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A Positive Monk Formula in the S^1-Equivariant Cohomology of Type A Peterson Varieties

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A POSITIVE MONK FORMULA IN THE S^1-EQUIVARIANT COHOMOLOGY OF TYPE A PETERSON VARIETIES

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ABSTRACT. **Peterson varieties** are a special class of Hessenberg varieties that have been extensively studied e.g. by Peterson, Kostant, and Rietsch, in connection with the quantum cohomology of the flag variety. In this manuscript, we develop a *generalized Schubert calculus*, and in particular a *positive* Chevalley-Monk formula, for the ordinary and Borel-equivariant cohomology of the Peterson variety Y in type A_{n-1}, with respect to a natural S^1-action arising from the standard action of the maximal torus on flag varieties. As far as we know, this is the first example of positive Schubert calculus beyond the realm of Kac-Moody flag varieties G/P.

Our main results are as follows. First, we identify a computationally convenient basis of $H^*_{S^1}(Y)$, which we call the basis of **Peterson Schubert classes**. Second, we derive a **manifestly positive, integral Chevalley-Monk formula** for the product of a cohomology-degree-2 Peterson Schubert class with an arbitrary Peterson Schubert class. Both $H^*_{S^1}(Y)$ and $H^*(Y)$ are generated in degree 2. Finally, by using our Chevalley-Monk formula we give explicit descriptions (via generators and relations) of both the S^1-equivariant cohomology ring $H^*_{S^1}(Y)$ and the ordinary cohomology ring $H^*(Y)$ of the type A_{n-1} Peterson variety. Our methods are both directly from and inspired by those of GKM (Goresky-Kottwitz-MacPherson) theory and classical Schubert calculus. We discuss several open questions and directions for future work.

CONTENTS

1. INTRODUCTION

The main results of this manuscript are

(1) a construction of a computationally convenient module basis of *Peterson Schubert classes* for the S^1-equivariant cohomology ring with $\mathbb{C}$ coefficients of the Peterson variety Y of Lie type A_{n-1}, obtained as the projections of a suitable subset of the well-known equivariant Schubert classes in $H^*_T(\mathcal{F}\ell ags(\mathbb{C}^n))$, and

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(2) a manifestly-positive and manifestly-integral [1] combinatorial Chevalley-Monk formula for the computation of certain products in $H^*_{S^1}(Y)$, obtained from explicit positive formulas for the restrictions of Peterson Schubert classes to the S^1-fixed points of Y.

Moreover, as a straightforward corollary of the above, we also obtain

(3) an explicit description, via generators and relations, of both the S^1-equivariant and ordinary cohomology rings of type A Peterson varieties.

As far as we know, this is the first example of an explicit, complete, and combinatorial Schubert calculus computation of equivariant or ordinary cohomology rings outside of the setting of partial flag varieties G/P. We use techniques both directly from and motivated by GKM (Goresky-Kottwitz-MacPherson) theory and the Schubert calculus of flag varieties. We view our results as the first steps in the development of a generalized equivariant Schubert calculus for Hessenberg varieties (of which Peterson varieties are a special case) and, more generally, for certain subspaces of GKM spaces.

We begin with some background and motivation. Hessenberg varieties arise in many areas of mathematics, including geometric representation theory, numerical analysis, mathematical physics, combinatorics, and algebraic geometry. Their geometry is complicated and subtle: for instance, many Hessenberg varieties are singular, and some are not even pure-dimensional. However, there is a close relationship between Hessenberg varieties and linear algebra, which allows for explicit analysis of their geometry and their connections to other fields. For instance, in the special case of Springer varieties [10, 27, 29], their associated cohomology rings carry natural representations of the symmetric group such that the top-dimensional cohomology is an irreducible representation. More generally, Hessenberg varieties have a paving by affines indexed by certain Young tableaux; the tableaux determine the dimension of the affines according to explicit and simple combinatorial conditions [31].

In this paper, we focus on the special case of **Peterson varieties**, the geometry and combinatorics of which are of particular interest and are the subject of active current research. Indeed, Kostant showed that Peterson varieties have a dense subvariety whose coordinate ring is isomorphic to the quantum cohomology of the flag variety [21]. Rietsch additionally proved that the quantum parameters can be realized as principal minors of certain Toeplitz matrices [25]. Furthermore, these Toeplitz matrices can be obtained using a particular Schubert decomposition of the flag variety intersected with the Peterson variety; this Schubert decomposition gives a paving by affines of the Peterson varieties [30, 31]. Much is still unknown about Peterson varieties. For example, this paper provides the first general computation (e.g. with generators and relations) of the ordinary and equivariant cohomology rings of Peterson varieties.

The goals and methods of this manuscript lie within the realm of **Schubert calculus** and **GKM theory**, both of which focus on explicit, combinatorial computations in (equivariant and ordinary) cohomology rings. We begin with a brief discussion of the former. Classical Schubert calculus is the study of the cohomology ring $H^*(\mathrm{Gr}(k,\mathbb{C}^n))$ of the Grassmannian of k-planes in $\mathbb{C}^n$; more specifically, it asks for the structure constants of $H^*(\mathrm{Gr}(k,\mathbb{C}^n))$ with respect to the *Schubert classes*, which are classes corresponding to *Schubert subvarieties* of $\mathrm{Gr}(k,\mathbb{C}^n)$ and also form an additive basis for the ring. These Schubert classes are also combinatorially natural in the following sense: in the Borel presentation of $H^*(\mathrm{Gr}(k,\mathbb{C}^n))$ as a quotient of a polynomial ring, it is possible to represent these classes by *Schur polynomials*, which are essential and ubiquitous in e.g. symmetric function theory. In the setting of $\mathrm{Gr}(k,\mathbb{C}^n)$, it is known that the structure constants mentioned above are both **positive** and **integral**. Thus, a natural and fundamental goal in classical Schubert

[1] Our structure constants will a priori be elements in $\mathbb{C}[t]$, a polynomial ring in one variable with $\mathbb{C}$ coefficients. In this setting, by "positive and integral" we will mean that the structure constants are polynomials $\sum_i a_i t^i$ with non-negative and integral coefficients $a_i \in \mathbb{Z}_{\geq 0}$.

calculus is to find and prove formulas for these structure constants which are *manifestly* positive and integral (e.g. by counting arguments).

Modern work in Schubert calculus encompasses the study of more general spaces, such as the generalized Kac-Moody flag varieties, as well as more general (ordinary or equivariant) cohomology theories, such as Borel-equivariant cohomology with various coefficient rings, ordinary and equivariant quantum cohomology, as well as ordinary and equivariant K-theory and quantum K-theory, among others. The main goal of modern Schubert calculus is still to prove that the relevant structure constants are positive in a suitable sense, and thence to obtain explicit, elegant, and/or computationally effective combinatorial formulas for these constants. Recently, efforts have been made to extend the ideas of Schubert calculus to cover even more general spaces (e.g. the work of Goldin-Tolman in the context of equivariant symplectic geometry [11]). This manuscript is another step in this direction, in that we develop a complete Schubert-calculus-type description of the equivariant and ordinary cohomology rings of a space which is not a G/P. Although our work and that of Goldin and Tolman are clearly related, they are different in nature; for instance, they always assume their spaces are manifolds, while Peterson varieties are in general singular. Nevertheless, both our methods and those of Goldin-Tolman depend heavily on GKM theory, to which we now turn.

GKM theory was presented by Goresky-Kottwitz-MacPherson [12] based on previous work of e.g. Chang-Skjelbred [7] and others. The original theory builds combinatorial tools to compute the T-equivariant cohomology ring of a T-space X that satisfies certain technical conditions. This influential theory and its many consequences have been extensively generalized and used since [4,11,13,15–17,19,20,22]. In particular, extensions of GKM theory apply to many of the generalized equivariant cohomology theories mentioned above. One of the powerful features of GKM theory is that it allows us to build convenient $H^*_T(\mathrm{pt})$-module generators for the equivariant cohomology $H^*_T(X)$ of the T-space X. (In an equivariant-symplectic-geometric context, the elements of such a basis can be given equivariant-Morse-theoretic interpretations in terms of the moment map for the Hamiltonian T-action.) In the case of $\mathrm{Gr}(k, \mathbb{C}^n)$ or $\mathcal{F}\ell ags(\mathbb{C}^n)$, the equivariant Schubert classes give precisely such a basis, thus allowing for effective use of GKM theory in both classical and modern Schubert calculus [20].

Unfortunately, classical GKM theory does *not* apply to our main objects of study, the type A Peterson varieties. Informally, this is because 'the torus is too small'. More precisely, we have the following. The Peterson variety Y is a subvariety of $\mathcal{F}\ell ags(\mathbb{C}^n)$. It is well-known that the torus action of $n \times n$ invertible diagonal matrices on $\mathcal{F}\ell ags(\mathbb{C}^n)$ satisfies the technical conditions required in GKM theory. However, the torus action of diagonal matrices does *not* preserve the Peterson variety $Y \subseteq \mathcal{F}\ell ags(\mathbb{C}^n)$. A circle subgroup of the torus does preserve Y, but this S^1-action on Y does *not* satisfy the GKM conditions. Nevertheless, we can explicitly analyze this S^1-action and its fixed points Y^{S^1}, and obtain our first main result (Theorem 4.12), which builds a **computationally effective** $H^*_{S^1}(\mathrm{pt})$**-module basis** for $H^*_{S^1}(Y)$. This basis satisfies certain crucial properties in GKM theory (also satisfied by the equivariant Schubert classes in $H^*_T(\mathcal{F}\ell ags(\mathbb{C}^n))$), namely:

(1) upper-triangularity (see Equations (4.1) and (4.2)) and
(2) minimality (see Equation (4.3)).

For precise statements and proof, see Theorem 4.12 and Proposition 5.13. In our situation, there is a natural map $H^*_T(\mathcal{F}\ell ags(\mathbb{C}^n) \to H^*_{S^1}(Y)$ induced by inclusions of tori and varieties. Our module basis of Theorem 4.12 additionally satisfies the property that

(3) each element of the basis is obtained as the image of an equivariant Schubert class in $H^*_T(\mathcal{F}\ell ags(\mathbb{C}^n))$.

Motivated by (3), we call our basis elements **Peterson Schubert classes**. They are indexed by subsets $\mathcal{A} \subseteq \{1, 2, \ldots, n-1\}$, and for the purposes of this section only, we denote by $p_{\mathcal{A}}$ the Peterson

Schubert class corresponding to $\mathcal{A}$. It turns out that the previous three conditions characterize our module basis $\{p_{\mathcal{A}}\}$ uniquely, in a suitable sense (Proposition 5.14).

We now describe our second main result, the Chevalley-Monk formula for Peterson varieties. As a preliminary step, we first prove in Proposition 6.2 that the subset of cohomology-degree-2 classes $p_i := p_{\{i\}}$ for $1 \leq i \leq n-1$ form a set of ring generators of $H^*_{S^1}(Y)$. Given this set of ring generators, our S^1**-equivariant Chevalley-Monk formula formula for Peterson varieties** (see Theorem 6.12, where we use slightly different notation) is a set of *explicit* formulas to compute the product of an arbitrary ring-generator class p_i with an arbitrary module-generator class $p_{\mathcal{A}}$. We have

$$p_i \cdot p_{\mathcal{A}} = c^{\mathcal{A}}_{i,\mathcal{A}} \cdot p_{\mathcal{A}} + \sum_{\mathcal{A} \subsetneq \mathcal{B} \text{ and } |\mathcal{B}|=|\mathcal{A}|+1} c^{\mathcal{B}}_{i,\mathcal{A}} \cdot p_{\mathcal{B}} \tag{1.1}$$

for any i and $\mathcal{A}$, where the structure constants $c^{\mathcal{A}}_{i,\mathcal{A}}, c^{\mathcal{B}}_{i,\mathcal{A}}$ can be explicitly computed as follows.

First,

- $c^{\mathcal{A}}_{i,\mathcal{A}} = 0$ if $i \not\in \mathcal{A}$,
- $c^{\mathcal{A}}_{i,\mathcal{A}} = (\mathcal{H}_{\mathcal{A}}(i) - i + 1)(i - \mathcal{T}_{\mathcal{A}}(i) + 1)t$ if $i \in \mathcal{A}$,

where the variable t is the cohomology degree 2 generator of $H^*_{S^1}(\mathrm{pt}) \cong \mathbb{C}[t]$. Additionally, for a subset $\mathcal{B} \subseteq \{1, 2, \ldots, n-1\}$ which is a disjoint union $\mathcal{B} = \mathcal{A} \cup \{k\}$, we have explicit formulas, for which we need some notation. Given any set $\mathcal{C} \subseteq \{1, 2, \ldots, n-1\}$ and any $k \in \mathcal{C}$, denote by $\mathcal{T}_{\mathcal{C}}(k)$ and $\mathcal{H}_{\mathcal{C}}(k)$ the unique integers such that $\mathcal{T}_{\mathcal{C}}(k) \leq k \leq \mathcal{H}_{\mathcal{C}}(k)$, the consecutive sequence $\{\mathcal{T}_{\mathcal{C}}(k), \mathcal{T}_{\mathcal{C}}(k)+1, \ldots, \mathcal{H}_{\mathcal{C}}(k)-1, \mathcal{H}_{\mathcal{C}}(k)\}$ is a subset of $\mathcal{C}$, and such that $\mathcal{T}_{\mathcal{C}}(k)-1 \not\in \mathcal{C}, \mathcal{H}_{\mathcal{C}}(k)+1 \not\in \mathcal{C}$. Then we have

- $c^{\mathcal{B}}_{i,\mathcal{A}} = 0$ if $i \not\in \{\mathcal{T}_{\mathcal{B}}(k), \mathcal{T}_{\mathcal{B}}(k)+1, \ldots, \mathcal{H}_{\mathcal{B}}(k)-1, \mathcal{H}_{\mathcal{B}}(k)\}$,
- if $k \leq i \leq \mathcal{H}_{\mathcal{B}}(k)$, then

$$c^{\mathcal{B}}_{i,\mathcal{A}} = (\mathcal{H}_{\mathcal{B}}(k) - i + 1) \cdot \begin{pmatrix} \mathcal{H}_{\mathcal{B}}(k) - \mathcal{T}_{\mathcal{B}}(k) + 1 \\ k - \mathcal{T}_{\mathcal{B}}(k) \end{pmatrix}$$

- if $\mathcal{T}_{\mathcal{B}}(k) \leq i \leq k-1$, then

$$c^{\mathcal{B}}_{i,\mathcal{A}} = (i - \mathcal{T}_{\mathcal{B}}(k) + 1) \cdot \binom{\mathcal{H}_{\mathcal{B}}(k) - \mathcal{T}_{\mathcal{B}}(k) + 1}{k - \mathcal{T}_{\mathcal{B}}(k) + 1}.$$

An immediate consequence of the formulas above is that the (non-zero) structure constants $c^{\mathcal{B}}_{i,\mathcal{A}}$ are both *positive* and *integral* in the appropriate sense. Moreover, our formula evidently has many of the desirable properties advertised above: it is explicit, easily computed, and both **manifestly positive** and **manifestly integral**.

Finally, since the cohomology degree 2 Peterson Schubert classes together with the pure equivariant class $t \in \mathbb{C}[t] \cong H^*_{S^1}(t)$ generate the ring $H^*_{S^1}(Y)$, our Chevalley-Monk formula completely determines the $H^*_{S^1}(\mathrm{pt})$-algebra structure of the S^1-equivariant cohomology $H^*_{S^1}(Y)$. In particular, we may explicitly describe $H^*_{S^1}(Y)$ as a ring with generators $\{p_{\mathcal{A}}\}$ and t satisfying precisely the relations (1.1), which we do in Corollary 6.14. Moreover, it can be seen that the forgetful map $H^*_{S^1}(Y) \to H^*(Y)$ takes the Peterson Schubert classes to a $\mathbb{C}$-basis of the ordinary cohomology $H^*(Y)$, and the cohomology degree 2 classes generate $H^*(Y)$ as a ring. Thus, as a straightforward consequence of our S^1-equivariant Chevalley-Monk formula, we obtain both a Chevalley-Monk formula for the ordinary cohomology $H^*(Y)$ of the Peterson variety (Corollary 6.16), as well as an explicit generators-and-relations description of $H^*(Y)$ (Corollary 6.17). We expect these results to lead to a rich array of further work.

The above discussion suggests the wide variety of mathematics related to, and touching upon, this work. Indeed, our intended audience consists of researchers interested in any subset of: Schubert calculus, combinatorics, equivariant algebraic topology, geometric representation theory, algebraic geometry, or symplectic geometry. For this reason we have attempted to keep exposition elementary and prerequisites to a minimum. In particular, we consistently use notation and terminology from type A. Similarly, we favor specificity to generality throughout. An exception to this rule is the appendix, where we prove a general lemma in Borel-equivariant cohomology with field coefficients, included here in this form to be of maximum use for our future work.

