. If G is not looped-Laman, then by Lemma 1, the edges of G/ij are not $(2,2)$-sparse, which implies that $G/ij - v_w$ cannot be $(2,0,2)$-graded-sparse for any loop v_w because the loops play no role in the $(2,2)$-sparsity condition for the edges.

If G is looped-Laman, then the edges will be $(2,2)$-sparse in any contraction G/ij. However, G/ij has $n-1$ vertices, $m-1$ edges and c loops, which implies that $m-1+c = 2n-1 = 2(n-1)+1$, so G/ij is not $(2,0)$-sparse as a looped graph. We have to show that there is *one* loop, which when removed, restores $(2,0)$-sparsity.

For a contradiction, we suppose the contrary: for any contraction G/ij, there is some subgraph $(G/ij)' = (V', E')$ of G/ij on n' vertices inducing m' edges and c' loops with $m' + c' \geq 2n' + 2$. As noted above $m' \leq 2n' - 2$. If $(G/ij)'$ does not contain i, the surviving endpoint of the contracted edge ij, then G was not looped-$(2,2)$, which is a contradiction. Otherwise, we consider the subgraph induced by $V' \cup \{i\}$ in G. By construction it has $n'+1$ vertices, $m'+1$ edges and c' loops. But then we have $m'+1+c' \geq 2n'+3 = 2(n'+1)+1$, contradicting $(2,0,2)$-graded-sparsity of G. $\square$

3 Natural realizations for $(2,2)$-tight and $(2,0,2)$-tight graphs

Both (k,ℓ)-sparse and (k,ℓ)-graded-sparse graphs form matroids, with the (k,ℓ)-tight and $(k,\boldsymbol{\ell})$-graded-tight graphs as the bases, which we define to be the (k,ℓ)-sparsity-matroid and the $(k,\boldsymbol{\ell})$-graded-sparsity matroid, respectively. Specialized to our case, we talk about the $(2,2)$- and $(2,3)$-sparsity matroids and the $(2,0,2)$- and $(2,0,3)$-graded-sparsity matroids, respectively.

In matroidal terms, the rigidity Theorems A and B state that the rigidity matrices for bar-joint and bar-slider frameworks are *representations* of the $(2,3)$-sparsity matroid and $(2,0,3)$-graded-sparsity matroid, respectives: linear independence among the rows of the matrix corresponds bijectively to independence in the associated combinatorial matroid for generic frameworks. The difficulty in the proof is that the pattern of the rigidity matrices $\mathbf{M}_{2,3}(G)$ and $\mathbf{M}_{2,0,3}(G)$ (see Figure 9) contain repeated variables that make the combinatorial analysis of the rank complicated.

By contrast, for the closely related $(2,2)$-sparsity-matroid and the $(2,0,2)$-graded-sparsity matroid, representation results are easier to obtain directly. The results of this section are representations of the $(2,2)$-sparsity- and $(2,0,2)$-graded-sparsity matroids which are *natural* in the sense that the matrices obtained have the same dimensions at the corresponding rigidity matrices and non-zero entries at the same positions. The $(2,2)$-sparsity-matroid case is due to Whiteley [20], but we include it here for completeness.

In the rest of this section, we give precise definitions of generic representations of matroids and then prove our representation results for the $(2,2)$-sparsity and $(2,0,2)$-graded-sparsity matroids.

The generic rank of a matrix. The matrices we define in this paper have as their non-zero entries *generic variables*, or formal polynomials over $\mathbb{R}$ or $\mathbb{C}$ in generic variables. We define such a matrix $\mathbf{M}$ is to be a *generic matrix*, and its *generic rank* is given by the largest number r for which $\mathbf{M}$ has an $r \times r$ matrix minor with a determinant that is formally non-zero.

Let $\mathbf{M}$ be a generic matrix in m generic variables $x_1,\dots,x_m$, and let $\mathbf{v} = (v_i) \in \mathbb{R}^m$ (or $\mathbb{C}^m$). We define a *realization* $\mathbf{M}(\mathbf{v})$ *of* $\mathbf{M}$ to be the matrix obtained by replacing the variable x_i with the corresponding number v_i. A vector $\mathbf{v}$ is defined to be a *generic point* if the rank of $\mathbf{M}(\mathbf{v})$ is equal to the generic rank of $\mathbf{M}$; otherwise $\mathbf{v}$ is defined to be a *non-generic* point.

We will make extensive use of the following well-known facts from algebraic geomety (see, e.g., [2]):

- The rank of a generic matrix $\mathbf{M}$ in m variables is equal to the maximum over $\mathbf{v} \in \mathbb{R}^m$ ($\mathbb{C}^m$) of the rank of all realizations $\mathbf{M}(\mathbf{v})$.
- The set of non-generic points of a generic matrix $\mathbf{M}$ is an algebraic subset of $\mathbb{R}^m$ ($\mathbb{C}^m$).
- The rank of a generic matrix $\mathbf{M}$ in m variables is at least as large as the rank of any specific realization $\mathbf{M}(\mathbf{v})$.

Generic representations of matroids. A *matroid* $\mathscr{M}$ on a ground set E is a combinatorial structure that captures properties of linear independence. Matroids have many equivalent definitions, which may be found in a monograph such as [15]. For our purposes, the most convenient formulation is in terms of *bases*: a matroid $\mathscr{M}$ on a finite ground set E is presented by its bases $\mathscr{B} \subset 2^E$, which satisfy the following properties:

- The set of bases $\mathscr{B}$ is not empty.
- All elements $B \in \mathscr{B}$ have the same cardinality, which is the *rank* of $\mathscr{M}$.
- For any two distinct bases $B_1, B_2 \in \mathscr{B}$, there are elements $e_1 \in B_1 - B_2$ and $e_2 \in B_2$ such that $B_2 + \{e_1\} - \{e_2\} \in \mathscr{B}$.

It is shown in [8] that the set of $(2,2)$-graphs form the bases of a matroid on the set of edges of K_n^2, the complete graph with edge multiplicity 2. In [9] we proved that the set of looped-$(2,2)$ graphs forms a matroid on the set of edges of $K_n^{2,2}$ a complete graph with edge multiplicity 2 and 2 distinct loops on every vertex.

Let $\mathscr{M}$ be a matroid on ground set E. We define a generic matrix $\mathbf{M}$ to be a *generic representation of* $\mathscr{M}$ if:

- There is a bijection between the rows of $\mathbf{M}$ and the ground set E.
- A subset of rows of $\mathbf{M}$ is attains the rank of the matrix $\mathbf{M}$ if and only if the corresponding subset of E is a basis of $\mathscr{M}$.

With the definitions complete, we prove the results of this section.

Natural representation of spanning trees. We begin with a standard lemma, also employed by Whiteley [20], about the linear representability of the well-known spanning tree matroid.

Let G be a graph. We define the matrix $\mathbf{M}_{1,1}(G)$ to have one column for each vertex $i \in V$ and one row for each edge $ij \in E$. The row ij has zeros in the columns not associated with i or j, a generic variable a_{ij} in the column for vertex i and $-a_{ij}$ in the column for vertex j. Figure 6(a) illustrates the pattern.

We define $\mathbf{M}^\bullet_{1,1}(G)$ to be the matrix obtained from $\mathbf{M}_{1,1}(G)$ by dropping any column. Lemma 3 shows that the ambiguity of the column to drop poses no problem for our purposes.

Lemma 3. *Let G be a graph on n vertices and $m = n-1$ edges. If G is a tree, then*

$$\det\left(\mathbf{M}^\bullet_{1,1}(G)\right) = \pm \prod_{ij \in E(G)} a_{ij}.$$

Otherwise $\det\left(\mathbf{M}^\bullet_{1,1}(G)\right) = 0$.

See [11, solution to Problem 4.9] for the proof.