We close with a discussion of avenues for further inquiry and a sampling of open questions. First, we intend to explore the relationship between our explicit presentation of the ordinary cohomology ring $H^*(Y)$ of type A Peterson varieties with conjectural presentations due to A. Mbirika. Mbirika's presentation is expressed in terms of 'partial symmetric functions' and Young tableaux, and directly generalizes the classical Borel presentation of $H^*(\mathcal{F}\ell ags(\mathbb{C}^n))$. We already have preliminary results which will be useful in this direction, including a Giambelli formula for the equivariant cohomology of Peterson varieties. Second, and as mentioned above, we view our results here as the first successful example of 'generalized Schubert calculus' which extends beyond the realm of Kac-Moody flag varieties G/P. In this manuscript, we heavily exploit the natural S^1-action on Y, obtained by restricting an $(S^1)^n$-action on a larger GKM space X (in this case $\mathcal{F}\ell ags(\mathbb{C}^n)$). We intend to explore the more general case in which a T'-space Y arises as a T'-invariant subspace of a T-space X which is GKM, for a subtorus T' of T. We have preliminary results which suggest that, under suitable hypotheses, there exist appropriate 'upper-triangular' module bases for $H^*_{T'}(Y)$ similar to those constructed in this manuscript. Finally, we conclude with several open questions which we hope to address in future work.

- The structure constants $c^{\mathcal{B}}_{i,\mathcal{A}}$ appearing in (1.1) are non-negative integers. Are the $c^{\mathcal{B}}_{i,\mathcal{A}}$ are some kind of intersection numbers for suitable geometric objects corresponding to the $p_{\mathcal{A}}$?
- In this manuscript, we restrict to Peterson varieties of Lie type A and to Borel-equivariant cohomology with $\mathbb{C}$ coefficients. Can our results can be generalized to
  - general Lie type,
  - general regular nilpotent Hessenberg varieties, and/or
  - other generalized equivariant cohomology theories (e.g. equivariant K-theory)?
- Brion and Carrell have announced a result of Peterson's which gives a presentation of the S^1-equivariant cohomology of the Peterson variety [5] which is different from ours. What is the relationship between our presentation and theirs?
- Are there Springer-type representations on S^1-equivariant cohomology for all or some Peterson varieties?

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Notation, terminology, and conventions.

n is a fixed but arbitrary positive integer.
G denotes the Lie group $GL(n, \mathbb{C})$.
B denotes the Borel subgroup of G consisting of upper-triangular matrices.
T is the compact maximal torus of the compact form $U(n, \mathbb{C})$ of G, consisting of unitary diagonal matrices.

$\{t_i - t_{i+1} : 1 \leq i \leq n-1\}$ is the set of positive simple roots of $\mathrm{Lie}(G)$.

$w \in S_n$ is expressed in one-line notation. Hence

$$w = (w(1), w(2), \ldots, w(n)) \in S_n$$

is the permutation on n letters sending i to $w(i)$. If $e_1, e_2, \ldots, e_n$ are the standard basis vectors of $\mathbb{C}^n$, then the permutation matrix w is related to the permutation $w \in S_n$ by $we_i = e_{w(i)}$ for all i.

s_i denotes the simple transposition in S_n that interchanges i and $i+1$ and acts as the identity on all other elements of $\{1, 2, \ldots, n\}$.

$s_i \cdot (t_j - t_{j+1})$, the action of the s_i on the positive simple roots $t_j - t_{j+1}$, is given by the action of s_i on the indices of the variables t_k.

$w < w'$ in the **Bruhat order** if for any (hence every) reduced-word decomposition of w', there exists a subword which equals w.

$\ell(w)$ is the length of $w \in S_n$ with respect to the Bruhat order, namely the minimal number k of simple transpositions needed to write $w = s_{i_1} s_{i_2} \cdots s_{i_k}$.

w_0 is the **unique maximal element** of S_n; it has the property that it is Bruhat-larger than every other element of the group.

$\mathbf{b} = (b_1, b_2, \ldots, b_{\ell(w)})$ denotes a reduced-word decomposition of a permutation w. Here $\mathbf{b}$ is the sequence of the indices of the simple transpositions whose product is w, so $w = s_{b_1} s_{b_2} \cdots s_{b_{\ell(w)}}$.

$[a_1, a_2]$ for integers a_1, a_2 with $a_1 \leq a_2$ denotes the set of consecutive integers $\{a_1, a_1+1, \ldots, a_2\}$.

Equivariant cohomology, in this manuscript, means **Borel-equivariant cohomlogy with $\mathbb{C}$ coefficients**.

Restriction refers to the natural map on (equivariant or ordinary) cohomology induced by an inclusion map of spaces $X_1 \hookrightarrow X_2$. In the setting when X_1 is the set of fixed points of X_2 under a group action, some manuscripts refer to this restriction map as a **localization**; we avoid this terminology to prevent confusion with other (e.g. Atiyah-Bott-Berline-Vergne) localization theories.

σ_w is the T-equivariant Schubert class in $H^*_T(G/B)$ corresponding to $w \in S_n$. We will abuse notation and denote also by σ_w the image of σ_w under the inclusion $H^*_T(G/B) \hookrightarrow H^*_T((G/B)^T)$.

$\mathrm{Sym}(\mathfrak{t}^*) \cong \mathbb{C}[t_1, t_2, \ldots, t_n]$ is identified with the T-equivariant cohomology $H^*_T(\mathrm{pt})$.

$\mathrm{Sym}(\mathrm{Lie}(S^1)^*) \cong \mathbb{C}[t]$ is identified with the S^1-equivariant cohomology $H^*_{S^1}(\mathrm{pt})$.

Y denotes the Peterson variety in $G/B \cong \mathcal{F}\ell ags(\mathbb{C}^n)$ of type A_{n-1}.

$\mathcal{H}_{\mathcal{A}}$ and $\mathcal{T}_{\mathcal{A}}$ denote integer functions as given in Definitions 5.4 and 5.5.

2. Peterson varieties, S^1-actions, and S^1-fixed points

In Sections 2.1 and 2.2 below, we very briefly introduce the main characters of this manuscript – both the spaces and the torus (or circle) actions on them. We refer the reader to [30] for a more leisurely account. Then in Section 2.3, we give an explicit combinatorial enumeration of the S^1-fixed points of the Peterson variety which will prove useful in the later sections.

2.1. Flag varieties, Hessenberg varieties, and Peterson varieties.

The **flag variety** (or **flag manifold**) is the complex homogeneous space G/B, which can also be described as the space of nested sequences of subspaces in $\mathbb{C}^n$. Let

$$\mathcal{F}\ell ags(\mathbb{C}^n) := \{V_\bullet = (V_1 \subseteq V_2 \subseteq \cdots V_{n-1} \subseteq \mathbb{C}^n) \mid \dim_{\mathbb{C}}(V_i) = i\}.$$

The group G acts naturally on $\mathcal{F}\ell ags(\mathbb{C}^n)$ by left multiplication, namely $g \cdot V_\bullet := (g \cdot V_i)_{i=1}^n$. The stabilizer of a fixed flag $V_\bullet$ is isomorphic to B; this provides the identification of G/B with $\mathcal{F}\ell ags(\mathbb{C}^n)$.

Hessenberg varieties (in type A) are subvarieties of $\mathcal{F}\ell ags(\mathbb{C}^n) \cong G/B$, specified by pairs consisting of an $n \times n$ complex matrix X and a Hessenberg function h, i.e. a nondecreasing function

$h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$. Given such an X and h, the Hessenberg variety $\mathcal{H}ess(X, h)$ is defined as

$$\mathcal{H}ess(X,h) := \{V_\bullet \in \mathcal{F}\ell ags(\mathbb{C}^n) \mid XV_i \subseteq V_{h(i)} \text{ for all } i = 1, \ldots, n\} \subseteq \mathcal{F}\ell ags(\mathbb{C}^n). \tag{2.1}$$

We say $\mathcal{H}ess(X, h)$ is a *regular nilpotent Hessenberg variety* if X is a principal nilpotent operator, i.e. X has a single Jordan block and its eigenvalue is zero. More concretely, if $E_{i,j}$ denotes the $n \times n$ matrix whose entries are zero except for a 1 in the $(i, j)^{th}$ place, then up to change of basis we may take

$$X = E_{1,2} + E_{2,3} + \cdots + E_{n-1,n}. \tag{2.2}$$

If X is a principal nilpotent operator and the Hessenberg function is given by $h(i) = i + 1$ for $1 \leq i \leq n - 1$ and $h(n) = n$ then $\mathcal{H}ess(X, h)$ is called a **Peterson variety** of Lie type A_{n-1}; we denote it by Y.

For example, if $n = 2$ then the Peterson variety is the full flag variety. If $n = 3$ then the Peterson variety consists of the following flags:

$$\left\{ V_1 = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle, V_2 = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle, V_3 = \mathbb{C}^3 \right\},$$

$$\left\{ V_1 = \left\langle \begin{pmatrix} a \\ 1 \\ 0 \end{pmatrix} \right\rangle, V_2 = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} a \\ 1 \\ 0 \end{pmatrix} \right\rangle, V_3 = \mathbb{C}^3 : \text{ for all } a \in \mathbb{C} \right\},$$

$$\left\{ V_1 = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle, V_2 = \left\langle \begin{pmatrix} 0 \\ b \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle, V_3 = \mathbb{C}^3 : \text{ for all } b \in \mathbb{C} \right\}, \text{ and}$$

$$\left\{ V_1 = \left\langle \begin{pmatrix} c \\ d \\ 1 \end{pmatrix} \right\rangle, V_2 = \left\langle \begin{pmatrix} d \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} c \\ d \\ 1 \end{pmatrix} \right\rangle, V_3 = \mathbb{C}^3 : \text{ for all } c, d \in \mathbb{C} \right\}.$$

2.2. **Torus actions on flag varieties and circle actions on Peterson varieties.** The flag variety $G/B \cong \mathcal{F}\ell ags(\mathbb{C}^n)$ is equipped with a natural $T \cong (S^1)^n$-action coming from usual left multiplication of cosets. This T-action has many useful properties: for instance, there are finitely many T-fixed points $wB \in G/B$, corresponding precisely to the permutation matrices $w \in S_n$.

However, this T-action does *not* restrict to the Hessenberg varieties in G/B, in the sense that an arbitrary Hessenberg variety is typically not preserved by the full T-action. However, not all is lost: a natural S^1 subgroup of the maximal torus T does preserve any Hessenberg variety $\mathcal{H}ess(X, h)$ whose matrix X is nilpotent and in Jordan canonical form. Consider the 1-dimensional subtorus

$$\left\{ \begin{bmatrix} t^n & 0 & \cdots & 0 \\ 0 & t^{n-1} & & 0 \\ & & & \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & & t \end{bmatrix} \middle| \; t \in \mathbb{C}, \; \|t\| = 1 \right\} \subseteq T^n \subseteq U(n, \mathbb{C}) \tag{2.3}$$

of the maximal torus T, which we henceforth denote S^1.

The following are straightforward consequences of (2.1) and (2.3); we leave proofs to the reader.

Fact 2.1. *The S^1 in* (2.3) *preserves any Hessenberg variety $\mathcal{H}ess(X, h)$ with X nilpotent and in Jordan form.*

Fact 2.2. *The S^1-fixed points of G/B are precisely the T-fixed points, i.e.* $(G/B)^T = (G/B)^{S^1}$.

Fact 2.3. *The S^1-fixed points in $\mathcal{H}ess(X,h)$ are the T-fixed points of G/B that lie in $\mathcal{H}ess(X,h)$, namely*

$$\mathcal{H}ess(X,h)^{S^1} = (G/B)^T \cap \mathcal{H}ess(X,h). \tag{2.4}$$

2.3. **Combinatorial enumeration of S^1-fixed points in the Peterson variety.** It is straightforward from the definitions to check that the S^1-fixed points (2.4) for regular nilpotent Hessenberg varieties in type A_{n-1} are the permutations w with $w^{-1}(i) \leq h(w^{-1}(i+1))$ for all $i < n$. In the case of Peterson varieties, this condition is equivalent to

$$w^{-1}(i) \leq w^{-1}(i+1) + 1 \quad \text{for all } 1 \leq i < n. \tag{2.5}$$

In particular, either $w^{-1}(i+1) > w^{-1}(i)$ or $w^{-1}(i+1) = w^{-1}(i) - 1$. This means that the entries in the one-line notation for w^{-1}, read from left to right, must either increase or, alternatively, decrease by exactly 1. The one-line notation for w^{-1} is therefore of the form

$$w^{-1} = (j_1, j_1 - 1, \ldots, 1, j_2, j_2 - 1, \ldots, j_1 + 1, \ldots, n, n-1, \ldots, j_m + 1), \tag{2.6}$$

where $1 \leq j_1 < j_2 < \cdots < j_m < n$ is any sequence of strictly increasing integers. It turns out that for our purposes the complement in $\{1, 2, \ldots, n-1\}$ of the set $\{j_1, j_2, \ldots, j_m\}$ will be more useful. Thus for each permutation $w \in S_n$ satisfying (2.5) we define the subset of $\{1, 2, \ldots, n-1\}$ given by

$$\mathcal{A} := \{i : w^{-1}(i) = w^{-1}(i+1) + 1 \text{ for } 1 \leq i \leq n-1\} \subseteq \{1, 2, \ldots, n-1\}.$$

Informally, $\mathcal{A}$ consists of those indices for which the one-line notation of w^{-1} decreases by 1. This argument shows that the permutations $w \in S_n$ satisfying (2.6) are in bijective correspondence with the set of subsets $\mathcal{A}$. Furthermore, note that the $n \times n$ permutation matrix associated to the w^{-1} above is block diagonal with blocks of size $j_1, (j_2 - j_1), \cdots, (n - j_m)$, each of which has 1's on the antidiagonal and 0 elsewhere. Thus a permutation w^{-1} of the form (2.6) is its own inverse: $w^{-1} = w$. Henceforth we denote by $w_{\mathcal{A}} \in S_n$ the permutation $w^{-1} = w =: w_{\mathcal{A}}$ corresponding as above to a subset $\mathcal{A} \subseteq \{1, 2, \ldots, n-1\}$.

Example 2.4. *Suppose $n = 7$ and $\mathcal{A} = \{1, 2, 3, 5\}$. Then*

$$w_{\mathcal{A}} = (4, 3, 2, 1, 6, 5, 7) \in S_7$$

and the corresponding 7×7 permutation matrix is given by

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

There is a natural decomposition of each set $\mathcal{A}$ into subsets corresponding to the block submatrices in the permutation matrix representation of $w_{\mathcal{A}}$. We make the following definition.

Definition 2.5. A **maximal consecutive (sub)string** of $\mathcal{A}$ is a set of consecutive integers $\{a_1, a_1 + 1, \ldots, a_1 + k\} \subseteq \mathcal{A}$ such that neither $a_1 - 1$ nor $a_1 + k + 1$ is in $\mathcal{A}$. Let $a_2 := a_1 + k$. We denote the corresponding maximal consecutive substring by $[a_1, a_2]$.

Any $\mathcal{A}$ uniquely decomposes into a disjoint union of maximal consecutive substrings

$$\mathcal{A} = [a_1, a_2] \cup [a_3, a_4] \cup \cdots \cup [a_{m-1}, a_m].$$

In Example 2.4, the maximal consecutive strings are $\{1, 2, 3\}$ and $\{5\}$.

Given a consecutive string $[a_j, a_{j+1}] = \{a_j, a_j+1, \ldots, a_{j+1}-1, a_{j+1}\}$, the element $w_{[a_j,a_{j+1}]}$ is the largest element with respect to Bruhat order in the subgroup $S_{[a_j,a_{j+1}]}$ of permutations of $\{a_j, a_j+1, \ldots, a_{j+1}-1, a_{j+1}\}$. We fix the following reduced-word decomposition of $w_{[a_j,a_{j+1}]}$:

$$w_{[a_j,a_{j+1}]} = \prod_{k=0}^{a_{j+1}-a_j} \left(\prod_{i=0}^{a_{j+1}-a_j-k} s_{a_j+i} \right). \tag{2.7}$$

Here we take the convention that a product is always composed from the left to the right, so $\prod_{i=0}^{k} \beta_i = \beta_0 \cdot \beta_1 \cdots \beta_k$ for any expressions β_i.

Example 2.6. *Continuing with Example 2.4, for the maximal consecutive string* $[1,3] := \{1,2,3\}$ *we have*

$$w_{[1,3]} = s_1 s_2 s_3 s_1 s_2 s_1.$$

A reduced-word decomposition for $w_{\mathcal{A}}$ is then obtained by taking the product of the $w_{[a_j,a_{j+1}]}$ for each of the maximal consecutive substrings $[a_j, a_{j+1}]$ of $\mathcal{A}$, ordered so that maximal consecutive substrings increase from left to right. (Simple transpositions commute if their indices differ by at least 2, so if $[a_j, a_{j+1}], [a'_j, a'_{j+1}]$ are disjoint maximal consecutive substrings of $\mathcal{A}$ then $w_{[a_j,a_{j+1}]}$ and $w_{[a'_j,a'_{j+1}]}$ commute.) In other words, suppose $\mathcal{A} = [a_1, a_2] \cup [a_3, a_4] \cup \cdots \cup [a_{m-1}, a_m]$ is a decomposition into maximal consecutive substrings of $\mathcal{A}$ with $a_1 < a_2 < \ldots < a_m$. Then we fix the reduced-word decomposition

$$w_{\mathcal{A}} = w_{[a_1,a_2]} w_{[a_3,a_4]} w_{[a_5,a_6]} \cdots w_{[a_{m-1},a_m]}. \tag{2.8}$$

Example 2.7. *Continuing further with Example 2.4, we have*

$$w_{\mathcal{A}} = w_{[1,3]} w_{[5,5]} = s_1 s_2 s_3 s_1 s_2 s_1 s_5.$$

3. GKM THEORY ON THE FLAG VARIETY AND RESTRICTION TO S^1-FIXED POINTS ON PETERSON VARIETIES

In this section, we describe the general framework used for our computations. Our main conceptual tool is the well-known GKM theory for T-spaces, as recounted in the introduction. Only two aspects of GKM theory are essential to our discussion: first, we use the injectivity of the restriction map to the equivariant cohomology of the torus-fixed points; and second, we use certain special classes, which we call **flow-up classes**, to build a natural module basis over the equivariant cohomology of a point for the equivariant cohomology of the T-space.

We begin by recalling well-known results. The flag variety $G/B \cong \mathcal{F}\ell ags(\mathbb{C}^n)$ is equipped with a natural T-action given by left multiplication on cosets; the fixed points are precisely the isolated points $wB \in G/B$ corresponding to the permutations $w \in S_n$. The T-equivariant inclusion $\imath : (G/B)^T \hookrightarrow G/B$ induces a ring homomorphism from the T-equivariant cohomology of G/B to that of its T-fixed points, i.e.

$$\imath^* : H_T^*(G/B) \hookrightarrow H_T^*((G/B)^T) \cong \bigoplus_{w \in S_n} H_T^*(\mathrm{pt}) \tag{3.1}$$

and it is well-known that $\imath^*$ is an injection. Note that the codomain of the restriction map (3.1) is a direct sum of polynomial rings $H_T^*(\mathrm{pt}) \cong \mathrm{Sym}(\mathfrak{t}^*)$. Since $\imath^*$ is injective, we may therefore uniquely specify elements of $H_T^*(G/B)$ as a list of polynomials in $\mathrm{Sym}(\mathfrak{t}^*) \cong \mathbb{C}[t_1, t_2, \ldots, t_n]$.

A classical result in Schubert calculus is that the T-equivariant cohomology ring $H_T^*(G/B)$ has an $H_T^*(\mathrm{pt})$-module basis given by the (T-equivariant) Schubert classes $\{\sigma_w\}_{w \in S_n}$ [1, 8]. By the above discussion, we may think of σ_w in terms of its image under ι^* in $H_T^*((G/B)^T)$, which in turn we view as a function $S_n \to \mathrm{Sym}(\mathfrak{t}^*)$. Let $\sigma_w(w') \in \mathrm{Sym}(\mathfrak{t}^*)$ denote the value of σ_w at $w' \in S_n$.

The Schubert classes σ_w satisfy certain computationally convenient properties with respect to the Bruhat order on S_n. First, they are **upper-triangular** in an appropriate sense, namely:

$$\sigma_w(v) = 0 \quad \text{if } v \not\geq w \tag{3.2}$$

and

$$\sigma_w(w) \neq 0. \tag{3.3}$$

Second, they are **minimal** among upper-triangular classes: if $\sigma_{w'}$ satisfies the equations (3.2) for w, then

$$\sigma_w(w) \text{ divides } \sigma_{w'}(w). \tag{3.4}$$

One of the main results of this manuscript is to construct a suitable additive $H^*_{S^1}(\mathrm{pt})$-module basis for the S^1-equivariant cohomology of Peterson varieties, similar to the Schubert classes in $H^*_T(G/B)$ in the sense that they satisfy analogous upper-triangularity and minimality conditions. This allows us to develop a theory of "generalized (S^1-equivariant) Schubert calculus" in the equivariant cohomology of Peterson varieties. Moreover, the module basis is obtained as a subset of the images of the Schubert classes σ_w in the S^1-equivariant cohomology of the Peterson variety, as we explain in Section 4. Here and below, we set the stage for this main result by developing the necessary preliminary tools and terminology.

Let Y denote the Peterson variety of type A_{n-1}. As seen in Section 2, the variety Y is naturally an S^1-space for a certain subtorus S^1 of T; moreover $Y^{S^1} = (G/B)^T \cap Y$. Recall that there is a natural forgetful map from T-equivariant cohomology to S^1-equivariant cohomology obtained by the inclusion map of groups $S^1 \hookrightarrow T$. These facts allow us to extend the map (3.1) to the commutative diagram

$$\begin{array}{ccc} H^*_T(G/B) & \longrightarrow & H^*_T((G/B)^T) \\ \downarrow & & \downarrow \\ H^*_{S^1}(G/B) & \longrightarrow & H^*_{S^1}((G/B)^T) \\ \downarrow & & \downarrow \\ H^*_{S^1}(Y) & \longrightarrow & H^*_{S^1}(Y^{S^1}). \end{array} \tag{3.5}$$

The images of the equivariant Schubert classes $\{\sigma_w\}$ under the composition of the natural maps $H^*_T(G/B) \to H^*_{S^1}(G/B) \to H^*_{S^1}(Y)$ are crucial to our discussion, so we make a definition.

Definition 3.1. Let σ_w be an equivariant Schubert class in $H^*_T(G/B)$. Let $p_w \in H^*_{S^1}(Y)$ be the image of σ_w under the ring map $H^*_T(G/B) \to H^*_{S^1}(Y)$ in (3.5). We call p_w the **Peterson Schubert class corresponding to** w.

We want to specify the Peterson Schubert class p_w by its image in $H^*_{S^1}(Y^{S^1})$ via the bottom horizontal arrow in (3.5). For this we need the following.

Theorem 3.2. *Let Y be the type A_{n-1} Peterson variety, equipped with the natural S^1-action defined by* (2.3)*. Then*

- $H^*_{S^1}(Y) \cong H^*_{S^1}(\mathrm{pt}) \otimes H^*(Y)$ *as* $H^*_{S^1}(\mathrm{pt})$*-modules, and*
- *the inclusion $Y^{S^1} \hookrightarrow Y$ induces a ring map*

$$\imath^* : H^*_{S^1}(Y) \to H^*_{S^1}(Y^{S^1})$$

which is injective.

Proof. It is well-known ([3, Chapter III, Section 14], [18, Chapter 6]) that if the ordinary cohomology of Y is concentrated in even degree, then the Leray-Serre spectral sequence for the fibration $Y \to Y \times_T ET \to BT$ collapses, which then implies the first conclusion of the theorem. Recall that the (complex) affine cells in a paving by affines[2] of a complex algebraic variety induce homology generators [9, 19.1.11]; in particular, since the cells are complex, they are even-dimensional and hence a complex variety with a paving by affines has ordinary cohomology only in even degree. Peterson varieties in type A_{n-1} admit a paving by affines [30]. Moreover, the abstract localization theorem [18, Theorem 11.4.4] states that the kernel of $\imath^*$ is the module of torsion elements in $H^*_T(Y)$. Since we have just seen that $H^*_T(Y)$ is a free $H^*_T(\mathrm{pt})$-module, the kernel must be 0, and $\imath^*$ is injective, as desired. □

The theorem above implies that we may think of $p_w \in H^*_{S^1}(Y)$ purely in terms of their images in $H^*_{S^1}(Y^{S^1})$, as in the case of equivariant Schubert classes in $H^*_T(G/B)$. Since the restriction map is injective, we will abuse notation and refer to the image of p_w in $H^*_{S^1}(Y^{S^1})$ also as p_w. The S^1-fixed points of Y are isolated so

$$H^*_{S^1}(Y^{S^1}) \cong \bigoplus_{w' \in Y^{S^1}} H^*_{S^1}(w') \cong \bigoplus_{w' \in Y^{S^1}} \mathbb{C}[t].$$

This means each p_w is a function $Y^{S^1} \to \mathbb{C}[t]$ just as in the case of G/B. Following our notation for G/B, if $w' \in Y^{S^1} \subseteq (G/B)^T$ is a fixed point, we denote by $p_w(w')$ the value of the restriction of p_w to w'.

Finally, we observe that the restrictions $p_w(w')$ may be computed using the restrictions $\sigma_w(w')$ of the equivariant Schubert classes on G/B and the maps in (3.5) .

Proposition 3.3. *Let Y be the type A_{n-1} Peterson variety and let p_w be a Peterson Schubert class corresponding to $w \in S_n$. Let $w' \in Y^{S^1} \subseteq (G/B)^T \cong S_n$. Then $p_w(w') \in H^*_{S^1}(\mathrm{pt})$ is the image of $\sigma_w(w') \in \mathrm{Sym}(\mathfrak{t}^*) \cong \mathbb{C}[t_1, t_2, \ldots, t_n]$ under the projection map*

$$\begin{aligned} \pi_{S^1} : \mathbb{C}[t_1, t_2, \ldots, t_n] &\longrightarrow \mathbb{C}[t] \\ t_i &\longmapsto (n-i+1)t. \end{aligned} \tag{3.6}$$

Proof. Recall that

$$H^*_T((G/B)^T) \cong \bigoplus_{w \in S_n} \mathrm{Sym}(\mathfrak{t}^*) \cong \bigoplus_{w \in S_n} \mathbb{C}[t_1, t_2, \ldots, t_n]$$

and

$$H^*_{S^1}((G/B)^T) \cong \bigoplus_{w \in S_n} \mathrm{Sym}(\mathrm{Lie}(S^1)^*) \cong \bigoplus_{w \in S_n} \mathbb{C}[t].$$

The top right arrow in (3.5) that sends $H^*_T((G/B)^T) \to H^*_{S^1}((G/B)^T)$ is induced from the projection map $\mathrm{Sym}(\mathfrak{t}^*) \to \mathrm{Sym}(\mathrm{Lie}(S^1)^*)$ coming from the inclusion $\mathrm{Lie}(S^1) \hookrightarrow \mathfrak{t}$. The definition of the subgroup S^1 in (2.3) implies that each t_i projects to $(n-i+1)t$. For the bottom right arrow in (3.5), we recall that $(G/B)^{S^1} = (G/B)^T$, as observed in Section 2. We then see that the map

$$\bigoplus_{w \in S_n} \mathbb{C}[t] \cong H^*_{S^1}((G/B)^{S^1}) \to H^*_{S^1}(Y^{S^1}) \cong \bigoplus_{w \in Y^{S^1}} \mathbb{C}[t]$$

is the identity on each component corresponding to $w \in Y^{S^1} \subseteq S_n \cong (G/B)^{S^1}$ and is 0 on each component corresponding to $w \in S_n \setminus Y^{S^1}$. More colloquially, it kills the components in the direct

[2] A paving by affines is like a cell decomposition, but the closure conditions on a paving by affines are weaker than for a cell decomposition.

sum associated to S^1-fixed points in G/B which do not appear in Y. Composition of the two arrows and commutativity of the diagram in (3.5) give the desired result. □

4. A $H^*_{S^1}(\mathrm{pt};\mathbb{Q})$-MODULE BASIS FOR THE S^1-EQUIVARIANT COHOMOLOGY OF PETERSON VARIETIES

As recounted in Section 3, the equivariant Schubert classes $\{\sigma_w\}_{w\in S_n}$ have properties which make them particularly convenient for Schubert-calculus computations. One of the main results of this manuscript is an explicit construction, in Theorem 4.12, of an $H^*_{S^1}(\mathrm{pt})$-module basis for the S^1-equivariant cohomology of Peterson varieties which also satisfies upper-triangularity and minimality conditions. As in classical Schubert calculus, this makes the basis especially useful for explicit computations; we exploit these properties to derive Monk formulas in Section 6.

First we make precise the conditions satisfied by our module basis of $H^*_{S^1}(Y)$. The upper-triangularity condition on Schubert classes is stated in terms of the Bruhat order on permutations $w \in S_n$ viewed as T-fixed points in G/B. Bruhat order restricts to Y^{S^1} since Y^{S^1} is a subset of $(G/B)^T \cong S_n$. We use this partial order, also called Bruhat order, on the S^1-fixed points of Y.

Next we define permutations $v_{\mathcal{A}} \in S_n$ which are naturally associated to each subset $\mathcal{A} \subseteq \{1, 2, \ldots, n-1\}$. We saw in Section 2.3 that Y^{S^1} is enumerated by the set of subsets $\mathcal{A}$ of $\{1, 2, \ldots, n-1\}$. We will see that the Peterson Schubert classes $p_{v_{\mathcal{A}}}$ associated to the permutations $v_{\mathcal{A}}$ form an additive $H^*_{S^1}(\mathrm{pt})$-module basis for $H^*_{S^1}(Y)$, thus playing a role analogous to Schubert classes in $H^*_T(G/B)$. We have the following.

Definition 4.1. Let $\mathcal{A} = \{j_1 < j_2 < \cdots j_m\}$ be a subset of $\{1, 2, \ldots, n-1\}$. We define the element $v_{\mathcal{A}} \in S_n$ to be the product of simple transpositions whose indices are in $\mathcal{A}$, in increasing order, i.e.

$$v_{\mathcal{A}} := s_{j_1} s_{j_2} \cdots s_{j_m} = \prod_{i=1}^{m} s_{j_i}.$$

Each subset $\mathcal{A} \subseteq \{1, 2, \ldots, n-1\}$ corresponds to a unique permutation of the form $v_{\mathcal{A}}$ so the collection of Peterson Schubert classes $\{p_{v_{\mathcal{A}}}\}$ for all subsets $\mathcal{A} \subseteq \{1, 2, \ldots, n-1\}$ gives rise to a collection of elements in $H^*_{S^1}(Y)$ in one-to-one correspondence with the S^1-fixed points of Y.

Our next tasks are to show that this collection $\{p_{v_{\mathcal{A}}}\}$ satisfies conditions analogous to (3.2) with respect to the (restricted) Bruhat order. We enumerate the conditions precisely.

(1) Upper-triangularity:

$$p_{v_{\mathcal{A}}}(w_{\mathcal{B}}) = 0 \quad \text{if } \; w_{\mathcal{B}} \not\geq v_{\mathcal{A}} \tag{4.1}$$

and

$$p_{v_{\mathcal{A}}}(w_{\mathcal{A}}) \neq 0. \tag{4.2}$$

(2) Minimality:

$$p_{v_{\mathcal{A}}}(w_{\mathcal{A}}) \text{ divides } p_w(w_{\mathcal{A}}) \text{ in } \mathbb{C}[t] \tag{4.3}$$

if p_w is any Peterson Schubert class satisfying the upper-triangularity condition (4.1) for $\mathcal{A}$.

We now prove that the $p_{v_{\mathcal{A}}}$ satisfy the upper-triangularity condition, which will naturally lead to our main Theorem 4.12; in the next section, we find that the collection $\{p_{v_{\mathcal{A}}}\}$ satisfies the minimality condition and is unique in an appropriate sense (Proposition 5.14).

Note that the definition of $v_{\mathcal{A}}$ immediately implies that

$$s_j < v_{\mathcal{A}} \quad \text{for all } j \in \mathcal{A}.$$

We record some basic facts below which will be important in what follows. The proofs are straightforward and left to the reader.