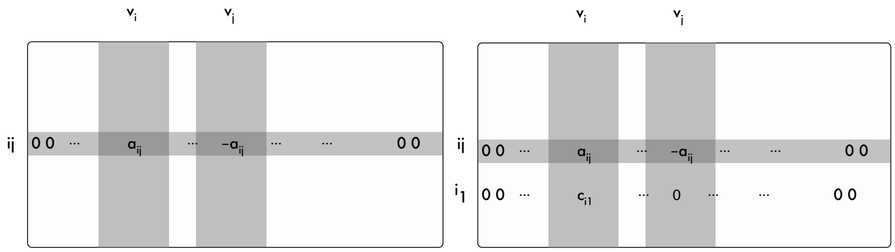


Fig. 6 The pattern of the matrices for trees and looped forests: (a) $\mathbf{M}_{1,1}(G)$; (b) $\mathbf{M}_{1,0,1}(G)$.

Natural representation of looped forests. In the setting of looped graphs, the object corresponding to a spanning tree is a forest in which every connected component spans exactly one loop. We define such a graph to be a *looped forest*. Looped forests are special cases of the *map-graphs* studied in our papers [6, 9, 18], which develop their combinatorial and matroidal properties.

Let G be a looped graph and define the matrix $\mathbf{M}_{1,0,1}(G)$ to have one column for each vertex $i \in V$. Each edge has a row corresponding to it with the same pattern as in $\mathbf{M}_{1,1}(G)$. Each loop i_j has a row corresponding to it with a variable c_{i_j} in the column corresponding to vertex i and zeros elsewhere. Figure 6(b) shows the pattern. Lemma 3 generalizes to the following.

Lemma 4. *Let G be a looped graph on n vertices and $c+m=n$ edges and loops. If G is a looped forest, then*

$$\det(\mathbf{M}_{1,0,1}(G)) = \pm\left(\prod_{\text{edges}\, ij\in E(G)} a_{ij}\right)\cdot\left(\prod_{\text{loops}\, i_j\in E(G)} c_{i_j}\right)$$

Otherwise $\det\left(\mathbf{M}^{\bullet}_{1,0,1}(G)\right) = 0.$

Proof. By the hypothesis of the lemma, $\mathbf{M}_{1,0,1}(G)$ is $n \times n$, so its determinant is well-defined.

If G is not a looped forest, then it has a vertex-induced subgraph G' on n' vertices spanning at least $n'+1$ edged and loops. The sub-matrix induced by the rows corresponding to edges and loops in G' has at least $n'+1$ rows by at most n' columns that are not all zero.

If G is a looped forest then $\mathbf{M}_{1,0,1}(G)$ can be arranged to have a block diagonal structure. Partition the vertices according to the $k \geq 1$ connected components $G_1, G_2, \ldots, G_k$ and arrange the columns so that $V(G_1), V(G_2), \ldots, V(G_k)$ appear in order. Then arrange the rows so that the $E(G_i)$ also appear in order. Thus the lemma follows from proving that if G is a tree with a loop on vertex i we have

$$\det(\mathbf{M}_{1,0,1}(G)) = \pm c_{i_1}\cdot\left(\prod_{\text{edges}\, ij\in E(G)} a_{ij}\right)$$

since we can multiply the determinants of the sub-matrices corresponding to the connected components.

To complete the proof, we expand the determinant along the row corresponding to the loop i_1. Since it has one non-zero entry, we have

$$\det(\mathbf{M}_{1,0,1}(G)) = \pm c_{i_1}\det(\mathbf{M}_{1,0,1}(G)[A,B])$$

where A is the set of rows correspond to the $n-1$ edges of G and B is the set of columns corresponding to all the vertices of G except for i. Since $\mathbf{M}_{1,0,1}(G)[A,B]$ has the same form at $\mathbf{M}^{\bullet}_{1,1}(G-\{i_j\})$ the claimed determinant formula follows from Lemma 3. □

The $(2,2)$-sparsity-matroid. Let G be a graph. We define the matrix $\mathbf{M}_{2,2}(G)$ to have two columns for each vertex $i \in V$ and one row for each edge $ij \in E$. The row ij has zeros in the columns not associated with i or J, variables (a_{ij}, b_{ij}) in the columns for vertex i and $(-a_{ij}, -b_{ij})$ in the columns for vertex j. Figure 7 illustrates the pattern.

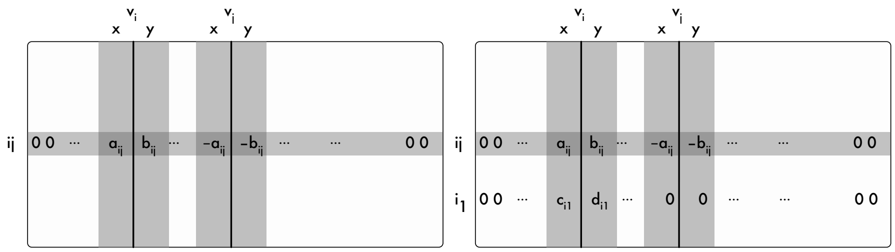


Fig. 7 The pattern of the matrices for $(2,2)$-graphs and looped-$(2,2)$ graphs: (a) $\mathbf{M}_{2,2}(G)$; (b) $\mathbf{M}_{2,0,2}(G)$.

Lemma 5 (Whiteley [20]). *The matrix $\mathbf{M}_{2,2}(K_n^2)$ is a generic representation of the $(2,2)$-sparsity matroid.*

The proof, which can be found in [20], is essentially the same as that used to prove the Matroid Union Theorem for linearly representable matroids (e.g., [1, Prop. 7.6.14]).

The $(2,0,2)$-graded-sparsity matroid. Let G be a looped graph and define the matrix $\mathbf{M}_{2,0,2}(G)$ to have two columns for each vertex, one row for each edge or self-loop. The rows for the edges are the same as in $\mathbf{M}_{2,2}(G)$. The row for a self-loop i_j (the jth loop on vertex i) has variables (c_{i_j}, d_{i_j}) in the columns for vertex i and zeros elsewhere. (See Figure 7(b).)

Lemma 6. *The matrix $\mathbf{M}_{2,0,2}(K_2^{2,2})$ is a generic representation of the $(2,0,2)$-graded-sparsity-matroid.*

Proof. We need to show that if G has n vertices, and $m+c=2n$ edges and loops, then the generic rank of $\mathbf{M}_{2,0,2}(G)$ is $2n$ if and only if G is a looped-$(2,2)$ graph.

Since $\mathbf{M}_{2,0,2}(G)$ is square, we expand the determinant around the $a_{\cdot}$ columns with the generalized Laplace expansion to get:

$$\sum \pm \det(\mathbf{M}_{2,0,2}(G)[A,X]) \cdot \det(\mathbf{M}_{2,0,2}(G)[B,Y])$$

where the sum is over all complementary sets of n rows X and Y. Since each smaller determinant has the form of $\mathbf{M}_{1,0,1}(G)$ from Lemma 4, the sum has a non-zero term if and only if G is the edge-disjoint union of two looped forests. Any non-zero term is a multilinear monomial that cannot generically cancel with any of the others, implying that the generic rank of $\mathbf{M}_{2,0,2}(G)$ is $2n$ if and only if G is the disjoint union of two looped forests.

The lemma then follows from the main theorems of our papers [9, 18], which show that G admits such a decomposition if and only if G is looped-$(2,2)$. □

4 Direction network realization

A *direction network* $(G, \mathbf{d})$ is a graph G together with an assignment of a direction vector $\mathbf{d}_{ij} \in \mathbb{R}^2$ to each edge. The *direction network realizability problem* [21] is to find a realization $G(\mathbf{p})$ of a direction network $(G, \mathbf{d})$.

A *realization* $G(\mathbf{p})$ of a direction network is an embedding of G onto a point set $\mathbf{p}$ such that $\mathbf{p}_i - \mathbf{p}_j$ is in the direction $\mathbf{d}_{ij}$. In a realization $G(\mathbf{p})$ of a direction network $(G, \mathbf{d})$, an edge ij is *collapsed* if $\mathbf{p}_i = \mathbf{p}_j$. A realization is collapsed if all the $\mathbf{p}_i$ are the same. A realization is *faithful*[1] if $ij \in E$ implies that $\mathbf{p}_i \neq \mathbf{p}_j$. In other words, a faithful parallel realization has no collapsed edges.