Fact 4.2. *The Bruhat-length of $v_{\mathcal{A}}$ is the size of the set $\mathcal{A}$, i.e. $\ell(v_{\mathcal{A}}) = |\mathcal{A}|$. In particular, the decomposition in Definition 4.1 is minimal-length.*

Fact 4.3. *If $\mathcal{A} = [a_k, a_{k+1}]$ is a maximal consecutive string, then the word in Definition 4.1 is the unique reduced word decomposition for $v_{[a_k,a_{k+1}]}$.*

Fact 4.4. *If $\mathcal{A} = [a_1, a_2] \cup [a_3, a_4] \cup \cdots \cup [a_{m-1}, a_m]$ is a decomposition of $\mathcal{A}$ into maximal consecutive substrings with $1 \leq a_1 < a_2 < \cdots < a_m < n$, then*

$$v_{\mathcal{A}} = v_{[a_1,a_2]} v_{[a_3,a_4]} \cdots v_{[a_{m-1},a_m]}.$$

Moreover, there exists exactly one subword of $v_{\mathcal{A}}$ which is equal to $v_{[a_i,a_{i+1}]}$.

Fact 4.5. *There exists exactly one reduced subword in the reduced word decomposition* (2.8) *of $w_{\mathcal{A}}$ which is equal to $v_{\mathcal{A}}$. (The proof uses uniqueness of the reduced word $v_{[a_i,a_{i+1}]}$ for each minimal string, and examination of the definition of $w_{\mathcal{A}}$.)*

The next lemma is the crucial observation which allows us to show that the Peterson Schubert classes $p_{v_{\mathcal{A}}}$ corresponding to these special Weyl group elements $v_{\mathcal{A}}$ are a $H^*_{S^1}(\mathrm{pt})$-module basis for $H^*_{S^1}(Y)$. The essence is that the Bruhat order on Y^{S^1} can be translated to the usual partial order on sets given by containment.

Lemma 4.6. *Let $\mathcal{A}, \mathcal{B}$ be subsets in $\{1, 2, \ldots, n-1\}$. Then $v_{\mathcal{A}} \leq w_{\mathcal{B}}$ if and only if $\mathcal{A} \subseteq \mathcal{B}$.*

Proof. If $\mathcal{A} \subseteq \mathcal{B}$ then $v_{\mathcal{A}} \leq w_{\mathcal{B}}$ by definition of $v_{\mathcal{A}}$ and $w_{\mathcal{B}}$. Now suppose that $v_{\mathcal{A}} \leq w_{\mathcal{B}}$. In particular this means that $s_i \leq v_{\mathcal{A}}$ for all $i \in \mathcal{A}$. Bruhat order is transitive so $s_i \leq w_{\mathcal{B}}$. By definition of $w_{\mathcal{B}}$ this means $i \in \mathcal{B}$. Hence $\mathcal{A} \subseteq \mathcal{B}$ as desired. □

We next develop tools to compute restrictions of $p_{v_{\mathcal{A}}}$ at various fixed points $w_{\mathcal{B}} \in Y^{S^1}$. These methods allow us to prove the upper-triangularity condition (4.1) with respect to the partial order on sets (equivalent to the restriction of Bruhat order by Lemma 4.6). We begin with terminology.

Definition 4.7. Given a permutation w, a choice of reduced-word decomposition $\mathbf{b} = (b_1, b_2, \ldots, b_{\ell(w)})$ of w, and an index $i \in \{1, 2, \ldots, \ell(w)\}$, define

$$r(i, \mathbf{b}) := s_{b_1} s_{b_2} \cdots s_{b_i - 1}(t_{b_i} - t_{b_i+1}). \tag{4.4}$$

By definition $r(i, \mathbf{b})$ is an element of $\mathrm{Sym}(\mathfrak{t}^*) \cong \mathbb{C}[t_1, t_2, \ldots, t_n]$ of the form $t_j - t_k$ for some j, k. Classical results also show that $r(i, \mathbf{b})$ is in fact a *positive* root, namely, it has the form $t_j - t_k$ for $j < k$ [2, Equation 4.1 and discussion]. These positive roots $r(i, \mathbf{b})$ are the building blocks of Billey's formula.

Theorem 4.8. *("Billey's formula", [2, Theorem 4]) Let $w \in S_n$. Fix a reduced word decomposition $w = s_{b_1} s_{b_2} \cdots s_{b_{\ell(w)}}$ and let $\mathbf{b} = (b_1, b_2, \ldots, b_{\ell(w)})$ be the sequence of its indices. Let $v \in S_n$. Then the value of the Schubert class σ_v at the T-fixed point w is given by*

$$\sigma_v(w) = \sum r(i_1, \mathbf{b}) r(i_2, \mathbf{b}) \cdots r(i_{\ell(v)}, \mathbf{b}) \tag{4.5}$$

where the sum is taken over subwords $s_{b_{i_1}} s_{b_{i_2}} \cdots s_{b_{i_{\ell(v)}}}$ of $\mathbf{b}$ that are reduced words for v.

We refer to an individual summand of the expression (4.5), corresponding to a single reduced subword $v = s_{b_{i_1}} s_{b_{i_2}} \cdots s_{b_{i_{\ell(v)}}}$ of w, as a **summand in Billey's formula**.

The following is a well-known consequence of the preceding discussion and theorem.

Fact 4.9. *Each summand in Billey's formula for $\sigma_v(w)$ is a degree $\ell(v)$ polynomial in the simple roots $\{t_i - t_{i+1}\}_{i=1}^{n-1}$ with* nonnegative *integer coefficients.*

This is because each $r(i, \mathbf{b})$ is a *positive* root, namely a non-negative integral linear combination of simple positive roots. Fact 4.9 is sometimes summarized by saying Billey's formula is *positive in the sense of Graham* [14]. This positivity implies that if any summand in Billey's formula for $\sigma_v(w)$ is nonzero, then the entire sum is nonzero. From this we derive the following.

Corollary 4.10. *Let $\mathcal{A} \subseteq \{1, 2, \ldots, n-1\}$. Then*

$$p_{v_{\mathcal{A}}}(w_{\mathcal{A}}) \neq 0.$$

Proof. We noted in Fact 4.5 that $v_{\mathcal{A}}$ can be found as a subword of $w_{\mathcal{A}}$. This implies that $\sigma_{v_{\mathcal{A}}}(w_{\mathcal{A}}) \neq 0$ by the positivity (in the sense of Graham) of Billey's formula. The projection $\mathrm{Sym}(\mathfrak{t}^*) \cong \mathbb{C}[t_1, t_2, \ldots, t_n] \to \mathbb{C}[t]$ sends each $t_i - t_{i+1}$ to t. Hence the image in $\mathbb{C}[t]$ of any nonzero polynomial in the $t_i - t_{i+1}$ with positive coefficients is also nonzero in $\mathbb{C}[t]$. □

The proof of the corollary above also shows the following.

Proposition 4.11. *Let $\mathcal{A}, \mathcal{B}$ be subsets of $\{1, 2, \ldots, n-1\}$. Then*

- *the restriction $p_{v_{\mathcal{A}}}(w_{\mathcal{B}})$ of the Peterson Schubert class $p_{v_{\mathcal{A}}}$ at any $w_{\mathcal{B}}$ has degree $\ell(v_{\mathcal{A}}) = |\mathcal{A}|$ as a polynomial in $\mathbb{C}[t]$,*
- *$p_{v_{\mathcal{A}}}$ has cohomology degree $2|\mathcal{A}|$, and*
- *$p_{v_{\mathcal{A}}}(w_{\mathcal{B}})$ is non-zero if $\sigma_{v_{\mathcal{A}}}(w_{\mathcal{B}})$ is non-zero.*

We may now prove our first main theorem.

Theorem 4.12. *Let Y be the Peterson variety of type A_{n-1} with the natural S^1-action defined by* (2.3). *For $\mathcal{A} \subseteq \{1, 2, \ldots, n-1\}$, let $v_{\mathcal{A}} \in S_n$ be the permutation given in Definition 4.1, and let $p_{v_{\mathcal{A}}}$ be the corresponding Peterson Schubert class in $H^*_{S^1}(Y)$. The classes $\{p_{v_{\mathcal{A}}} : \mathcal{A} \subseteq \{1, 2, \ldots, n-1\}\}$ in $H^*_{S^1}(Y)$*

- *form an $H^*_{S^1}(\mathrm{pt})$-module basis for $H^*_{S^1}(Y)$, and*
- *satisfy the upper-triangularity conditions*

$$p_{v_{\mathcal{A}}}(w_{\mathcal{B}}) = 0 \quad \text{if } \mathcal{B} \not\supseteq \mathcal{A}, \tag{4.6}$$

and

$$p_{v_{\mathcal{A}}}(w_{\mathcal{A}}) \neq 0. \tag{4.7}$$

Proof. We begin with a proof of the upper-triangularity condition (4.6). Recall that $v_{\mathcal{A}} \leq w_{\mathcal{B}}$ if and only if $\mathcal{A} \subseteq \mathcal{B}$ by Lemma 4.6. The image of zero under the map π_{S^1} of Proposition 3.3 is still zero, so it suffices to show that $\sigma_{v_{\mathcal{A}}}(w_{\mathcal{B}}) = 0$ if $v_{\mathcal{A}} \not\leq w_{\mathcal{B}}$. This follows from the upper-triangularity of equivariant Schubert classes (3.2) (or can be proven directly from Billey's formula).

The assertion that $p_{v_{\mathcal{A}}}(w_{\mathcal{A}}) \neq 0$ is the content of Corollary 4.10.

We now show that assertions (4.6) and (4.7) imply that the $\{p_{v_{\mathcal{A}}}\}$, ranging over subsets $\mathcal{A}$ of $\{1, 2, \ldots, n-1\}$, are $H^*_{S^1}(\mathrm{pt})$-linearly independent. Suppose $\sum_{\mathcal{A}} c_{\mathcal{A}} p_{v_{\mathcal{A}}} = 0 \in H^*_{S^1}(Y)$ for $c_{\mathcal{A}} \in H^*_{S^1}(\mathrm{pt})$. If any subset $\mathcal{A}$ has $c_{\mathcal{A}} \neq 0$, then there must exist a minimal such, say $\mathcal{B}$. Evaluating at $w_{\mathcal{B}}$, we conclude that

$$\sum_{\mathcal{A}} c_{\mathcal{A}} \cdot p_{v_{\mathcal{A}}}(w_{\mathcal{B}}) = 0. \tag{4.8}$$

By hypothesis on $\mathcal{B}$, we have $c_{\mathcal{A}} = 0$ for all $\mathcal{A} \subsetneq \mathcal{B}$. On the other hand, by (4.6) we know $p_{v_{\mathcal{A}}}(w_{\mathcal{B}}) = 0$ for all $\mathcal{A} \not\subseteq \mathcal{B}$. Hence the equality (4.8) simplifies to

$$c_{\mathcal{B}} \cdot p_{v_{\mathcal{B}}}(w_{\mathcal{B}}) = 0.$$

From (4.7) and the fact that $H^*_{S^1}(\mathrm{pt}) \cong \mathbb{C}[t]$ is an integral domain, we conclude $c_{\mathcal{B}} = 0$, a contradiction. Hence the $\{p_{v_{\mathcal{A}}}\}$ are linearly independent over $H^*_{S^1}(\mathrm{pt})$.

Facts 4.2 and 4.9 show that for any $w \in Y^{S^1}$ the degree of the polynomial $p_{v_{\mathcal{A}}}(w)$ is $|\mathcal{A}|$. The polynomial variable t has cohomology degree 2 so the cohomology degree of $p_{v_{\mathcal{A}}}$ in $H^*_{S^1}(Y)$ is

$2|\mathcal{A}|$. Since the $p_{v_{\mathcal{A}}}$ are enumerated precisely by the subsets $\mathcal{A}$ of $\{1, 2, \ldots, n-1\}$, we conclude that there are $\binom{n-1}{j}$ distinct Peterson Schubert classes $p_{v_{\mathcal{A}}}$ of cohomology degree precisely $2j$. A result of Sommers and Tymoczko [28] states that $\binom{n-1}{j}$ is precisely the $2j$-th Betti number of $H^*(Y)$. Hence by Proposition A.1 the $\{p_{v_{\mathcal{A}}}\}$ form an $H^*_{S^1}(\mathrm{pt})$-module basis for $H^*_{S^1}(Y)$, as desired. □

5. Combinatorial formulas for restrictions of Peterson Schubert classes to S^1-fixed points

In this section we explicitly evaluate the restrictions of a Peterson Schubert class $p_{v_{\mathcal{A}}}$ at certain S^1-fixed points $w_{\mathcal{B}}$. This will give a closed-form expression for the values $p_{v_{\mathcal{A}}}(w_{\mathcal{B}})$ needed in Section 6. We also use these results to show that the module basis $\{p_{v_{\mathcal{A}}}\}$ satisfies the minimality condition (4.3), and is the unique (in an appropriate sense) set of Peterson Schubert classes in $H^*_{S^1}(Y)$ satisfying both the upper-triangularity and minimality conditions.

Our formulas will arise from a careful analysis of Billey's formula, although our main interest is not in the $\sigma_{v_{\mathcal{A}}}(w_{\mathcal{B}})$ but rather their images $p_{v_{\mathcal{A}}}(w_{\mathcal{B}})$ via the projection map π_{S^1} in (3.6). This motivates us to establish the following terminology.

Definition 5.1. Let $v, w \in S_n$ and let $\sigma_v(w) = \sum r(i_1, \mathbf{b})r(i_2, \mathbf{b}) \cdots r(i_{\ell(v)}, \mathbf{b})$ be Billey's formula for the restriction. Using the projection π_{S^1} of Proposition 3.3, we refer to the expression

$$p_v(w) = \sum \pi_{S^1}(r(i_1, \mathbf{b}))\pi_{S^1}(r(i_2, \mathbf{b})) \cdots \pi_{S^1}(r(i_{\ell(v)}, \mathbf{b}))$$

as **Billey's formula for** p_w.

We will proceed by first explicitly computing the projection to $H^*_{S^1}(Y)$ of each of the factors $r(i, \mathbf{b})$ in each of the summands of Billey's formula for σ_w. From this, we derive concrete, explicit expressions for the terms in Billey's formula for $p_{v_{\mathcal{A}}}(w_{\mathcal{B}})$.

We begin with the special case when $\mathcal{A}$ consists of a single maximal consecutive string. Before stating the lemma, we recall that the positive roots of $G = GL(n, \mathbb{C})$ have the form

$$t_j - t_{k+1}$$

for $j < k+1$, and each such root may be expressed as a sum of positive simple roots as follows:

$$t_j - t_{k+1} = (t_j - t_{j+1}) + (t_{j+1} - t_{j+2}) + \cdots + (t_k - t_{k+1}).$$

The **length** of the positive root $t_j - t_{k+1}$ is $k - j + 1$. Recall that Proposition 3.3 showed that

$$\pi_{S^1}(t_j - t_{k+1}) = (k + 1 - j)t.$$

Lemma 5.2. *Let $\mathcal{A} = [a_1, a_2] \subseteq \{1, 2, \ldots, n-1\}$ consist of a single maximal consecutive string, let $w_{\mathcal{A}}$ be the corresponding Weyl group element, and let $\mathbf{b} = (b_1, \ldots, b_{\ell(w_{\mathcal{A}})})$ be the reduced word decomposition of $w_{\mathcal{A}}$ given in* (2.8). *Fix an index b_m for some $1 \leq m \leq \ell(w_{\mathcal{A}})$. Then*

(1) *the index b_m equals i for some $a_1 \leq i \leq a_2$,*
(2) *$r(m, \mathbf{b})$ is a positive root of length $i - a_1 + 1$, and*
(3) *$\pi_{S^1}(r(m, \mathbf{b})) = (i - a_1 + 1)t$.*

In particular, the projection $\pi_{S^1}(r(m, \mathbf{b}))$ and the length of the positive root $r(m, \mathbf{b})$ depend only on the value of the index b_m and not on its position m in $\mathbf{b}$.

Proof. The first claim is immediate from the fact that $\mathcal{A} = [a_1, a_2]$ and the definition of $w_{\mathcal{A}}$. We prove the latter two claims by induction on the length of the consecutive string. The base case is when $\mathcal{A} = [a] = \{a\}$ is a singleton set, $\ell(w_{\mathcal{A}}) = 1$, and $w_{\mathcal{A}} = s_a$ is a single simple transposition. In this case the only possible choice of b_m is $m = 1$ and $b_m = a$. Moreover $r(m = 1, \{a\})$ is $t_a - t_{a+1}$, which is a positive root of length 1. By Proposition 3.3, the root $t_a - t_{a+1}$ maps to t, so

$$\pi_{S^1}(r(1, \{a\})) = (a - a + 1)t = 1 \cdot t$$

as desired.

Now suppose that the consecutive string is $[a_1, a_2]$ with $a_2 > a_1$ and that the lemma holds for the consecutive string $[a_1, a_2 - 1]$. By definition of $w_{[a_1,a_2]}$ and $w_{[a_1,a_2-1]}$ in (2.7), we have

$$w_{[a_1,a_2]} = s_{a_1} s_{a_1+1} \cdots s_{a_2} w_{[a_1,a_2-1]}. \tag{5.1}$$

Let $\mathbf{b}_{[a_1,a_2-1]}$ and $\mathbf{b}_{[a_1,a_2]}$ be the reduced word decompositions of $w_{[a_1,a_2-1]}$ and $w_{[a_1,a_2]}$, respectively, given by (2.7). Then

$$\mathbf{b}_{[a_1,a_2]} = \{b_1 = a_1, b_2 = a_1 + 1, \cdots, b_{a_2-a_1+1} = a_2\} \cup \mathbf{b}_{[a_1,a_2-1]} \tag{5.2}$$

where the union is of *ordered* sequences. We now prove the lemma holds for the first $a_2 - a_1 + 1$ indices in $\mathbf{b}_{[a_1,a_2]}$, i.e. for b_m when $1 \leq m \leq a_2 - a_1 + 1$. Direct calculation shows that for any such b_m we have

$$r(m, \mathbf{b}_{[a_1,a_2]}) = s_{a_1} s_{a_1+1} \cdots s_{b_m-1}(t_{b_m} - t_{b_m+1}) = t_{a_1} - t_{b_m+1}$$

which by definition of π_{S^1} projects to

$$\pi_{S^1}(r(m, \mathbf{b}_{[a_1,a_2]})) = (b_m - a_1 + 1)t.$$

This proves the result for the first $a_2 - a_1 + 1$ indices appearing in $\mathbf{b}_{[a_1,a_2]}$.

We now prove the result for indices $b_m \in \mathbf{b}_{[a_1,a_2]}$ with $m > a_2 - a_1 + 1$. By observations (5.1) and (5.2), the m-th element $b_m = i$ in $\mathbf{b}_{[a_1,a_2]}$ for $m > a_2 - a_1 + 1$ is the $(m-(a_2-a_1+1))$-th element in $\mathbf{b}_{[a_1,a_2-1]}$ and

$$r(m, \mathbf{b}_{[a_1,a_2]}) = s_{a_1} s_{a_1+1} \cdots s_{a_2} r(m - (a_2 - a_1 + 1), \mathbf{b}_{[a_1,a_2-1]}).$$

By the inductive assumption $r(m - (a_2 - a_1 + 1), \mathbf{b}_{[a_1,a_2-1]})$ is a positive root $t_j - t_{j+i-a_1+1}$ of length $i - a_1 + 1$. Note that i, j, and $j + i - a_1$ are all in $[a_1, a_2 - 1]$ by definition of $w_{[a_1,a_2-1]}$. A computation shows

$$r(m, \mathbf{b}_{[a_1,a_2]}) = s_{a_1} s_{a_1+1} \cdots s_{a_2}(t_j - t_{j+i-a_1+1}) = t_{j+1} - t_{j+i-a_1+2}.$$

So $r(m, \mathbf{b}_{[a_1,a_2]})$ is also a positive root of length $i-a_1+1$, and this length depends only on the value i of the index b_m and not on its location in $\mathbf{b}_{[a_1,a_2]}$. By definition of π_{S^1} we see $\pi_{S^1}(r(m, \mathbf{b}_{[a_1,a_2]})) = (i - a_1 + 1)t$. □

Example 5.3. *Let $w = w_{[1,3]} = s_1 s_2 s_3 s_1 s_2 s_1$. Then we have*

j	1	2	3	4	5	6
$r(j, \mathbf{b}_{[1,3]})$	$t_1 - t_2$	$t_1 - t_3$	$t_1 - t_4$	$t_2 - t_3$	$t_2 - t_4$	$t_3 - t_4$
$\pi_{S^1}(r(j, \mathbf{b}_{[1,3]}))$	t	$2t$	$3t$	t	$2t$	t

The above lemma says the maximal consecutive substring containing $i \in \mathcal{A}$ determines the corresponding factor in each summand of Billey's formula. This motivates the following definitions.

Definition 5.4. Fix $\mathcal{A} \subseteq \{1, 2, \ldots, n-1\}$. Let $\mathcal{H}_\mathcal{A} : \mathcal{A} \to \mathcal{A}$ be the function such that

$\mathcal{H}_\mathcal{A}(j) =$ the maximal element ("the head") in the maximal consecutive substring of $\mathcal{A}$ containing j.

Definition 5.5. Fix $\mathcal{A} \subseteq \{1, 2, \ldots, n-1\}$. Let $\mathcal{T}_\mathcal{A} : \mathcal{A} \to \mathcal{A}$ be the function such that

$\mathcal{T}_\mathcal{A}(j) =$ the minimal element ("the tail") in the maximal consecutive substring of $\mathcal{A}$ containing j.

Example 5.6. *For example if $\mathcal{A} = \{1, 2, 3, 5, 6\}$ then*

j	1	2	3	5	6
$\mathcal{T}_\mathcal{A}(j)$	1	1	1	5	5
$\mathcal{H}_\mathcal{A}(j)$	3	3	3	6	6

Using these functions, we may describe the $p_{v_\mathcal{A}}(w_\mathcal{B})$ concretely. Building on the previous lemma, we obtain the following expression for the summands in Billey's formula for $p_{v_\mathcal{A}}$.

Lemma 5.7. *Let $\mathcal{A}, \mathcal{B} \subseteq \{1, \ldots, n-1\}$. If $\mathcal{A} \subseteq \mathcal{B}$ then each summand in Billey's formula for $p_{v_\mathcal{A}}(w_\mathcal{B})$ is*

$$\left(\prod_{j \in \mathcal{A}} (j - \mathcal{T}_\mathcal{B}(j) + 1) \right) t^{|\mathcal{A}|}. \tag{5.3}$$

In particular, all summands in Billey's formula for $p_{v_\mathcal{A}}(w_\mathcal{B})$ are equal.

Proof. Let $\mathbf{b} = (b_1, b_2, \ldots, b_{\ell(w_\mathcal{B})})$ be the reduced-word decomposition for $w_\mathcal{B}$ given in (2.8). Let $J = \{i_1, i_2, \ldots, i_{|J|}\} \subseteq [1, \ell(w_\mathcal{B})]$ be a choice of subset of indices of $\mathbf{b}$ so that $s_{b_{i_1}} s_{b_{i_2}} \cdots s_{b_{i_{|J|}}} = v_\mathcal{A}$. Then there is an equality of sets $\{b_{i_1}, b_{i_2}, \ldots, b_{i_{|J|}}\} = \mathcal{A}$ by Facts 4.3 and 4.4. The image under π_{S^1} of the summand in Billey's formula for $\sigma_{v_\mathcal{A}}(w_\mathcal{B})$ corresponding to the subword specified by J is a product of terms $\pi_{S^1}(r(j, \mathbf{b}))$ for $j \in J$. Lemma 5.2 implies that this product is

$$\prod_{j \in \mathcal{A}} ((j - \mathcal{T}_\mathcal{B}(j) + 1)t) = \left(\prod_{j \in \mathcal{A}} (j - \mathcal{T}_\mathcal{B}(j) + 1) \right) t^{|\mathcal{A}|}$$

as desired. □

Example 5.8. *Suppose $n \geq 6, \mathcal{A} = \{1, 2, 3, 5, 6\}$ and $\mathcal{B} = \{1, 2, 3, 4, 5, 6\}$.*

- *Each summand in Billey's formula for $p_{v_\mathcal{A}}(w_\mathcal{A})$ is $(3!) \cdot (2!)t^5 = 12t^5$.*
- *Each summand in Billey's formula for $p_{v_\mathcal{A}}(w_\mathcal{B})$ is $(3!) \cdot (6 \cdot 5)t^5 = 180t^5$.*
- *Each summand in Billey's formula for $p_{v_\mathcal{B}}(w_\mathcal{B})$ is $(6!)t^6$.*

Since all the summands in Billey's formula for $p_{v_\mathcal{A}}(w_\mathcal{B})$ are equal, we conclude the following.

Proposition 5.9. *Let $\mathcal{A} \subseteq \mathcal{B} \subseteq \{1, 2, \ldots, n-1\}$. Let $p_{v_\mathcal{A}}$ be the Peterson Schubert class corresponding to $v_\mathcal{A}$ and $w_\mathcal{B}$ the permutation corresponding to $\mathcal{B}$. Then*

$$p_{v_\mathcal{A}}(w_\mathcal{B}) = (\textit{number of distinct subwords of } w_\mathcal{B} \textit{ equal to } v_\mathcal{A}) \left(\prod_{j \in \mathcal{A}} (j - \mathcal{T}_\mathcal{B}(j) + 1) \right) t^{|\mathcal{A}|}. \tag{5.4}$$

Example 5.10. *Continuing the previous example, suppose $n \geq 7, \mathcal{A} = \{1, 2, 3, 5, 6\}$ and $\mathcal{B} = \{1, 2, 3, 4, 5, 6\}$. The reader can check that*

- $p_{v_\mathcal{A}}(w_\mathcal{A}) = (3!) \cdot (2!)t^5 = 12t^5$.
- $p_{v_\mathcal{A}}(w_\mathcal{B}) = \binom{6}{3} \cdot (3!) \cdot (6 \cdot 5)t^5 = 3600t^5$.
- $p_{v_\mathcal{B}}(w_\mathcal{B}) = (6!)t^6$.

Remark 5.11. *In Section 6, we give explicit formulas for counting the number of ways to find $v_\mathcal{A}$ in $w_\mathcal{B}$ for special cases of $\mathcal{B}$ and $\mathcal{A}$ relevant for the equivariant Chevalley-Monk formula.*

We can now give an explicit combinatorial formula for the value of $p_{v_\mathcal{A}}$ at the fixed point $w_\mathcal{A}$.

Corollary 5.12.

$$p_{v_\mathcal{A}}(w_\mathcal{A}) = \prod_{i \in \mathcal{A}} (i - \mathcal{T}_\mathcal{A}(i) + 1) t^{\ell(v_\mathcal{A})}. \tag{5.5}$$

Proof. We observed in Fact 4.5 that exactly one subword of $w_\mathcal{A}$ is a reduced word decomposition of $v_\mathcal{A}$. The desired result is now a corollary of the previous proposition. □

Next we show that the Peterson Schubert classes $\{p_{v_\mathcal{A}}\}$ satisfy the minimality condition (4.3).

Proposition 5.13. *Let p_w be a Peterson Schubert class for $w \in S_n$ and suppose that p_w satisfies*

$$p_w(w_\mathcal{B}) = 0 \quad \textit{for all} \;\; w_\mathcal{B} \not\geq w_\mathcal{A},$$

and

$$p_w(w_{\mathcal{A}}) \neq 0.$$

Then

$$p_{v_{\mathcal{A}}}(w_{\mathcal{A}}) \mid p_w(w_{\mathcal{A}}) \quad in \quad H^*_{S^1}(\mathrm{pt}).$$

Proof. Let $w \in S_n$ as above. We first claim that $s_i \leq w$ if and only if $i \in \mathcal{A}$. To see this, observe that if $p_w(w_{\mathcal{A}}) \neq 0$, then by Billey's formula, any reduced word for $w_{\mathcal{A}}$ contains a subword which equals w. In particular, if $s_i \leq w$ then s_i also appears in every reduced word for $w_{\mathcal{A}}$. Thus $i \in A$.

To show the converse, we argue by contradiction. Suppose there exists $i \in \mathcal{A}$ with $s_i \not\leq w$. By the above argument, this implies that there exists a proper subset $\mathcal{C} \subsetneq \mathcal{A}$ such that w is generated by $\{s_i : i \in \mathcal{C} \subsetneq \mathcal{A}\}$. Denote this subgroup by $S_{\mathcal{C}}$. Since $w_{\mathcal{C}}$ is by definition the longest word in $S_{\mathcal{C}}$ and $w \in S_{\mathcal{C}}$, it follows that $w \leq w_{\mathcal{C}}$. Billey's formula implies $p_w(w_{\mathcal{C}}) \neq 0$, but $w_{\mathcal{C}} \not\geq w_{\mathcal{A}}$ since $\mathcal{C} \subsetneq \mathcal{A}$, contradicting the upper-triangularity assumption on p_w. Hence if $i \in \mathcal{A}$ then $s_i \leq w$.

Now let $\mathbf{b}$ be the reduced word decomposition for $w_{\mathcal{A}}$ given in (2.8). Lemma 5.2 states that the projection $\pi_{S^1}(r(j, \mathbf{b}))$ of each factor of Billey's formula depends only on the root b_j and not on the location j in $\mathbf{b}$. Since $s_i \leq w$ for each $i \in \mathcal{A}$, we conclude that the product

$$\prod_{i \in \mathcal{A}} ((i - \mathcal{T}_{\mathcal{A}}(i) + 1)\, t)$$

divides each summand in Billey's formula for $p_w(w_{\mathcal{A}})$. On the other hand, Corollary 5.12 shows that $p_{v_{\mathcal{A}}}(w_{\mathcal{A}}) = \prod_{i \in \mathcal{A}}((i - \mathcal{T}_{\mathcal{A}}(i) + 1)t)$. Hence each summand in Billey's formula for $p_w(v_{\mathcal{A}})$ is divisible by $p_{v_{\mathcal{A}}}(w_{\mathcal{A}})$. Since each summand is divisible by $p_{v_{\mathcal{A}}}(w_{\mathcal{A}})$, so is the sum $p_w(v_{\mathcal{A}})$. □

Finally, we prove that the classes $\{p_{v_{\mathcal{A}}}\}$ are uniquely specified among all Peterson Schubert classes by their upper-triangularity properties and their values at the appropriate $w_{\mathcal{A}}$. We emphasize that the uniqueness statement given below in Proposition 5.14 is at the level of cohomology classes in $H^*_{S^1}(Y)$ and not at the level of elements $w \in S_n$. More specifically, since the projection $H^*_T(G/B) \to H^*_{S^1}(Y)$ is *not* one-to-one, there may exist multiple $w \in S_n$ such that $p_w = p_{v_{\mathcal{A}}}$. This latter subtlety is explored further in Proposition 5.16.

Proposition 5.14. *Let $\mathcal{A} \subseteq \{1, 2, \ldots, n-1\}$. Suppose $w \in S_n$ is a permutation such that the corresponding Peterson Schubert class p_w satisfies the upper-triangularity condition for $\mathcal{A}$, i.e.*

$$p_w(w_{\mathcal{B}}) = 0 \quad \text{for all} \quad w_{\mathcal{B}} \not\geq w_{\mathcal{A}},$$

and agrees with $p_{v_{\mathcal{A}}}$ at $w_{\mathcal{A}}$, i.e.

$$p_w(w_{\mathcal{A}}) = p_{v_{\mathcal{A}}}(w_{\mathcal{A}}).$$

Then $p_w = p_{v_{\mathcal{A}}}$.

Proof. Any Peterson Schubert class p_w is a homogeneous-degree class in cohomology. The restriction of p_w at $w_{\mathcal{A}}$ agrees with that of $p_{v_{\mathcal{A}}}$. By Proposition 5.9 the class $p_{v_{\mathcal{A}}}$ has cohomology degree $2|\mathcal{A}|$. Hence both p_w and $p_w - p_{v_{\mathcal{A}}}$ have cohomology degree $2|\mathcal{A}|$.