In this section, in preparation for the main result, we give a new derivation of the Parallel Redrawing Theorem of Whiteley.

Theorem C (Generic direction network realization (Whiteley [20–22])). *Let $(G, \mathbf{d})$ be a generic direction network, and let G have n vertices and $2n-3$ edges. Then $(G, \mathbf{d})$ has a (unique, up to translation and rescaling) faithful realization if and only if G is a Laman graph.*

Roadmap. Here is an outline of the proof.

- We formally define the direction network realization problem as a linear system $\mathbf{P}(G, \mathbf{d})$ and prove that its generic rank is equivalent to that of $\mathbf{M}_{2,2}(G)$. (Lemma 7 and Lemma 8.)
- We show that if a solution to the realization problem $\mathbf{P}(G, \mathbf{d})$ collapses an edge vw, the solution space is equivalent to the solution space of $\mathbf{P}_{vw}(G, \mathbf{d})$, a linear system in which $\mathbf{p}_v$ is replaced with $\mathbf{p}_w$. The combinatorial interpretation of this algebraic result is that the realizations of $(G/vw, \mathbf{d})$ are in bijective correspondence with those of $(G, \mathbf{d})$. (Lemma 12 and Corollary 13.)
- We then state and prove a *genericity condition* for direction networks $(G, \mathbf{d})$ where G is $(2,2)$-sparse and has $2n-3$ edges: the set of $\mathbf{d}$ such that $(G, \mathbf{d})$ and all contracted networks $(G/ij, \mathbf{d})$ is open and dense in $\mathbb{R}^{2m}$. (Lemma 14.)
- The final step in the proof is to show that for a Laman graph, if there is a collapsed edge in a generic realization, then the whole realization is collapsed by the previous steps and obtain a contradiction. (Proof of Theorem C.)

4.1 Direction network realization as a linear system

Let $(G, \mathbf{d})$ be a direction network. We define the linear system $\mathbf{P}(G, \mathbf{d})$ to be

$$\left\langle \mathbf{p}_i - \mathbf{p}_j, \mathbf{d}_{ij}^{\perp} \right\rangle = 0 \text{ for all } ij \in E \tag{1}$$

where the $\mathbf{p}_i$ are the unknowns. From the definition of a realization (p. 12, above the statement of Theorem C), every realization $G(\mathbf{p})$ of $(G, \mathbf{d})$, $\mathbf{p}$ is a solution of $\mathbf{P}(G, \mathbf{d})$.

If the entries of $\mathbf{d}$ are generic variables, then the solutions to $\mathbf{P}(G, \mathbf{d})$ are polynomials in the entries of $\mathbf{d}$. We start by describing $\mathbf{P}(G, \mathbf{d})$ in matrix form.

Lemma 7. *Let $(G, \mathbf{d})$ be a direction network. Then the solutions $\mathbf{p}$ of the system $\mathbf{P}(G, \mathbf{d})$ are solutions to the matrix equation*

$$\mathbf{M}_{2,2}(G)\mathbf{p} = \mathbf{0}$$

[1] Whiteley [21] calls this condition "proper."

Proof. Bilinearity of the inner product implies that (1) is equivalent to

$$\left\langle \mathbf{p}_i, \mathbf{d}_{ij}^{\perp} \right\rangle + \left\langle \mathbf{p}_j, -\mathbf{d}_{ij}^{\perp} \right\rangle = 0$$

which in matrix form is $\mathbf{M}_{2,2}(G)$. $\square$

The matrix form of $\mathbf{P}(G,\mathbf{d})$ leads to an immediate connection to the $(2,2)$-sparsity-matroid.

Lemma 8. *Let G be a graph on n vertices with $m \leq 2n-2$ edges. The generic rank of $\mathbf{P}(G,\mathbf{d})$ (with the $2n$ variables in $\mathbf{p} = (\mathbf{p}_1, \cdots, \mathbf{p}_n)$ as the unknowns) is m if and only if G is $(2,2)$-sparse. In particular, the rank is $2n-2$ if and only if G is a $(2,2)$-graph.*

Proof. Apply Lemma 7 and then Lemma 5. $\square$

An immediate consequence of Lemma 8 that we will use frequently is the following.

Lemma 9. *Let G be $(2,2)$-sparse. Then the set of edge direction assignments $\mathbf{d} \in \mathbb{R}^{2m}$ such that the direction network realization system $\mathbf{P}(G,\mathbf{d})$ has rank m is the (open, dense) complement of an algebraic subset of $\mathbb{R}^{2m}$.*

Proof. By Lemma 8 any $\mathbf{d} \in \mathbb{R}^{2m}$ for which the rank drops is a common zero of the $m \times m$ minors of the generic matrix $\mathbf{M}_{2,2}(G)$, which are polynomials. $\square$

Because of Lemma 9, when we work with $\mathbf{P}(G,\mathbf{d})$ as a system with numerical directions, we may select directions $\mathbf{d} \in \mathbb{R}^{2m}$ such that $\mathbf{P}(G,\mathbf{d})$ has full rank when G is $(2,2)$-sparse. We use this fact repeatedly below.

Translation invariance of $\mathbf{P}(G,\mathbf{d})$. Another simple property is that solutions to the system $\mathbf{P}(G,\mathbf{d})$ are preserved by translation.

Lemma 10. *The space of solutions to the system $\mathbf{P}(G,\mathbf{d})$ is preserved by translation.*

Proof. Let $\mathbf{t}$ be a vector in $\mathbb{R}^2$. Then $\left\langle (\mathbf{p}_i + \mathbf{t}) - (\mathbf{p}_j + \mathbf{t}), \mathbf{d}_{ij}^{\perp} \right\rangle = \left\langle \mathbf{p}_i - \mathbf{p}_j, \mathbf{d}_{ij}^{\perp} \right\rangle$. $\square$

4.2 Realizations of direction networks on $(2,2)$-graphs

There is a simple characterization of realizations of generic direction networks on $(2,2)$-graphs: they are all collapsed.

Lemma 11. *Let G be a $(2,2)$-graph on n vertices, and let $\mathbf{d}_{ij}$ be directions such that the system $\mathbf{P}(G,\mathbf{d})$ has rank $2n-2$. (This is possible by Lemma 9.) Then the (unique up to translation) realization of G with directions $\mathbf{d}_{ij}$ is collapsed.*

Proof. By hypothesis the system $\mathbf{P}(G,\mathbf{d})$ is homogeneous of rank $2n-2$. Factoring out translations by moving the variables giving associated with $\mathbf{p}_1$ to the right, we have a unique solution for each setting of the value of $\mathbf{p}_1$. Since a collapsed realization satisfies the system, it is the only one. $\square$

4.3 Realizations of direction networks on Laman graphs

In the rest of this section we complete the proof of Theorem C.

The contracted direction network realization problem. Let $(G,\mathbf{d})$ be a direction network, with realization system $\mathbf{P}(G,\mathbf{d})$, and let vw be an edge of G. We define the *vw-contracted realization system* $\mathbf{P}_{vw}(G,\mathbf{d})$ to be the linear system obtained by replacing $\mathbf{p}_v$ with $\mathbf{p}_w$ in $\mathbf{P}(G,\mathbf{d})$.

Combinatorial interpretation of $\mathbf{P}_{vw}(G)$. We relate $\mathbf{P}(G/vw,\mathbf{d})$ and $\mathbf{P}_{vw}(G,\mathbf{d})$ in the following lemma.

Lemma 12. *Let $(G,\mathbf{d})$ be a generic direction network. Then for any edge vw the system $\mathbf{P}_{vw}(G,\mathbf{d})$ is the same as the system $\mathbf{P}(G/vw,\mathbf{d})$, and the generic rank of $\mathbf{P}_{vw}(G,\mathbf{d})$ is the same as that of $\mathbf{M}_{2,2}(G/vw)$.*

Proof. By definition, in the system $\mathbf{P}_{vw}(G,\mathbf{d})$:

- The point $\mathbf{p}_v$ disappears
- Every occurrence of $\mathbf{p}_v$ is replaced with $\mathbf{p}_w$

Combinatorially, this corresponds to contracting over the edge vw in G, which shows that $\mathbf{P}_{vw}(G,\mathbf{d})$ is the same system as $\mathbf{P}(G/vw,\mathbf{d})$. An application of Lemma 8 to $\mathbf{P}(G/vw,\mathbf{d})$ shows that its rank is equivalent to that of $\mathbf{M}_{2,2}(G/vw)$. $\square$

Since the replacement of $\mathbf{p}_v$ with $\mathbf{p}_w$ is the same as setting $\mathbf{p}_v = \mathbf{p}_w$, we have the following corollary to Lemma 12.

Corollary 13. *Let $(G,\mathbf{d})$ be a direction network and ij an edge in G. If in every solution $\mathbf{p}$ of $\mathbf{P}(G,\mathbf{d})$, $\mathbf{p}_i = \mathbf{p}_j$, then $\mathbf{p}$ is a solution to $\mathbf{P}(G,\mathbf{d})$ if and only if $\mathbf{p}'$ obtained by dropping $\mathbf{p}_i$ from $\mathbf{p}$ is a solution to $\mathbf{P}(G/ij,\mathbf{d})$.*

A genericity condition. The final ingredient we need is the following genericity condition.

Lemma 14. *Let G be a Laman graph on n vertices. Then the set of directions $\mathbf{d} \in \mathbb{R}^{2m}$ such that:*

- *The system $\mathbf{P}(G,\mathbf{d})$ has rank $2n-3$*
- *For all edges $ij \in E$, the system $\mathbf{P}(G/ij,\mathbf{d})$ has rank $2(n-1)-2$*

is open and dense in $\mathbb{R}^{2m}$.

Proof. By Lemma 1 all the graphs G/ij are $(2,2)$-graphs and since G is Laman, all the graphs appearing in the hypothesis are $(2,2)$-sparse, so we may apply Lemma 9 to each of them separately. The set of $\mathbf{d}$ failing the requirements of the lemma is thus the union of finitely many closed algebraic sets in $\mathbb{R}^{2m}$ of measure zero. Its complement is open and dense, as required. $\square$

Proof of Theorem C. We first assume that G is not Laman. In this case it has an edge-induced subgraph G' that is a $(2,2)$-graph by results of [8]. This means that for generic directions $\mathbf{d}$, the system $\mathbf{P}(G,\mathbf{d})$ has a subsystem corresponding to G' to which Lemma 11 applies. Thus any realization of $(G,\mathbf{d})$ has a collapsed edge.

For the other direction, we assume, without loss of generality, that G is a Laman graph. We select directions $\mathbf{d}$ meeting the criteria of Lemma 14 and consider the direction network $(G,\mathbf{d})$.

Since $\mathbf{P}(G,d)$ has $2n$ variables and rank $2n-3$, we move $\mathbf{p}_1$ to the right to remove translational symmetry and one other variable, say, x_2, where $\mathbf{p}_2 = (x_2,y_2)$. The system has full rank, so for each setting of $\mathbf{p}_1$ and x_2 we obtain a unique solution. Set $\mathbf{p}_1 = (0,0)$ and $x_2 = 1$ to get a solution $\hat{\mathbf{p}}$ of $\mathbf{P}(G,\mathbf{d})$ where $\mathbf{p}_1 \neq \mathbf{p}_2$.

We claim that $G(\hat{\mathbf{p}})$ is faithful. Supposing the contrary, for a contradiction, we assume that some edge $ij \in E$ is collapsed in $G(\hat{\mathbf{p}})$. Then the equation $\mathbf{p}_i = \mathbf{p}_j$ is implied by $\mathbf{P}(G,\mathbf{d})$. Applying Corollary 13, we see that after removing $\hat{\mathbf{p}}_i$ from $\hat{\mathbf{p}}$, we obtain a solution to $\mathbf{P}(G/ij,\mathbf{d})$. But then by Lemma 1, G/ij is a $(2,2)$-graph. Because $\mathbf{d}$ was selected (using Lemma 14) so that $\mathbf{P}(G/ij,\mathbf{d})$ has full rank, Lemma 11 applies to $(G/ij,\mathbf{d})$, showing that every edge is collapsed in $G(\hat{\mathbf{p}})$. We have now arrived at a contradiction: G is connected, and by construction $\mathbf{p}_1 \neq \mathbf{p}_2$, so some edge is not collapsed in $G(\hat{\mathbf{p}})$. $\square$

Remarks on genericity. The proof of Theorem C shows why each of the two conditions in Lemma 14 are required. The first, that $\mathbf{P}(G,\mathbf{d})$ have full rank, ensures that there is a unique solution up to translation. The second condition, that for each edge ij the system $\mathbf{P}(G,\mathbf{d})$ has full rank, rules out sets of directions that are only realizable with collapsed edges.

The second condition in the proof is necessary by the following example: let G be a triangle and assign two of its edges the horizontal direction and the other edge the vertical direction. It is easy to check that the resulting $\mathbf{P}(G,\mathbf{d})$ has full rank, but it is geometrically evident that the edges of a non-collapsed triangle require either one or three directions. This example is ruled out by the contraction condition in Lemma 14, since contracting the vertical edge results in a rank-deficient system with two vertices and two copies of an edge in the same direction.

5 Direction-slider network realization

A *direction-slider network* $(G,\mathbf{d},\mathbf{n},\mathbf{s})$ is a looped graph, together with assignments of:

- A direction $\mathbf{d}_{ij} \in \mathbb{R}^2$ to each edge $ij \in E$.
- A *slider*, which is an affine line $\langle \mathbf{n}_{i_j}, \mathbf{x} \rangle = s_{i_j}$ in the plane, to each loop $i_j \in E$.

A *realization* $G(\mathbf{p})$ of a direction-slider network is an embedding of G onto the point set $\mathbf{p}$ such that:

- Each edge ij is drawn in the direction $\mathbf{d}_{ij}$.
- For each loop i_j on a vertex i, the point $\mathbf{p}_i$ is on the line $\langle \mathbf{n}_{i_j}, \mathbf{x} \rangle = s_{i_j}$.

As in the definitions for direction networks in the previous section, an edge ij is *collapsed* in a realization $G(\mathbf{p})$ if $\mathbf{p}_i = \mathbf{p}_j$. A realization $G(\mathbf{p})$ is *faithful* if none of the edges of G are collapsed.

The main result of this section is:

Theorem D (Generic direction-slider network realization). *Let $(G,\mathbf{d},\mathbf{n},\mathbf{s})$ be a generic direction-slider network. Then $(G,\mathbf{d},\mathbf{n},\mathbf{s})$ has a (unique) faithful realization if and only if G is a looped-Laman graph.*

Roadmap. The approach runs along the lines of previous section. However, because the system $\mathbf{S}(G,\mathbf{d},\mathbf{n},\mathbf{s})$ is inhomogeneous, we obtain a contradiction using unsolvability instead of a unique collapsed realization. The steps are:

- Formulate the direction-slider realization problem as a linear system and relate the rank of the parallel sliders realization system to the representation of the $(2,0,2)$-sparsity-matroid to show the generic rank of the realization system is given by the rank of the graph G in the $(2,0,2)$-matroid. (Lemma 16)

- Connect graph theoretic contraction over an edge ij to the edge being collapsed in all realizations of the direction-slider network: show that when $\mathbf{S}(G,\mathbf{d},\mathbf{n},\mathbf{s})$ implies that some edge ij is collapsed in all realizations means that it is equivalent to $\mathbf{S}(G/ij,\mathbf{d},\mathbf{n},\mathbf{s})$. (Lemma 18 and Corollary 19)
- Show that for looped graphs with combinatorially independent edges and one too many loops, the system $\mathbf{S}(G/ij,\mathbf{d},\mathbf{n},\mathbf{s})$ is generically not solvable. (Lemma 17).
- Show that if G is looped-Laman, then there are generic directions and sliders for $\mathbf{M}_{2,0,2}(G)$ so that the contraction of any edge leads to an unsolvable system. (Lemma 20.)
- Put the above tools together to show that for a looped-Laman graph, the realization problem is generically solvable, and the (unique solution) does not collapse any edges.