Theorem 4.12 says the $\{p_{v_{\mathcal{A}}}\}$ form a $H^*_{S^1}(\mathrm{pt})$-basis for $H^*_{S^1}(Y)$, so there are $c_{\mathcal{B}} \in H^*_{S^1}(\mathrm{pt})$ with

$$p_w - p_{v_{\mathcal{A}}} = \sum_{\mathcal{B}} c_{\mathcal{B}} \cdot p_{v_{\mathcal{B}}}.$$

Suppose that some $c_{\mathcal{B}} \neq 0$. Let $\mathcal{A}'$ be a minimal set with $c_{\mathcal{A}'} \neq 0$, meaning there is no $\mathcal{B}$ with $\mathcal{B} \subsetneq \mathcal{A}'$ with $c_{\mathcal{B}} \neq 0$. The upper-triangularity properties of the $p_{v_{\mathcal{B}}}$ imply

$$(p_w - p_{v_{\mathcal{A}}})(w_{\mathcal{A}'}) = c_{\mathcal{A}'} \cdot p_{v_{\mathcal{A}'}}(w_{\mathcal{A}'}).$$

By assumption on $\mathcal{A}'$, Corollary 5.12, and the fact that $H^*_{S^1}(\mathrm{pt})$ is an integral domain, the right hand side of the above equality must be nonzero. Hence the left hand side must also be non-zero. By the upper-triangularity conditions on $p_w - p_{v_{\mathcal{A}}}$ and since $p_w(w_{\mathcal{A}}) = p_{v_{\mathcal{A}}}(w_{\mathcal{A}})$, we conclude that $\mathcal{A} \subsetneq \mathcal{A}'$. In particular $2|\mathcal{A}'| > 2|\mathcal{A}|$ and consequently the cohomology degree of $p_{v_{\mathcal{A}'}}$ is strictly greater than the cohomology degree of $p_w - p_{v_{\mathcal{A}}}$. Moreover, any $H^*_{S^1}(\mathrm{pt})$-multiple of $p_{v_{\mathcal{A}'}}$ must also be of cohomology degree strictly greater than $p_w - p_{v_{\mathcal{A}}}$. Hence we achieve a contradiction if any $c_{\mathcal{A}'} \neq 0$. We conclude all coefficients are zero and that $p_w - p_{v_{\mathcal{A}}} = 0$, as was to be shown. □

Remark 5.15. *This proposition implies we may use the notation $p_{\mathcal{A}}$ instead of $p_{v_{\mathcal{A}}}$ to denote without ambiguity the Peterson Schubert class in $H^*_{S^1}(Y)$ that satisfies the upper-triangularity and minimality conditions for $w_{\mathcal{A}}$. To maintain consistency, we will not change notation in this paper.*

As discussed above, Proposition 5.14 does not imply uniquess at the level of permutations in S_n. Indeed, it is not difficult to verify that if $\mathcal{A} = [a_1, a_2]$ is a single consecutive string, then

$$p_{(v_{\mathcal{A}})^{-1}} = p_{v_{\mathcal{A}}}$$

as cohomology classes in $H^*_{S^1}(Y)$, although for most choices of such $\mathcal{A}$ the two permutations are different. We now prove that this is essentially the only other permutation w with $p_w = p_{v_{\mathcal{A}}}$.

Proposition 5.16. *Let $\mathcal{A} = [a_1, a_2] \subseteq \{1, 2, \ldots, n-1\}$ be a maximal consecutive string with at least two elements. Then $(v_{\mathcal{A}})^{-1}$ is the only permutation $w \neq v_{\mathcal{A}}$ with $p_w = p_{v_{\mathcal{A}}}$.*

Proof. Suppose $w \neq v_{\mathcal{A}}$ and $p_w = p_{v_{\mathcal{A}}}$. Then $w < w_{\mathcal{A}}$ since $p_w(w_{\mathcal{A}})$ is nonzero; in particular $w > s_i$ only if $i \in \mathcal{A}$. On the other hand, for any $\mathcal{B} \subsetneq \mathcal{A}$, we must have $w \not< w_{\mathcal{B}}$ since $p_w(w_{\mathcal{B}})$ is zero for all $\mathcal{B} \subsetneq \mathcal{A}$ by assumption; in particular for all $i \in \mathcal{A}$ the simple transposition s_i must appear in a reduced word decomposition of w, i.e. $s_i < w$ if $i \in \mathcal{A}$. Since $p_w(w_{\mathcal{A}}) = p_{v_{\mathcal{A}}}(w_{\mathcal{A}})$ we conclude that $\ell(w) = \ell(v_{\mathcal{A}})$. Hence w is a permutation of the simple transpositions s_i for all $i \in \mathcal{A}$. The Peterson Schubert class p_w corresponding to any such w satisfies the upper-triangularity condition for $\mathcal{A}$ so it suffices to find w that satisfy the minimality condition. By Proposition 5.9, this is equivalent to finding w that appear exactly once as a subword of $w_{\mathcal{A}}$.

We induct on the size $|\mathcal{A}|$ of $\mathcal{A}$. Let $\mathcal{A} = \{a_1, a_1+1\}$. There are exactly two words of length two in the letters s_{a_1}, s_{a_1+1}. By direct calculation $p_{s_{a_1}s_{a_1+1}}(w_{\mathcal{A}}) = p_{s_{a_1+1}s_{a_1}}(w_{\mathcal{A}})$. Hence the claim holds if $|\mathcal{A}| = 2$.

Now suppose the claim holds when $|\mathcal{A}| = j-1$ and let $|\mathcal{A}| = j$. Exactly one of $s_{a_2}s_{a_2-1}$ and $s_{a_2-1}s_{a_2}$ is a subword of w. The simple transposition s_j commutes with s_{a_2} if $j \in \{a_1, a_1+1, \ldots, a_2-2\}$. Hence either $w = s_{a_2}w'$ or $w = w's_{a_2}$ depending on the relative position of s_{a_2-1} and s_{a_2}. We treat each case separately. Recall also that the simple reflection s_{a_2} appears exactly once in $w_{\mathcal{A}}$ and that $w_{\mathcal{A}} = s_{a_1}s_{a_1+1}\cdots s_{a_2}w_{[a_1,a_2-1]}$.

Suppose $w = s_{a_2}w'$. If $w' \neq v_{[a_1,a_2-1]}$ or if $w' \neq (v_{[a_1,a_2-1]})^{-1}$, then by the inductive hypothesis there are at least two subwords of $w_{[a_1,a_2-1]}$ that equal w', which in turn implies there are at least two subwords of $w_{\mathcal{A}}$ equal to w. This contradicts the assumption on w, so either $w' = v_{[a_1,a_2-1]}$ or $w' = (v_{[a_1,a_2-1]})^{-1}$. Now suppose $w' = v_{[a_1,a_2-1]}$. Then there are at least two subwords of $w_{\mathcal{A}}$ that equal w, namely the subword corresponding to $s_{a_1}s_{a_1+1}\cdots s_{a_2-2}s_{a_2}s_{a_2-1}$ and the subword corresponding to $s_{a_2}s_{a_1}s_{a_1+1}\cdots s_{a_2-2}s_{a_2-1}$, which again contradicts the hypothesis on w. Finally suppose $w' = (v_{[a_1,a_2-1]})^{-1}$. Then $w = (v_{[a_1,a_2]})^{-1}$, and a direct calculation shows that $w_{\mathcal{A}}$ has a unique subword that equals $(v_{[a_1,a_2-1]})^{-1}$. Hence $w = (v_{[a_1,a_2]})^{-1}$ is the only word of the form $s_{a_2}w'$ that satisfies our hypotheses.

Now suppose $w = w's_{a_2}$. If $w' \neq v_{[a_1,a_2-1]}$ then by definition of $v_{[a_1,a_2-1]}$ and assumption on w', the indices of the simple transpositions in a reduced word decomposition of w' are not strictly increasing. In particular there exists an index j such that $s_{j+1}s_{j+2}\cdots s_{a_2}$ is a subword of w and $s_js_{j+1}s_{j+2}\cdots s_{a_1}$ is not a subword of w. Since each of $s_{a_1}, s_{a_1+1}, \ldots, s_{j-1}$ commutes with any of the $s_{j+1}, s_{j+2}, \cdots, s_{a_2}$ and since s_j commutes with all of $s_{j+2}, s_{j+3}, \cdots, s_{a_2}$ we may write w as

$$w = s_{j+1}s_{j+2}\cdots s_{a_2}w''$$

where w'' is a permutation of the transpositions $s_{a_1}, s_{a_1+1}, \ldots, s_j$. The assumption on w implies $j \neq a_2 - 1$, so there is at least one way to insert $s_{j+1}, \ldots, s_{a_2-1}$ into w'' so that it is neither $v_{[a_1,a_2-1]}$ nor

$(v_{[a_1,a_2-1]})^{-1}$. Applying the inductive hypothesis to this word, we conclude that w'' is a subword of $w_{[a_1,a_2-1]}$ in at least two ways. This in turn implies that w occurs as a subword of $w_{\mathcal{A}}$ in at least two ways, contradicting the assumption on w. Hence $w = v_{[a_1,a_2-1]}s_{a_1} = v_{[a_1,a_2]}$ is the only word of the form $w's_{a_2}$ that satisfies our hypotheses, completing the proof. □

If $\mathcal{A}$ has k maximal consecutive substrings of size at least two, Lemma 6.7 below shows that there are 2^k different Peterson Schubert classes p_w with $p_w = p_{v_{\mathcal{A}}}$. These Peterson Schubert classes correspond to all possible choices of either $v_{[a_i,a_{i+1}]}$ or $(v_{[a_i,a_{i+1}]})^{-1}$ on each maximal substring.

6. A MANIFESTLY-POSITIVE EQUIVARIANT MONK FORMULA FOR PETERSON VARIETIES

One of the central problems of modern Schubert calculus is to find concrete combinatorial formulas for the (ordinary or equivariant) structure constants in the (ordinary or equivariant, generalized) cohomology rings, with respect to the special module basis of Schubert classes. In line with this general philosophy, we therefore ask for concrete combinatorial methods to compute products $p_{v_{\mathcal{A}}} \cdot p_{v_{\mathcal{B}}}$ of Peterson Schubert classes $\{p_{v_{\mathcal{A}}}\}$, which we showed in Section 4 form an $H^*_{S^1}(\mathrm{pt})$-module basis for $H^*_{S^1}(Y)$.

In this section, we partly achieve this goal: we prove an S^1**-equivariant Chevalley-Monk formula** (also called a **Monk formula**) in the S^1-equivariant cohomology of the Peterson variety, i.e. we obtain an explicit, combinatorial formula for the product of an *arbitrary Peterson Schubert class* with a *Peterson Schubert class of cohomology degree* 2. As a word of caution, we note that the terminology in the literature is ambiguous. For instance, in the Schubert calculus of the classical Grassmanian, the term "Chevalley-Monk formula" refers to a formula for the product of an arbitrary Schubert class with an arbitrary cohomology degree 2 class (the 'single-box' class), while a "Pieri formula" refers to a formula for the product of an arbitrary Schubert class with an arbitrary "special" Schubert class (the 'single-row' classes), which generate the cohomology ring but may have cohomology degree ≥ 2. In other cases, the use of terminology seems to depend on the relative importance ascribed by the authors to the two possible definitions of the subset of 'special classes': either 'degree 2' or 'generate cohomology ring'. This results in ambiguity in cases when the two definitions agree. For instance, in the case of the flag variety, "Chevalley" is sometimes used to refer to formulas for products with 'single-box' classes [32], sometimes "Pieri" or "Pieri-Chevalley" refers to formulas for products with 'single-box' classes [24], and sometimes "Pieri" is used for formulas with 'single-row' classes [23, 26]. We adhere to the *Iowa convention*, a standardization of terminology negotiated at a small Schubert calculus workshop in 2009 at the University of Iowa: we refer to formulas for multiplication by cohomology-degree-2 classes as *Chevalley-Monk* (or *Monk*) formulas, while we refer to formulas for multiplication by "special classes" of degree ≥ 2 as *Pieri* formulas.

We also prove that our Monk formula completely determines the S^1-equivariant cohomology $H^*_{S^1}(Y)$ of the Peterson variety, namely that the cohomology-degree-2 classes generate $H^*_{S^1}(Y)$ as a ring. Moreover, we show that our Monk formula is quite simple in that "most terms are zero" (made precise below), and that the structure constants in our Monk formula are **non-negative** and **integral**, either literally or in the sense of Graham, depending on the polynomial degree of the structure constant. This yields an explicit description via generators and relations of $H^*_{S^1}(Y)$. Finally, we give analogues of the above results in the context of the *ordinary* cohomology $H^*(Y)$ of the Peterson variety.

We begin with a definition for notational convenience.

Definition 6.1. Let p_i denote the class $p_{s_i} \in H^*_{S^1}(Y)$, i.e. the Peterson Schubert class $p_{v_{\mathcal{A}}}$ for the singleton $\mathcal{A} = \{i\}$.

From Proposition 4.11, the set of $\{p_i\}_{i=1}^{n-1}$ are exactly the cohomology degree 2 classes among the Peterson Schubert classes. We now prove that these, together with one more degree 2 class coming from $H^*_{S^1}(\mathrm{pt})$, are in fact *ring* generators for $H^*_{S^1}(Y)$. Recall that the $H^*_{S^1}(\mathrm{pt})$-module structure of $H^*_{S^1}(Y)$ comes from the ring map $\pi^*_{BS^1} : H^*_{S^1}(\mathrm{pt}) \to H^*_{S^1}(Y)$ induced from the projection π_{BS^1} in the fiber bundle $Y \to Y \times_{S^1} ES^1 \stackrel{\pi_{BS^1}}{\longrightarrow} BS^1$. In particular we view the equivariant element $t \in \mathbb{C}[t] \cong H^*_{S^1}(\mathrm{pt})$ of cohomology degree 2 also as an element of $H^*_{S^1}(Y)$. We have the following.

Proposition 6.2. *Let Y be the type A_{n-1} Peterson variety, equipped with the natural S^1-action defined by* (2.3). *The Peterson Schubert classes $\{p_i : i = 1, \ldots, n-1\}$ of cohomology degree 2 together with the pure equivariant degree 2 class $t \in H^*_{S^1}(Y)$ generate the S^1-equivariant cohomology $H^*_{S^1}(Y)$ as a ring.*

Proof. It is well-known that $H^*_T(G/B)$ is generated in degree 2, as is $H^*_{S^1}(G/B)$. Since the restriction map $H^*_{S^1}(G/B) \to H^*_{S^1}(Y)$ is surjective, the same holds true for $H^*_{S^1}(Y)$. We have already seen that the $\{p_{v_\mathcal{A}}\}_{\mathcal{A}\subseteq\{1,2,\ldots,n-1\}}$ are a $H^*_{S^1}(\mathrm{pt})$-module basis, and in particular the subspace of $H^*_{S^1}(Y)$ of degree 2 is $\mathbb{C}$-spanned by $\{p_i\}_{i=1}^{n-1}$ and the single 'pure equivariant' class t. The result follows. □

Monk's formula is an explicit relationship between ring generators and module generators. More precisely, the fact that the set $\{p_{v_\mathcal{A}}\}$ form a module basis for $H^*_{S^1}(Y)$ implies that for any p_i and $p_{v_\mathcal{A}}$ there exist structure constants $c^{\mathcal{B}}_{i,\mathcal{A}} \in H^*_{S^1}(\mathrm{pt}) \cong \mathbb{C}[t]$ such that

$$p_i \cdot p_{v_\mathcal{A}} = \sum_{\mathcal{B}} c^{\mathcal{B}}_{i,\mathcal{A}} \cdot p_{v_\mathcal{B}}. \tag{6.1}$$

Our main theorem of this section provides a simple combinatorial formula for the $c^{\mathcal{B}}_{i,\mathcal{A}}$. Its proof has several steps which occupy the rest of this section.

We begin by proving that a simple condition on the subsets $\mathcal{B}$ guarantees that the corresponding structure constants $c^{\mathcal{B}}_{i,\mathcal{A}}$ are zero. This allows us to refine the summation on the right hand side of (6.1) and to obtain some simple formulas for structure constants, as below.

Proposition 6.3. *Let $\mathcal{A} \subseteq \{1, 2, \ldots, n-1\}$ and $i \in \{1, 2, \ldots, n-1\}$. Then*

$$p_i \cdot p_{v_\mathcal{A}} = c^{\mathcal{A}}_{i,\mathcal{A}} p_{v_\mathcal{A}} + \sum_{\mathcal{A}\subsetneq\mathcal{B} \text{ and } |\mathcal{B}|=|\mathcal{A}|+1} c^{\mathcal{B}}_{i,\mathcal{A}} \cdot p_{v_\mathcal{B}}, \tag{6.2}$$

where

(1) $c^{\mathcal{A}}_{i,\mathcal{A}} = p_i(w_\mathcal{A})$ *and*
(2) *if $\mathcal{A} \subsetneq \mathcal{B}$ and $|\mathcal{B}| = |\mathcal{A}| + 1$, then*

$$c^{\mathcal{B}}_{i,\mathcal{A}} = (p_i(w_\mathcal{B}) - p_i(w_\mathcal{A}))\frac{p_{v_\mathcal{A}}(w_\mathcal{B})}{p_{v_\mathcal{B}}(w_\mathcal{B})}. \tag{6.3}$$

Proof. For simplicity, in this argument we use the polynomial degree of the Peterson Schubert classes instead of the cohomology degree. (Recall that the cohomology degree is double the polynomial degree.)

Note that the degree of p_i is 1, so by Proposition 4.11 the left hand side of (6.1) is homogeneous of degree $|\mathcal{A}| + 1$. Since each $c^{\mathcal{B}}_{i,\mathcal{A}}$ is a polynomial in $\mathbb{C}[t]$, the term $c^{\mathcal{B}}_{i,\mathcal{A}} p_{v_\mathcal{B}}$ in the right hand side of (6.1) has degree at least $|\mathcal{B}|$. The degree of the right hand side agrees with that of the left, and the $\{p_{v_\mathcal{A}}\}$ are $\mathbb{C}[t]$-linearly independent, so $c^{\mathcal{B}}_{i,\mathcal{A}} = 0$ if $|\mathcal{B}| > |\mathcal{A}| + 1$. In other words

$$p_i \cdot p_{v_\mathcal{A}} = \sum_{|\mathcal{B}|\leq|\mathcal{A}|+1} c^{\mathcal{B}}_{i,\mathcal{A}} \cdot p_{v_\mathcal{B}}.$$

We now claim that $c^{\mathcal{B}}_{i,\mathcal{A}} = 0$ for any $\mathcal{B}$ with $\mathcal{A} \not\subseteq \mathcal{B}$. We argue by contradiction. Suppose there exists some $\mathcal{B}$ with $\mathcal{B} \not\supseteq \mathcal{A}$ such that $c^{\mathcal{B}}_{i,\mathcal{A}} \neq 0$. Then there is a minimal such; denote it $\mathcal{A}'$. Evaluate the equation (6.1) at $w_{\mathcal{A}'}$. Since $\mathcal{A} \not\subseteq \mathcal{A}'$ the left hand side is zero. The minimality assumption on $\mathcal{A}'$

implies that $c_{i,\mathcal{A}}^{\mathcal{B}} = 0$ if $\mathcal{B} \subsetneq \mathcal{A}'$ while the upper-triangularity property of Peterson Schubert classes implies that $p_{v_{\mathcal{B}}}(w_{\mathcal{A}'}) = 0$ if $\mathcal{B} \not\subseteq \mathcal{A}'$. Hence

$$0 = c_{i,\mathcal{A}}^{\mathcal{A}'} \cdot p_{v_{\mathcal{A}'}}(w_{\mathcal{A}'}).$$

Since $H^*_{S^1}(\mathrm{pt}) \cong \mathbb{C}[t]$ is an integral domain, either $c_{i,\mathcal{A}}^{\mathcal{A}'}$ or $p_{v_{\mathcal{A}'}}(w_{\mathcal{A}'})$ is zero. By Corollary 5.12 we conclude $c_{i,\mathcal{A}}^{\mathcal{B}} = 0$ if $\mathcal{A} \not\subseteq \mathcal{B}$. This proves (6.2).

To prove the formula for $c_{i,\mathcal{A}}^{\mathcal{A}}$ we evaluate (6.2) at the fixed point $w_{\mathcal{A}}$. If $\mathcal{B}$ satisfies $\mathcal{A} \subsetneq \mathcal{B}$ then $w_{\mathcal{B}} > w_{\mathcal{A}}$ so $p_{v_{\mathcal{B}}}(w_{\mathcal{A}}) = 0$. We conclude

$$p_i(w_{\mathcal{A}}) p_{v_{\mathcal{A}}}(w_{\mathcal{A}}) = c_{i,\mathcal{A}}^{\mathcal{A}} \cdot p_{v_{\mathcal{A}}}(w_{\mathcal{A}}).$$

Since $p_{v_{\mathcal{A}}}(w_{\mathcal{A}}) \neq 0$ and $H^*_{S^1}(\mathrm{pt}) \cong \mathbb{C}[t]$ is an integral domain, we conclude

$$c_{i,\mathcal{A}}^{\mathcal{A}} = p_i(w_{\mathcal{A}}).$$

To prove the last claim, suppose that $\mathcal{B}$ is such that $\mathcal{A} \subsetneq \mathcal{B}$ and $|\mathcal{B}| = |\mathcal{A}| + 1$. Evaluating (6.2) at the fixed point $w_{\mathcal{B}}$ we obtain

$$p_i(w_{\mathcal{B}}) \cdot p_{v_{\mathcal{A}}}(w_{\mathcal{B}}) = c_{i,\mathcal{A}}^{\mathcal{A}} \cdot p_{v_{\mathcal{A}}}(w_{\mathcal{B}}) + \sum_{\mathcal{A} \subsetneq \mathcal{B} \text{ and } |\mathcal{B}| = |\mathcal{A}|+1} c_{i,\mathcal{A}}^{\mathcal{B}} \cdot p_{v_{\mathcal{B}}}(w_{\mathcal{B}}).$$

The previous claim showed $c_{i,\mathcal{A}}^{\mathcal{A}} = p_i(w_{\mathcal{A}})$. If $\mathcal{B}' \neq \mathcal{B}$ is another subset in the sum above, the upper-triangularity condition on the Peterson Schubert classes implies $p_{v_{\mathcal{B}'}}(w_{\mathcal{B}}) = 0$. Hence

$$p_i(w_{\mathcal{B}}) \cdot p_{v_{\mathcal{A}}}(w_{\mathcal{B}}) = c_{i,\mathcal{A}}^{\mathcal{A}} \cdot p_{v_{\mathcal{A}}}(w_{\mathcal{B}}) + c_{i,\mathcal{A}}^{\mathcal{B}} \cdot p_{v_{\mathcal{B}}}(w_{\mathcal{B}}).$$

By Corollary 5.12, we know $p_{v_{\mathcal{B}}}(w_{\mathcal{B}}) \neq 0$, so we may solve for $c_{i,\mathcal{A}}^{\mathcal{B}}$ to obtain (6.3), as desired. □

Next we compute explicitly the expression for $c_{i,\mathcal{A}}^{\mathcal{B}}$ in (6.3). We need some preliminary lemmas.

Lemma 6.4. *Suppose $i \in \{1, 2, \ldots, n-1\}$ and $\mathcal{A} \subseteq \{1, 2, \ldots, n-1\}$.*

- *If $i \notin \mathcal{A}$ then $p_i(w_{\mathcal{A}}) = 0$.*
- *If $i \in \mathcal{A}$ then*

$$p_i(w_{\mathcal{A}}) = (\mathcal{H}_{\mathcal{A}}(i) - i + 1)\,(i - \mathcal{T}_{\mathcal{A}}(i) + 1)\,t. \tag{6.4}$$

Proof. If i is not contained in $\mathcal{A}$ then s_i does not appear in $w_{\mathcal{A}}$ and so $p_i(w_{\mathcal{A}}) = 0$. Now suppose $i \in \mathcal{A}$. We saw in Lemma 5.2 that each summand in Billey's formula for $p_i(w_{\mathcal{A}})$ is $(i - \mathcal{T}_{\mathcal{A}}(i) + 1)t$. On the other hand s_i appears exactly $\mathcal{H}_{\mathcal{A}}(i) - i + 1$ times in the reduced word for $w_{\mathcal{A}}$ given in equation (2.8), by inspection. Equation (6.4) now follows from Proposition 5.9. □

The previous lemma lets us further refine the vanishing conditions for $c_{i,\mathcal{A}}^{\mathcal{B}}$. We begin with terminology.

Definition 6.5. Given any index k and any subset $\mathcal{A} \subseteq \{1, 2, \ldots, n-1\}$ containing k, we refer to $[\mathcal{T}_{\mathcal{A}}(k), \mathcal{H}_{\mathcal{A}}(k)]$ as **the maximal consecutive substring of $\mathcal{A}$ which contains k**.

Let $\mathcal{A} \subseteq \{1, 2, \ldots, n-1\}$. If $\mathcal{B}$ is a subset such that $\mathcal{A} \subsetneq \mathcal{B}$ and $|\mathcal{B}| = |\mathcal{A}| + 1$ then there exists $k \in \{1, 2, \ldots, n-1\}$ with $k \notin \mathcal{A}$ and $\mathcal{B} = \mathcal{A} \cup \{k\}$. Exactly one of the following occurs:

1. a maximal consecutive substring in $\mathcal{A}$ is lengthened, from either $[k+1, \mathcal{H}_{\mathcal{B}}(k)]$ or $[\mathcal{T}_{\mathcal{B}}(k), k-1]$ to $[\mathcal{T}_{\mathcal{B}}(k), \mathcal{H}_{\mathcal{B}}(k)]$, with either $\mathcal{T}_{\mathcal{B}}(k) = k$ or $\mathcal{H}_{\mathcal{B}}(k) = k$ respectively;
2. two maximal consecutive substrings in $\mathcal{A}$ are merged, namely $[\mathcal{T}_{\mathcal{B}}(k), k-1]$ and $[k+1, \mathcal{H}_{\mathcal{B}}(k)]$ are both in $\mathcal{A}$ and become $[\mathcal{T}_{\mathcal{B}}(k), \mathcal{H}_{\mathcal{B}}(k)]$ in $\mathcal{B}$; or
3. the new index k is itself a maximal consecutive substring $\{k\} = [k, k]$ in $\mathcal{B}$.

Conversely, all but one of the maximal consecutive strings of $\mathcal{B}$ is a maximal consecutive string in $\mathcal{A}$. Summarizing, the maximal consecutive strings of $\mathcal{A}$ that differ from the maximal consecutive strings in $\mathcal{B}$ are

$$\{\mathcal{T}_{\mathcal{B}}(k), \mathcal{T}_{\mathcal{B}}(k)+1, \ldots, k-1\} \subseteq \mathcal{A} \quad \text{and} \quad \{k+1, k+2, \ldots, \mathcal{H}_{\mathcal{B}}(k)\} \subseteq \mathcal{A},$$

of lengths

$$k-1-\mathcal{T}_{\mathcal{B}}(k)+1 = k-\mathcal{T}_{\mathcal{B}}(k) \quad \text{and} \quad \mathcal{H}_{\mathcal{B}}(k)-k-1+1 = \mathcal{H}_{\mathcal{B}}(k)-k$$

respectively. (The first string is empty if $k = \mathcal{T}_{\mathcal{B}}(k)$ and the second string is empty if $\mathcal{H}_{\mathcal{B}}(k) = k$.)

Lemma 6.6. *Suppose $\mathcal{B}$ is the disjoint union $\mathcal{B} = \mathcal{A} \cup \{k\}$. If either one of the following conditions hold:*

- *$i \notin \mathcal{B}$, or*
- *$i \in \mathcal{B}$, and i and k are not contained in the same maximal consecutive substring in $\mathcal{B}$, namely $\mathcal{T}_{\mathcal{B}}(i) = \mathcal{T}_{\mathcal{A}}(i)$ and $\mathcal{H}_{\mathcal{B}}(i) = \mathcal{H}_{\mathcal{A}}(i)$,*

then $c_{i,\mathcal{A}}^{\mathcal{B}} = 0$.

Proof. In the first case $i \notin \mathcal{B}$ and so $i \notin \mathcal{A}$; hence both $p_i(w_{\mathcal{B}}) = 0$ and $p_i(w_{\mathcal{A}}) = 0$. In the second case $p_i(w_{\mathcal{A}}) = p_i(w_{\mathcal{B}})$. The claim now follows from Equation (6.3). □

The above lemma suggests that the information needed to compute $c_{i,\mathcal{A}}^{\mathcal{B}}$ is contained in the maximal consecutive substring of $\mathcal{B}$ containing i, and that we should be able to "ignore" all other maximal consecutive substrings of $\mathcal{B}$. The next two lemmas make this idea precise. We call two disjoint consecutive strings **adjacent** if their union is again a consective string. The next lemma asserts that if two disjoint subsets $\mathcal{B}, \mathcal{B}'$ contain no adjacent maximal consecutive substrings, then the Peterson Schubert class corresponding to $\mathcal{B} \cup \mathcal{B}'$ is simply the product of the classes corresponding to $\mathcal{B}$ and $\mathcal{B}'$ respectively.

Lemma 6.7. *Let $\mathcal{B}$ and $\mathcal{B}'$ be disjoint subsets of $\{1, 2, \ldots, n-1\}$. Suppose that $\mathcal{B}$ and $\mathcal{B}'$ contain* no *adjacent maximal consecutive substrings, i.e. there exists no $j \in \mathcal{B}, j' \in \mathcal{B}'$ such that $|j - j'| = 1$. Then*

$$p_{v_{\mathcal{B}\cup\mathcal{B}'}} = p_{v_{\mathcal{B}}} p_{v_{\mathcal{B}'}}. \tag{6.5}$$

Proof. We prove that for all $\mathcal{A} \subseteq \{1, 2, \ldots, n-1\}$ the restrictions in (6.5) agree at $w_{\mathcal{A}}$:

$$p_{v_{\mathcal{B}\cup\mathcal{B}'}}(w_{\mathcal{A}}) = p_{v_{\mathcal{B}}}(w_{\mathcal{A}}) \cdot p_{v_{\mathcal{B}'}}(w_{\mathcal{A}}).$$

We take cases. Suppose $\mathcal{B} \cup \mathcal{B}' \not\subseteq \mathcal{A}$, which implies $\mathcal{B} \not\subseteq \mathcal{A}$ or $\mathcal{B}' \not\subseteq \mathcal{A}$. By the upper-triangularity property of Peterson Schubert classes, both the right and left sides of Equation (6.5) are zero. Hence the equality holds.

Now let $\mathcal{B} \cup \mathcal{B}' \subseteq \mathcal{A}$. If $[b_i, b_{i+1}] \subseteq \mathcal{B} \cup \mathcal{B}'$ is a maximal consecutive substring then every reduced word for $v_{\mathcal{B}\cup\mathcal{B}'}$ contains $v_{[b_i,b_{i+1}]}$ as a reduced subword by definition of $v_{\mathcal{B}\cup\mathcal{B}'}$. (Fact 4.3 tells us that there is a unique reduced word for $v_{[b_i,b_{i+1}]}$.) No two distinct maximal consecutive strings $[b_i, b_{i+1}]$ and $[b_j, b_{j+1}]$ are adjacent in $\mathcal{B} \cup \mathcal{B}'$ so all simple transpositions in $v_{[b_i,b_{i+1}]}$ commute with all simple transpositions in $v_{[b_j,b_{j+1}]}$. Comparing lengths of the permutations, we conclude that each reduced word for $v_{\mathcal{B}\cup\mathcal{B}'}$ can be partitioned into a unique subword that equals $v_{\mathcal{B}}$ and a unique subword that equals $v_{\mathcal{B}'}$.

Let $\mathbf{b}$ be the reduced word for $w_{\mathcal{A}}$ given in (2.8). The previous discussion implies that $b_{j_1} b_{j_2} \cdots b_{j_{|\mathcal{B}|+|\mathcal{B}'|}}$ is a reduced subword of $\mathbf{b}$ that equals $v_{\mathcal{B}\cup\mathcal{B}'}$ if and only if $b_{j_1} b_{j_2} \cdots b_{j_{|\mathcal{B}|+|\mathcal{B}'|}}$ contains exactly one subword $b_{k_1} b_{k_2} \cdots b_{k_{|\mathcal{B}|}}$ that equals $v_{\mathcal{B}}$ and exactly one subword $b_{k'_1} b_{k'_2} \cdots b_{k'_{|\mathcal{B}'|}}$ that equals $v_{\mathcal{B}'}$. Conversely, the product (in the ordering induced from $\mathbf{b}$) of each pair of reduced subwords $b_{k_1} b_{k_2} \cdots b_{k_{|\mathcal{B}|}} = v_{\mathcal{B}}$ and $b_{k'_1} b_{k'_2} \cdots b_{k'_{|\mathcal{B}'|}} = v_{\mathcal{B}'}$ of $\mathbf{b}$ is a reduced subword $b_{j_1} b_{j_2} \cdots b_{j_{|\mathcal{B}|+|\mathcal{B}'|}}$ of $\mathbf{b}$

equalling $v_{\mathcal{B}\cup\mathcal{B}'}$. This implies that the number of terms in Billey's formula for $p_{v_{\mathcal{B}\cup\mathcal{B}'}}(w_{\mathcal{A}})$ is precisely the product of the number of terms in Billey's formula for $p_{v_{\mathcal{B}}}(w_{\mathcal{A}})$ and $p_{v_{\mathcal{B}'}}(w_{\mathcal{A}})$. By Proposition 5.9, we need only show that each summand in Billey's formula for $p_{v_{\mathcal{B}\cup\mathcal{B}'}}(w_{\mathcal{A}})$ is the product of a summand in Billey's formula for $p_{v_{\mathcal{B}}}(w_{\mathcal{A}})$ and another for $p_{v_{\mathcal{B}'}}(w_{\mathcal{A}})$.