5.1 Direction-slider realization as a linear system.

Let $(G,\mathbf{d},\mathbf{n},\mathbf{s})$ be a direction-slider network. We define the system of equations $\mathbf{S}(G,\mathbf{d},\mathbf{n},\mathbf{s})$ to be:

$$\left\langle \mathbf{p}_i - \mathbf{p}_j, \mathbf{d}_{ij}^{\perp} \right\rangle = 0 \text{ for all edges } ij \in E \tag{2}$$

$$\left\langle \mathbf{p}_i, \mathbf{n}_{i_j} \right\rangle = s_{i_j} \text{ for all loops } i_j \in E \tag{3}$$

From the definition, it is immediate that the realizations of $(G,\mathbf{d},\mathbf{n},\mathbf{s})$ are exactly the solutions of $\mathbf{S}(G,\mathbf{d},\mathbf{n},\mathbf{s})$. The matrix form of $\mathbf{S}(G,\mathbf{d},\mathbf{n},\mathbf{s})$ gives the connection to the $(2,0,2)$-sparsity matroid.

Lemma 15. *Let $(G,\mathbf{d},\mathbf{n},\mathbf{s})$ be a direction slider network. The solutions to the system $\mathbf{S}(G,\mathbf{d},\mathbf{n},\mathbf{s})$ are exactly the solutions to the matrix equation*

$$\mathbf{M}_{2,0,2}(G)\mathbf{p} = (\mathbf{0},\mathbf{s})^{\mathsf{T}}$$

Proof. Similar to the proof of Lemma 7 for the edges of G. The slider are already in the desired form. □

As a consequence, we obtain the following two lemmas.

Lemma 16. *Let G be a graph on n vertices with $m \leq 2n$ edges. The generic rank of $\mathbf{S}(G,\mathbf{d},\mathbf{n},\mathbf{s})$ (with the $\mathbf{p}_i$ as the $2n$ unknowns) is m if and only if G is $(2,0,2)$-sparse. In particular, it is $2n$ if and only if G is a looped-$(2,2)$ graph.*

Proof. Apply Lemma 15 and then Lemma 6. □

We need, in addition, the following result on when $\mathbf{S}(G,\mathbf{d},\mathbf{n},\mathbf{s})$ has no solution.

Lemma 17. *Let G be a looped-$(2,2)$ graph and let G' be obtained from G by adding a single loop i_j to G. Then the set of edge direction assignments and slider lines $(\mathbf{d},\mathbf{n},\mathbf{s}) \in \mathbb{R}^{2m+3c}$ such that the direction-slider network realization system $\mathbf{S}(G',\mathbf{d},\mathbf{n},\mathbf{s})$ has no solution is the (open, dense) complement of an algebraic subset of $\mathbb{R}^{2m+3c}$.*

Proof. By Lemma 15 and Lemma 16, the solution $\mathbf{p} = \hat{\mathbf{p}}$ to the generic matrix equation

$$\mathbf{M}_{2,0,2}(G)\mathbf{p} = (\mathbf{0},\mathbf{s})^{\mathsf{T}}$$

has as its entries non-zero formal polynomials in the entries of $\mathbf{d}$, $\mathbf{n}$, and $\mathbf{s}$. In particular, the entries of $\hat{\mathbf{p}}_i$ are non-zero. This implies that for the equation

$$\mathbf{M}_{2,0,2}(G')\mathbf{p} = (\mathbf{0},\mathbf{s})^{\mathsf{T}}$$

to be solvable, the solution will have to be $\hat{\mathbf{p}}$, and $\hat{\mathbf{p}}_i$ will have to satisfy the additional equation

$$\left\langle \mathbf{n}_{i_j}, \hat{\mathbf{p}}_i \right\rangle = s_{i_j}$$

Since the entries of $\mathbf{n}_{i_j}$ and s_{i_j} are generic and don't appear at in $\hat{\mathbf{p}}_i$, the system $\mathbf{S}(G',\mathbf{d},\mathbf{n},\mathbf{s})$ is solvable only when either the rank of $\mathbf{M}_{2,0,2}(G)$ drops, which happens only for closed algebraic subset of $\mathbb{R}^{2m+3c}$ or when $\mathbf{n}_{i_j}$ and s_{i_j} satisfy the above equation, which is also a closed algebraic set. (Geometrically, the latter condition says that the line of the slider corresponding to the loop i_j is in the pencil of lines through $\hat{\mathbf{p}}_i$.) $\square$

Contracted systems. Let $vw \in E$ be an edge. We define $\mathbf{S}_{vw}(G,\mathbf{d},\mathbf{n},\mathbf{s})$, the contracted realization system, which is obtained by replacing $\mathbf{p}_v$ with $\mathbf{p}_w$ in $\mathbf{S}(G,\mathbf{d},\mathbf{n},\mathbf{s})$. The contracted system has two fewer variables and one fewer equation (corresponding to the edge vw).

The proof of Lemma 12 is identical to the proof of the analogous result for direction-slider networks.

Lemma 18. *Let $(G,\mathbf{d},\mathbf{n},\mathbf{s})$ be a generic direction-slider network. Then for any edge vw the system $\mathbf{P}_{vw}(G,\mathbf{d})$ is the same as the system $\mathbf{P}(G/vw,\mathbf{d},\mathbf{n},\mathbf{s})$, and the generic rank of $\mathbf{P}_{vw}(G,\mathbf{d},\mathbf{n},\mathbf{s})$ is the same as that of $\mathbf{M}_{2,0,2}(G/vw)$.*

The following is the direction-slider analogue of Corollary 13.

Corollary 19. *Let $(G,\mathbf{d},\mathbf{n},\mathbf{s})$ be a direction-slider network and ij an edge in G. If in all solutions $\mathbf{p}$ of $\mathbf{P}(G,\mathbf{d},\mathbf{n},\mathbf{s})$ $\mathbf{p}_i = \mathbf{p}_j$, then $\mathbf{p}$ is a solution to $\mathbf{P}(G,\mathbf{d},\mathbf{n},\mathbf{s})$ if and only if $\mathbf{p}'$ obtained by dropping $\mathbf{p}_i$ from $\mathbf{p}$ is a solution to $\mathbf{P}(G/ij,\mathbf{d},\mathbf{n},\mathbf{s})$.*

A genericity condition. The following lemma, which is the counterpart of Lemma 14, captures genericity for direction-slider networks.

Lemma 20. *Let G be a looped-Laman subgraph. The set of directions and slider lines such that:*

- *The system $\mathbf{S}(G,\mathbf{d},\mathbf{n},\mathbf{s})$ has rank $2n$ (and thus has a unique solution)*
- *For all edges $ij \in E$, the system $\mathbf{S}(G/ij,\mathbf{d},\mathbf{n},\mathbf{s})$ has no solution*

is open and dense in $\mathbb{R}^{2m+3c}$.

Proof. Because a looped-Laman graph is also a looped-$(2,2)$ graph, Lemma 6 and Lemma 16 imply that $\det(\mathbf{M}_{2,0,2}(G))$ which is a polynomial in the entries of $\mathbf{d}$ and $\mathbf{n}$ is not constantly zero, and so for any values of $\mathbf{s}$, the generic system $\mathbf{S}(G,\mathbf{d},\mathbf{n},\mathbf{s})$ has a unique solution $\hat{\mathbf{p}}$ satisfying

$$\mathbf{M}_{2,0,2}(G)\hat{\mathbf{p}} = (\mathbf{0},\mathbf{s})^{\mathsf{T}}$$

The generic directions and slider lines are the ones in the complement of the zero set of $\det(\mathbf{M}_{2,0,2}(G))$, and the non-generic set has measure zero.

By the combinatorial Lemma 2, each edge contraction G/ij has the combinatorial form required by Lemma 20. By Lemma 20, for each of m contractions, the set of directions and slider lines such that the contracted system $\mathbf{S}(G/ij,\mathbf{d},\mathbf{n},\mathbf{s})$ is an algebraic set of measure zero.

The proof follows from the fact that set of directions and slider lines for which the conclusion fails is the union of a finite number of measure-zero algebraic sets: $\det(\mathbf{M}_{2,0,2}(G)) = 0$ is one non-generic set and each application of Lemma 20 gives another algebraic set to avoid. Since the union of finitely many measure zero algebraic sets is itself a measure zero algebraic set, the intersection of the complements is non-empty. □

Proof of Theorem D. With all the tools in place, we give the proof of our direction-slider network realization theorem.

Proof of Theorem D. If G is not looped-Laman, then by Lemma 11 applied on a $(2,2)$-tight subgraph, G has no faithful realization.

Now we assume that G is looped-Laman. Assign generic directions and sliders as in Lemma 20. By Lemma 16, the system $\mathbf{S}(G,\mathbf{d},\mathbf{n},\mathbf{s})$ has rank $2n$ and thus a unique solution. For a contradiction, we suppose that some edge ij is collapsed. Then by Lemma 18 and Corollary 19 this system has a non-empty solution space equivalent to the contracted system $\mathbf{S}(G/ij,\mathbf{d},\mathbf{n},\mathbf{s})$. However, since we picked the directions and sliders as in Lemma 20, $\mathbf{S}(G/ij,\mathbf{d},\mathbf{n},\mathbf{s})$ has no solution, leading to a contradiction. □

6 Axis-parallel sliders

An *axis-parallel direction-slider network* is a direction network in which each slider is either vertical or horizontal. The combinatorial model for axis-parallel direction-slider networks is defined to be a looped graph in which each loop is colored either red or blue, indicating slider direction. A *color-looped-Laman graph* is a looped graph with colored loops that is looped-Laman, and, in addition, admits a coloring of its edges into red and blue forests so that each monochromatic tree spans exactly one loop of its color. Since the slider directions of an axis-parallel direction-slider network are given by the combinatorial data, it is formally defined by the tuple $(G,\mathbf{d},\mathbf{s})$. The realization problem for axis-parallel direction-slider networks is simply the specialization of the slider equations to $x_i = s_{i_j}$, where $\mathbf{p}_i = (x_i, y_i)$, for vertical sliders and $y_i = s_{i_j}$ for horizontal ones.

We prove the following extension to Theorem D.

Theorem G (Generic axis-parallel direction-slider network realization). *Let $(G,\mathbf{d},\mathbf{s})$ be a generic axis-parallel direction-slider network. Then $(G,\mathbf{d},\mathbf{s})$ has a (unique) faithful realization if and only if G is a color looped-Laman graph.*

The proof of Theorem G is a specialization of the arguments in the previous section to the axis-parallel setting. The modifications we need to make are:

- Specialize the $(2,0,2)$-matroid realization Lemma 6 to the case where in each row corresponding to a slider $i_j \in E$ one of c_{i_j} and d_{i_j} is zero and the other is one. This corresponds to the slider direction equations in the realization system for an axis-parallel direction-slider network.
- Specialize the genericity statement Lemma 20

Otherwise the proof of Theorem D goes through word for word. The rest of the section gives the detailed definitions and describes the changes to the two key lemmas.

Color-looped-$(2,2)$ and color-looped-Laman graphs. A *color-looped-(2,2) graph* is a looped graph with colored loops that is looped-$(2,2)$, in addition, admits a coloring of its edges into two forests so that each monochromatic tree spans exactly one loop of its color.

A *color-looped-Laman graph* is a looped graph with colored loops that is looped-Laman, and, in addition, admits a coloring of its edges into red and blue forests so that each monochromatic tree spans exactly one loop of its color.

Figure 8 shows examples. The difference between these definitions and the ones of looped-$(2,2)$ and looped-Laman graphs is that they are defined in terms of *both* graded sparsity counts *and* a specific decomposition of the edges, depending on the colors of the loops.

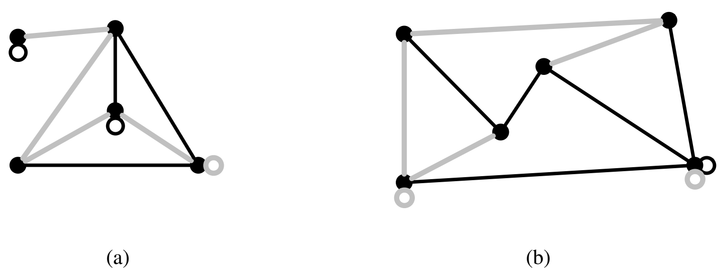


Fig. 8 Examples of color-looped graphs, shown with forests certifying the color-looped property: (a) a color-looped $(2,2)$-graph; (b) a color-looped Laman graph. The colors red and blue are represented by gray and black respectively.

Realizing the $(2,0,2)$-graded-sparsity matroid for color-looped graphs. Recall that the matrix $\mathbf{M}_{2,0,2}(G)$ (see Figure 7(b)) realizing the $(2,0,2)$-sparsity matroid has a row for each slider loop $i_j \in E$ with generic entries c_{i_j} and d_{i_j} in the two columns associated with vertex i. For the color-looped case, we specialize to the matrix $\mathbf{M}^{\mathrm{c}}_{2,0,2}(G)$, which has the same pattern as $\mathbf{M}_{2,0,2}(G)$, except:

- $c_{i_j} = 1$ and $d_{i_j} = 0$ for red loops $i_j \in E$
- $c_{i_j} = 0$ and $d_{i_j} = 1$ for blue loops $i_j \in E$

The extension of the realization Lemma 6 to this case is the following.

Lemma 21. *Let G be a color-looped graph on n vertices with $m + c = 2n$. The matrix $\mathbf{M}^{\mathrm{c}}_{2,0,2}(G)$ has generic rank $2n$ if and only if G is color-looped-$(2,2)$.*

Proof. Modify the proof of Lemma 6 to consider only decompositions into looped forests in which each loop is assigned its correct color. The definition of color-looped-$(2,2)$ graphs implies that one exists if and only if G is color-looped-$(2,2)$. As in the uncolored case, the determinant is generically non-zero exactly when the required decomposition exists. $\square$

Genericity for axis-parallel sliders. In the axis-parallel setting, our genericity condition is the following.

Lemma 22. *Let G be a color-looped-Laman subgraph. The set of directions and slider lines such that:*

- *The system* $\mathbf{S}(G,\mathbf{d},\mathbf{n},\mathbf{s})$ *has rank* $2n$ *(and thus has a unique solution)*
- *For all edges* $ij \in E$*, the system* $\mathbf{S}(G/ij,\mathbf{d},\mathbf{n},\mathbf{s})$ *has no solution*

is open and dense in $\mathbb{R}^{2m+3c}$.

Proof. Similar to the proof of Lemma 20, except using Lemma 21. □

7 Generic rigidity via direction network realization

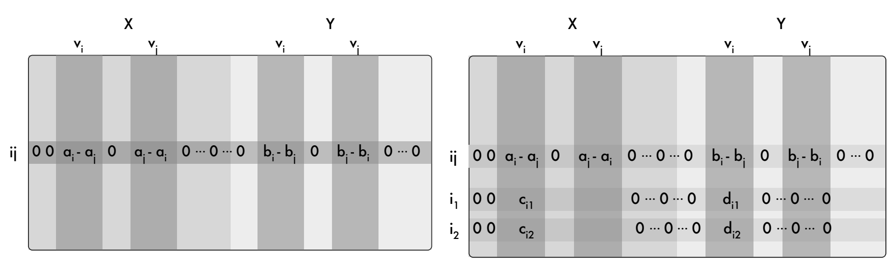
Having proven our main results on direction and direction-slider network realization, we change our focus to the rigidity theory of bar-joint and bar-slider frameworks.

7.1 Bar-joint rigidity

In this section, we prove the Maxwell-Laman Theorem, following Whiteley [21]:

Theorem A (Maxwell-Laman Theorem: Generic bar-joint rigidity [7, 14]). *Let* $(G,\boldsymbol{\ell})$ *be a generic abstract bar-joint framework. Then* $(G,\boldsymbol{\ell})$ *is minimally rigid if and only if G is a Laman graph.*

The difficult step of the proof is to show that a generic bar-joint framework $G(\mathbf{p})$ with $m = 2n-3$ edges is *infinitesimally rigid*, that is the generic rank of the rigidity matrix $\mathbf{M}_{2,3}(G)$, shown in Figure 9(a) has rank $2n-3$ if and only if G is a Laman graph. We will deduce this as a consequence of Theorem C.



(a) (b)

Fig. 9 The pattern of the rigidity matrices: (a) the matrix $\mathbf{M}_{2,3}(G)$ for bar-joint rigidity; (b) the matrix $\mathbf{M}_{2,0,3}(G)$ for bar-slider framework.

Proof of the Maxwell-Laman Theorem A.

Proof of the Maxwell-Laman Theorem A. Let G be a Laman graph. We need to show that the rank of the rigidity matrix $\mathbf{M}_{2,3}(G)$ is $2n-3$ for a generic framework $G(\mathbf{p})$. We will do this by constructing a point set $\hat{\mathbf{p}}$ for which the rigidity matrix has full rank.

Define a generic direction network $(G,\mathbf{d})$ with its underlying graph G. Because $\mathbf{d}$ is generic, the rank of $\mathbf{M}_{2,2}(G)$ is $2n-3$ for these directions d_{ij}, by Lemma 5.

By Theorem C, there is a point set $\hat{\mathbf{p}}$ such that $\hat{\mathbf{p}}_i \neq \hat{\mathbf{p}}_j$ for all $ij \in E$ and $\hat{\mathbf{p}}_i - \hat{\mathbf{p}}_j = \alpha_{ij}\mathbf{d}_{ij}$ for some non-zero real number α_{ij}. Replacing a_{ij} by $\alpha_{ij}(a_i - a_j)$ and b_{ij} by $\alpha_{ij}(b_i - b_j)$ in $\mathbf{M}_{2,2}(G)$ and scaling each row $1/\alpha_{ij}$ we obtain the rigidity matrix $\mathbf{M}_{2,3}(G)$. It follows that $\mathbf{M}_{2,3}(G)$ has rank $2n-3$ as desired. □

Remarks on Tay's proof of the Maxwell-Laman Theorem [19]. In [19], Tay gives a proof of the Maxwell-Laman Theorem based on so-called *proper* 3T2 *decompositions* of Laman graphs (see [18] for a detailed discussion). The key idea is to work with what Tay calls a "generalized framework" that may have collapsed edges; in the generalized rigidity matrix Tay defines, collapsed edges are simply assigned directions. Tay then starts with a generalized framework in which all edges are collapsed for which it is easy to prove the generalized rigidity matrix has full rank and then uses a 3T2 decomposition to explicitly perturb the vertices so that the rank of the generalized rigidity matrix is maintained as the endpoints of collapsed edges are pulled apart. At the end of the process, the generalized rigidity matrix coincides with the Laman rigidity matrix.

In light of our genericity Lemma 14, we can simplify Tay's approach. Let G be a Laman graph, $\mathscr{D} \subset \mathbb{R}^{2m}$ is the set of directions for which $\mathbf{M}_{2,2}(G)$ has full rank, and $\mathscr{P} \subset \mathscr{D}$ as $\mathscr{P} = \{\mathbf{d} \in \mathscr{D} : \exists \mathbf{p} \in \mathbb{R}^{2n}\, \forall ij \in E\, \mathbf{d}_{ij} = \mathbf{p}_i - \mathbf{p}_j\}$; i.e., $\mathscr{P}$ is the subset of $\mathscr{D}$ arising from the difference set of some planar point set. From the definition of $\mathscr{P}$ and arguments above, if $\mathbf{d} \in \mathscr{P}$ any realization of $(G, \mathbf{d})$ interpreted as a framework will be infinitesimally rigid.

Lemma 14 says that $\mathscr{P}$ is dense in $\mathscr{D}$ (and indeed $\mathbb{R}^{2m}$) if and only if G is a Laman graph. In the language of Tay's generalized frameworks, then, Lemma 14 gives a short, existential proof that a full rank generalized framework can be perturbed into an infinitesimally rigid framework without direct reference to Theorem C. By making the connection to Theorem C explicit, we obtain a canonical infinitesimally rigid realization that can be found using only linear algebra.

7.2 Slider-pinning rigidity

In this section we develop the theory of slider pinning rigidity and prove a Laman-type theorem for it.

Theorem B (Generic bar-slider rigidty). *Let $(G, \ell, \mathbf{n}, \mathbf{s})$ be a generic bar-slider framework. Then $(G, \ell, \mathbf{n}, \mathbf{s})$ is minimally rigid if and only if G is looped-Laman.*

We begin with the formal definition of the problem.

The slider-pinning problem. An abstract *bar-slider framework* is a triple $(G, \boldsymbol{\ell}, \boldsymbol{s})$ where $G = (V, E)$ is a graph with n vertices, m edges and c self-loops. The vector $\boldsymbol{\ell}$ is a vector of m positive squared edge-lengths, which we index by the edges E of G. The vector $\boldsymbol{s}$ specifies a line in the Euclidean plane for each self-loop in G, which we index as i_j for the jth loop at vertex i; lines are given by a normal vector $\mathbf{n}_{i_j} = (c_{i_j}, d_{i_j})$ and a constant e_{i_j}.

A *realization* $G(\mathbf{p})$ is a mapping of the vertices of G onto a point set $\mathbf{p} \in (\mathbb{R}^2)^n$ such that:

$$||\mathbf{p}_i - \mathbf{p}_j||^2 = \boldsymbol{\ell}_{ij} \quad \text{for all edges } ij \in E \tag{4}$$

$$\langle \mathbf{p}_i, \mathbf{n}_{i_j} \rangle = e_{i_j} \text{ for all self-loops } i_j \in E \tag{5}$$

In other words, $\mathbf{p}$ respects all the edge lengths and assigns every self-loop on a vertex to a point on the line specified by the corresponding slider.

Continuous slider-pinning. The *configuration space* $\mathscr{C}(G) \subset (\mathbb{R}^2)^n$ of a bar-slider framework is defined as the space of real solutions to equations (4) and (5):

$$\mathscr{C}(G) = \{\mathbf{p} \in (\mathbb{R}^2)^n : G(\mathbf{p}) \text{ is a realization of } (G, \boldsymbol{\ell}, \boldsymbol{s})\}$$

A bar-slider framework $G(\mathbf{p})$ is *slider-pinning rigid* (shortly, *pinned*) if $\mathbf{p}$ is an isolated point in the configuration space $\mathscr{C}(G)$ and *flexible* otherwise. It is minimally pinned if it is pinned but fails to remain so if any edge or loop is removed.

Infinitesimal slider-pinning. Pinning-rigidity is a difficult condition to establish algorithmically, so we consider instead the following linearization of the problem. Let $G(\mathbf{p})$ be an axis-parallel bar-slider framework with m edges and c sliders. The *pinned rigidity matrix* (shortly rigidity matrix) $\mathbf{M}_{2,0,3}(G(\mathbf{p}))$ is an $(m+c) \times 2n$ matrix that has one row for each edge $ij \in E$ and self-loop $i_j \in E$, and one column for each vertex of G. The columns are indexed by the coordinate and the vertex, and we think of them as arranged into two blocks of n, one for each coordinate. The rows corresponding to edges have entries $a_i - a_j$ and $b_i - b_j$ for the x- and y-coordinate columns of vertex i, respectively. The x- and y-coordinate columns associated with vertex j contain the entries $a_j - a_i$ and $b_j - b_i$; all other entries are zero. The row for a loop i_j contains entries c_{i_j} and d_{i_j} in the x- and y-coordinate columns for vertex i; all other entries are zero. Figure 9(b) shows the pattern.

If $\mathbf{M}(G(\mathbf{p}))$ has rank $2n$ (the maximum possible), we say that $G(\mathbf{p})$ is *infinitesimally slider-pinning rigid* (shortly infinitesimally pinned); otherwise it is *infinitesimally flexible*. If $G(\mathbf{p})$ is infinitesimally pinned but fails to be so after removing any edge or loop from G, then it is *minimally infinitesimally pinned*.

The pinned rigidity matrix arises as the differential of the system given by (1) and (5). Its rows span the normal space of $\mathscr{C}$ at $\mathbf{p}$ and the kernel is the tangent space $T_{\mathbf{p}}\mathscr{C}(G)$ at $\mathbf{p}$. With this observation, we can show that infinitesimal pinning implies pinning.

Lemma 23. *Let $G(\mathbf{p})$ be a bar-slider framework. If $G(\mathbf{p})$ is infinitesimally pinned, then $G(\mathbf{p})$ is pinned.*

In the proof, we will need the *complex configuration space* $\mathscr{C}_{\mathbb{C}}(G)$ of G, which is the solution space to the system (1) and (5) in $(\mathbb{C}^2)^n$. The rigidity matrix has the same form in this setting.

Proof. Since $\mathbf{M}(G(\mathbf{p}))$ has $2n$ columns, if its rank is $2n$, then its kernel is the just the zero vector. By the observation above, this implies that the tangent space $T_{\mathbf{p}}\mathscr{C}_{\mathbb{C}}(G)$ is zero-dimensional. A fundamental result result of algebraic geometry [2, p. 479, Theorem 8] says that the irreducible components of $\mathscr{C}_{\mathbb{C}}(G)$ through $\mathbf{p}$ have dimension bounded by the dimension of the tangent space at $\mathbf{p}$.

It follows that $\mathbf{p}$ is an isolated point in the complex configuration space and, by inclusion, in the real configuration space. □

7.3 Generic bar-slider frameworks

Although Lemma 23 shows that infinitesimal pinning implies pinning, the converse is not, in general, true. For example, a bar-slider framework that is combinatorially a triangle with one loop on each vertex is pinned, but not infinitesimally pinned, in a realization where the sliders are tangent to the circumcircle.

For *generic* bar-slider frameworks, however, pinning and infinitesimal pinning coincide. A realization $G(\mathbf{p})$ bar-slider framework is generic if the rigidity matrix attains its maximum rank at $\mathbf{p}$; i.e., $\operatorname{rank}(\mathbf{M}(\mathbf{p})) \geq \operatorname{rank}(\mathbf{M}(\mathbf{p}))$ for all $\mathbf{q} \in \mathbb{R}^{2n}$.

We reformulate genericity in terms of the *generic pinned rigidity matrix* $\mathbf{M}(G)$, which is defined to have the same pattern as the pinned rigidity matrix, but with entries that are formal polynomials in variables a_i, b_i, c_{i_j}, and d_{i_j}. The rank of the generic rigidity matrix is defined as the largest integer r for which there is an $r \times r$ minor of $\mathbf{M}(G)$ which is not *identically zero* as a formal polynomial.

A *graph* G is defined to be *generically infinitesimally rigid* if its generic rigidity matrix $\mathbf{M}(G)$ has rank $2n$ (the maximum possible).

7.4 Proof of Theorem B

We are now ready to give the proof of our Laman-type Theorem B for bar-slider frameworks.

Theorem B (Generic bar-slider rigidty). *Let $(G, \ell, \mathbf{n}, \mathbf{s})$ be a generic bar-slider framework. Then $(G, \ell, \mathbf{n}, \mathbf{s})$ is minimally rigid if and only if G is looped-Laman.*

Proof. Let G be looped-Laman. We will construct a point set $\hat{\mathbf{p}}$, such that the bar-slider framework $G(\hat{\mathbf{p}})$ is infinitesimally pinned.

Fix a generic direction-slider network $(G, \mathbf{d}, \mathbf{n}, \mathbf{s})$ with underlying graph G. By Lemma 6, $\mathbf{M}_{2,0,2}(G)$ has rank $2n$. Applying Theorem D, we obtain a point set $\hat{\mathbf{p}}$ with $\hat{\mathbf{p}}_i \neq \hat{\mathbf{p}}_j$ for all edges $ij \in E$ and $\mathbf{p}_i - \mathbf{p}_j = \alpha_{ij}\mathbf{d}_{ij}$. Substituting in to $\mathbf{M}_{2,0,2}(G)$ and rescaling shows the rank of $\mathbf{M}_{2,0,3}(G)$ is $2n$. □

References

1. Brylawski, T.: Constructions. In: White, N. (ed.) Theory of Matroids, Encyclopedia of Mathematics and Its Applications, chap. 7, pp. 127–223. Cambridge University Press (1986)
2. Cox, D.A., Little, J., O'Shea, D.: Ideals, Varieties and Algorithms. Undergraduate texts in Mathematics, second edn. Springer Verlag, New York (1997)
3. Fekete, Z.: Source location with rigidity and tree packing requirements. Operations Research Letters **34**(6), 607–612 (2006)
4. Gluck, H.: Almost all simply connected closed surfaces are rigid. Lecture Notes in Matehmatics **438**, 225–239 (1975)
5. Graver, J., Servatius, B., Servatius, H.: Combinatorial rigidity, *Graduate Studies in Mathematics*, vol. 2. American Mathematical Society (1993)
6. Haas, R., Lee, A., Streinu, I., Theran, L.: Characterizing sparse graphs by map decompositions. Journal of Combinatorial Mathematics and Combinatorial Computing **62**, 3–11 (2007)
7. Laman, G.: On graphs and rigidity of plane skeletal structures. Journal of Engineering Mathematics **4**, 331–340 (1970)
8. Lee, A., Streinu, I.: Pebble game algorihms and sparse graphs. Discrete Mathematics **308**(8), 1425–1437 (2008). DOI 10.1016/j.disc.2007.07.104
9. Lee, A., Streinu, I., Theran, L.: Graded sparse graphs and matroids. Journal of Universal Computer Science **13**(10) (2007)

10. Lee, A., Streinu, I., Theran, L.: The slider-pinning problem. In: Proceedings of the 19th Canadian Conference on Computational Geometry (CCCG'07) (2007)
11. Lovász, L.: Combinatorial problems and exercises. second edn. AMS Chelsea Publishing, Providence, RI (2007)
12. Lovász, L., Yemini, Y.: On generic rigidity in the plane. SIAM J. Algebraic and Discrete Methods **3**(1), 91–98 (1982)
13. Lovász, L.: Matroid matching and some applications. Journal of Combinatorial Theory, Series (B) **28**, 208–236 (1980)
14. Maxwell, J.C.: On the calculation of the equilibrium and stiffness of frames. Philos. Mag. **27**, 294 (1864)
15. Oxley, J.G.: Matroid theory. The Clarendon Press Oxford University Press, New York (1992)
16. Recski, A.: Matroid Theory and Its Applications in Electric Network Theory and in Statics. Springer Verlag (1989)
17. Saxe, J.B.: Embeddability of weighted graphs in k-space is strongly np-hard. In: Proc. of 17th Allerton Conference in Communications, Control, and Computing, pp. 480–489. Monticello, IL (1979)
18. Streinu, I., Theran, L.: Sparsity-certifying graph decompositions. Graphs and Combinatorics (2009). Accepted, to appear. ArXiv: 0704.0002
19. Tay, T.S.: A new proof of Laman's theorem. Graphs and Combinatorics **9**, 365–370 (1993)
20. Whiteley, W.: The union of matroids and the rigidity of frameworks. SIAM J. Discrete Math. **1**(2), 237–255 (1988)
21. Whiteley, W.: A matroid on hypergraphs, with applications in scene analysis and geometry. Discrete and Computational Geometry (1989)
22. Whiteley, W.: Some matroids from discrete applied geometry. In: Bonin, J., Oxley, J.G., Servatius, B. (eds.) Matroid Theory, *Contemporary Mathematics*, vol. 197, pp. 171–311. American Mathematical Society (1996)
23. Whiteley, W.: Rigidity and scene analysis. In: Goodman, J.E., O'Rourke, J. (eds.) Handbook of Discrete and Computational Geometry, chap. 60, pp. 1327–1354. CRC Press, Boca Raton New York (2004)