Using Lemma 5.2 and the above discussion, we conclude that the summand in Billey's formula for $p_{v_{\mathcal{B}\cup\mathcal{B}'}}(w_{\mathcal{A}})$ corresponding to $b_{j_1}b_{j_2}\cdots b_{j_{|\mathcal{B}|+|\mathcal{B}'|}}$ is

$$\prod_{i=1}^{|\mathcal{B}|+|\mathcal{B}'|} \pi_{S^1}(r(j_i,\mathbf{b})) = \prod_{i\in\mathcal{B}\cup\mathcal{B}'}(i-\mathcal{T}_{\mathcal{B}\cup\mathcal{B}'}(i)+1).$$

Since $\mathcal{B},\mathcal{B}'$ contain no adjacent maximal consecutive strings, for any $i\in\mathcal{B}\cup\mathcal{B}'$, precisely one of the following hold: either $i\in\mathcal{B}$ and $\mathcal{T}_{\mathcal{B}\cup\mathcal{B}'}(i)=\mathcal{T}_{\mathcal{B}}(i)$ or $i\in\mathcal{B}'$ and $\mathcal{T}_{\mathcal{B}\cup\mathcal{B}'}(i)=\mathcal{T}_{\mathcal{B}'}(i)$. Hence we may compute

$$\begin{aligned}\prod_{i\in\mathcal{B}\cup\mathcal{B}'}(i-\mathcal{T}_{\mathcal{B}\cup\mathcal{B}'}(i)+1) &= \prod_{i\in\mathcal{B}}(i-\mathcal{T}_{\mathcal{B}}(i)+1)\prod_{i\in\mathcal{B}'}(i-\mathcal{T}_{\mathcal{B}'}(i)+1)\\ &= \prod_{i=1}^{|\mathcal{B}|}\pi_{S^1}(r(k_i,\mathbf{b}))\prod_{i=1}^{|\mathcal{B}'|}\pi_{S^1}(r(k_i',\mathbf{b})).\end{aligned}$$

Hence each summand in Billey's formula for the left side of (6.5) may be written as a product of a summand in Billey's formula for $p_{v_{\mathcal{B}}}(w_{\mathcal{A}})$ and another for $p_{v_{\mathcal{B}'}}(w_{\mathcal{A}})$. The claim follows. □

As observed in Section 2.3, any subset of $\{1,2,\ldots,n-1\}$ decomposes into a series of non-adjacent maximal consecutive substrings. The above lemma indicates that the Peterson Schubert class associated to each set $\mathcal{A}$ may be computed in terms of the classes corresponding to its maximal consecutive substrings. This allows us to derive the following simplification of one of the expressions appearing in Equation (6.3).

Lemma 6.8. *Suppose $\mathcal{B}\subseteq\{1,2,\ldots,n-1\}$ is a disjoint union $\mathcal{A}\cup\{k\}$. Then*

$$\frac{p_{v_{\mathcal{A}}}(w_{\mathcal{B}})}{p_{v_{\mathcal{B}}}(w_{\mathcal{B}})} = \frac{p_{v_{[\mathcal{T}_{\mathcal{B}}(k),k-1]}}(w_{\mathcal{B}})p_{v_{[k+1,\mathcal{H}_{\mathcal{B}}(k)]}}(w_{\mathcal{B}})}{p_{v_{[\mathcal{T}_{\mathcal{B}}(k),\mathcal{H}_{\mathcal{B}}(k)]}}(w_{\mathcal{B}})} = \frac{p_{v_{[\mathcal{T}_{\mathcal{B}}(k),k-1]\cup[k+1,\mathcal{H}_{\mathcal{B}}(k)]}}(w_{\mathcal{B}})}{p_{v_{[\mathcal{T}_{\mathcal{B}}(k),\mathcal{H}_{\mathcal{B}}(k)]}}(w_{\mathcal{B}})}.$$

Proof. Suppose that $\mathcal{A}$ decomposes into maximal consecutive substrings as

$$\mathcal{A}=[\mathcal{T}_{\mathcal{B}}(k),k-1]\cup[k+1,\mathcal{H}_{\mathcal{B}}(k)]\cup[a_1,a_2]\cup[a_3,a_4]\cup\cdots\cup[a_{m-1},a_m]$$

and that $\mathcal{B}$ decomposes into maximal consecutive substrings as

$$\mathcal{B}=[\mathcal{T}_{\mathcal{B}}(k),\mathcal{H}_{\mathcal{B}}(k)]\cup[a_1,a_2]\cup[a_3,a_4]\cup\cdots\cup[a_{m-1},a_m].$$

The previous lemma showed that

$$p_{v_{\mathcal{B}}}(w_{\mathcal{B}}) = p_{v_{[\mathcal{T}_{\mathcal{B}}(k),\mathcal{H}_{\mathcal{B}}(k)]}}(w_{\mathcal{B}})\cdot p_{v_{[a_1,a_2]}}(w_{\mathcal{B}})\cdot p_{v_{[a_3,a_4]}}(w_{\mathcal{B}})\cdots p_{v_{[a_{m-1},a_m]}}(w_{\mathcal{B}})$$

and similarly for $p_{v_{\mathcal{A}}}$. The claim follows. □

As a consequence of the above, for the purposes of computing the right hand side of Equation (6.3), we may assume without loss of generality that $\mathcal{B}$ consists of a single consecutive string $[\mathcal{T}_{\mathcal{B}}(k),\mathcal{H}_{\mathcal{B}}(k)]$ and $\mathcal{A}=\mathcal{B}\setminus\{k\})$ for any $k\in\mathcal{B}$. We can now give a combinatorial and explicit expression for both factors in Equation (6.3).

Lemma 6.9. *Let $\mathcal{A}=[\mathcal{T}_{\mathcal{B}}(k),k-1]\cup[k+1,\mathcal{H}_{\mathcal{B}}(k)]$ and $\mathcal{B}=[\mathcal{T}_{\mathcal{B}}(k),\mathcal{H}_{\mathcal{B}}(k)]$. Then*

$$p_{v_{\mathcal{A}}}(w_{\mathcal{B}}) = \frac{(\mathcal{H}_{\mathcal{B}}(k)-\mathcal{T}_{\mathcal{B}}(k)+1)!}{k-\mathcal{T}_{\mathcal{B}}(k)+1}\binom{\mathcal{H}_{\mathcal{B}}(k)-\mathcal{T}_{\mathcal{B}}(k)+1}{k-\mathcal{T}_{\mathcal{B}}(k)}t^{\mathcal{H}_{\mathcal{B}}(k)-\mathcal{T}_{\mathcal{B}}(k)}.$$

In particular,

$$\frac{p_{v_{\mathcal{A}}}(w_{\mathcal{B}})}{p_{v_{\mathcal{B}}}(w_{\mathcal{B}})} = \frac{1}{k - \mathcal{T}_{\mathcal{B}}(k) + 1} \cdot \binom{\mathcal{H}_{\mathcal{B}}(k) - \mathcal{T}_{\mathcal{B}}(k) + 1}{k - \mathcal{T}_{\mathcal{B}}(k)} \frac{1}{t}. \tag{6.6}$$

Proof. We apply Billey's formula to compute $p_{v_{\mathcal{A}}}(w_{\mathcal{B}})$. Recall that

$$v_{\mathcal{B}} := s_{\mathcal{T}_{\mathcal{B}}(k)} s_{\mathcal{T}_{\mathcal{B}}(k)+1} \cdots s_{\mathcal{H}_{\mathcal{B}}(k)} \text{ and } v_{\mathcal{A}} := s_{\mathcal{T}_{\mathcal{B}}(k)} s_{\mathcal{T}_{\mathcal{B}}(k)+1} \cdots s_{k-1} \widehat{s_k} s_{k+1} \cdots s_{\mathcal{H}_{\mathcal{B}}(k)}.$$

By Lemma 5.7, we conclude that each summand in Billey's formula for $p_{v_{\mathcal{A}}}(w_{\mathcal{B}})$ equals

$$\frac{(\mathcal{H}_{\mathcal{B}}(k) - \mathcal{T}_{\mathcal{B}}(k) + 1)!}{k - \mathcal{T}_{\mathcal{B}}(k) + 1}\ t^{\mathcal{H}_{\mathcal{B}}(k) - \mathcal{T}_{\mathcal{B}}(k)}.$$

By Proposition 5.9, we need next to compute the number of distinct ways that $v_{\mathcal{A}}$ appears as a reduced subword of $w_{\mathcal{B}}$. First, by construction, the element $v_{\mathcal{A}}$ is equal to

$$v_{[\mathcal{T}_{\mathcal{B}}(k), k-1]} \cdot v_{[k+1, \mathcal{H}_{\mathcal{B}}(k)]}. \tag{6.7}$$

(By definition $v_{\emptyset} = 1$.) Moreover, both factors appear in every reduced-word decomposition of $v_{\mathcal{A}}$ and each factor has a *unique* reduced word decomposition (see Fact 4.3), in which the indices in $\{\mathcal{T}_{\mathcal{B}}(k), \ldots, k-1\}$ are listed in increasing order, as are the indices in $\{k+1, \ldots, \mathcal{H}_{\mathcal{B}}(k)\}$. Since the two factors correspond to non-adjacent maximal consecutive strings, each simple transposition appearing in $v_{[\mathcal{T}_{\mathcal{B}}(k), k-1]}$ commutes with each such of $v_{[k+1, \mathcal{H}_{\mathcal{B}}(k)]}$. Hence the set of reduced-word decompositions of $v_{\mathcal{A}}$ are in bijective correspondence with orderings of the set $\mathcal{A}$ such that the elements $\{\mathcal{T}_{\mathcal{B}}(k), \ldots, k-1\}$ appear in increasing order, as do the elements $\{k+1, \ldots, \mathcal{H}_{\mathcal{B}}(k)\}$.

Let $\mathbf{b}$ be the reduced word decomposition for $w_{\mathcal{B}}$ given by (2.8). We wish to find subwords of $\mathbf{b}$ which equal $v_{\mathcal{A}}$. The index $\mathcal{H}_{\mathcal{B}}(k)$ appears only once, and as observed above, the indices $\{k+1, \ldots, \mathcal{H}_{\mathcal{B}}(k)\}$ must appear in increasing order. We conclude that there is only one subword of $\mathbf{b}$ which equals $v_{\{k+1, \ldots, \mathcal{H}_{\mathcal{B}}(k)\}}$. If $k = \mathcal{T}_{\mathcal{B}}(k)$ this unique subword determines the factorization, and the formula of the claim reduces to 1. (In the special case when $k = \mathcal{H}_{\mathcal{B}}(k)$, the set $\{k + 1, \ldots, \mathcal{H}_{\mathcal{B}}(k)\}$ is empty and this discussion is vacuous.)

Suppose $k > \mathcal{T}_{\mathcal{B}}(k)$. Note that the indices $\{\mathcal{T}_{\mathcal{B}}(k), \ldots, k-1\}$ appear in the first $\mathcal{H}_{\mathcal{B}}(k) - k + 2$ factors of (2.8) and no others. A reduced word for $v_{[\mathcal{T}_{\mathcal{B}}(k), k-1]}$ is a choice of the indices $\{\mathcal{T}_{\mathcal{B}}(k), \ldots, k-1\}$ from any of these factors, in increasing order; in other words, the reduced words for $v_{[\mathcal{T}_{\mathcal{B}}(k), k-1]}$ in $w_{\mathcal{B}}$ correspond bijectively with ordered partitions of $\mathcal{H}_{\mathcal{B}}(k) - k + 2$ into $k - \mathcal{T}_{\mathcal{B}}(k)$ nonnegative parts. This is given by the binomial coefficient

$$\binom{\mathcal{H}_{\mathcal{B}}(k) - \mathcal{T}_{\mathcal{B}}(k) + 1}{k - \mathcal{T}_{\mathcal{B}}(k)} = \frac{(\mathcal{H}_{\mathcal{B}}(k) - \mathcal{T}_{\mathcal{B}}(k) + 1)!}{(k - \mathcal{T}_{\mathcal{B}}(k))!(\mathcal{H}_{\mathcal{B}}(k) - k + 1)!}.$$

By Proposition 5.9 we conclude

$$p_{v_{\mathcal{A}}}(w_{\mathcal{B}}) = \frac{(\mathcal{H}_{\mathcal{B}}(k) - \mathcal{T}_{\mathcal{B}}(k) + 1)!}{k - \mathcal{T}_{\mathcal{B}}(k) + 1} \binom{\mathcal{H}_{\mathcal{B}}(k) - \mathcal{T}_{\mathcal{B}}(k) + 1}{k - \mathcal{T}_{\mathcal{B}}(k)} t^{\mathcal{H}_{\mathcal{B}}(k) - \mathcal{T}_{\mathcal{B}}(k)},$$

as desired. Formula (6.6) follows immediately from the above equality and Corollary 5.12. □

Remark 6.10. *As may be seen from the proof of the lemma above, the number of distinct subwords of the reduced word decomposition* $\mathbf{b}$ *of* $w_{\mathcal{B}}$ *given in* (2.8) *that equal* $v_{\mathcal{A}}$, *is also equal to the number of Young diagrams that fit in a* $(\mathcal{H}_{\mathcal{B}}(k) - k + 1) \times (k - \mathcal{T}_{\mathcal{B}}(k))$ *box. We do not know whether this is a coincidence or, given the prevalence of Young diagrams in Schubert calculus, intrinsic to the product structure of the ring.*

We proceed with a computation of the rest of Equation (6.3). Here we assume that i satisfies $\mathcal{T}_{\mathcal{B}}(k) \leq i \leq \mathcal{H}_{\mathcal{B}}(k)$, since otherwise $c_{i,\mathcal{A}}^{\mathcal{B}}$ vanishes by Lemma 6.6.

Lemma 6.11. *Let* $\mathcal{A} = [\mathcal{T}_{\mathcal{B}}(k), k-1] \cup [k+1, \mathcal{H}_{\mathcal{B}}(k)]$ *and* $\mathcal{B} = [\mathcal{T}_{\mathcal{B}}(k), \mathcal{H}_{\mathcal{B}}(k)]$. *Let i be an index with* $\mathcal{T}_{\mathcal{B}}(k) \leq i \leq \mathcal{H}_{\mathcal{B}}(k)$. *Then*

$$p_i(w_{\mathcal{B}}) - p_i(w_{\mathcal{A}}) = \begin{cases} (\mathcal{H}_{\mathcal{B}}(k) - k + 1)(i - \mathcal{T}_{\mathcal{B}}(k) + 1)t & \textit{if } \mathcal{T}_{\mathcal{B}}(k) \leq i \leq k-1 \text{ and} \\ (\mathcal{H}_{\mathcal{B}}(k) - i + 1)(k - \mathcal{T}_{\mathcal{B}}(k) + 1)t & \textit{if } k \leq i \leq \mathcal{H}_{\mathcal{B}}(k). \end{cases} \tag{6.8}$$

Proof. First suppose $\mathcal{T}_{\mathcal{B}}(k) \leq i \leq k-1$. Then Equation (6.4) yields

$$p_i(w_{\mathcal{A}}) = (i - \mathcal{T}_{\mathcal{B}}(k) + 1)(k - 1 - i + 1)t \quad \text{and} \quad p_i(w_{\mathcal{B}}) = (i - \mathcal{T}_{\mathcal{B}}(k) + 1)(\mathcal{H}_{\mathcal{B}}(k) - i + 1)t,$$

hence we have, as desired,

$$p_i(w_{\mathcal{B}}) - p_i(w_{\mathcal{A}}) = (\mathcal{H}_{\mathcal{B}}(k) - k + 1)(i - \mathcal{T}_{\mathcal{B}}(k) + 1)t.$$

Now suppose $i = k$. Since $k \notin \mathcal{A}$ the transposition s_k never appears in $w_{\mathcal{A}}$. Thus we have

$$p_k(w_{\mathcal{A}}) = 0,$$

and we compute

$$p_k(w_{\mathcal{B}}) = (\mathcal{H}_{\mathcal{B}}(k) - k + 1)(k - \mathcal{T}_{\mathcal{B}}(k) + 1)t,$$

which also agrees with Equation (6.8). Finally suppose $k + 1 \leq i \leq \mathcal{H}_{\mathcal{B}}(k)$. Then

$$p_i(w_{\mathcal{A}}) = (\mathcal{H}_{\mathcal{B}}(k) - i + 1)(i - k - 1 + 1) \quad \text{and} \quad p_i(w_{\mathcal{B}}) = (\mathcal{H}_{\mathcal{B}}(k) - i + 1)(i - \mathcal{T}_{\mathcal{B}}(k) + 1)t,$$

so as desired

$$p_i(w_{\mathcal{B}}) - p_i(w_{\mathcal{A}}) = (\mathcal{H}_{\mathcal{B}}(k) - i + 1)(k - \mathcal{T}_{\mathcal{B}}(k) + 1)t.$$

□

We may now state and prove our main theorem, **the S^1-equivariant Chevalley-Monk formula for type A Peterson varieties**, which gives a "manifestly positive" combinatorial formula for the non-negative, integral structure constants $c_{i,\mathcal{A}}^{\mathcal{B}}$. We have the following.

Theorem 6.12. *(***"The S^1-equivariant Chevalley-Monk formula for Peterson varieties."***) Let Y be the Peterson variety of type A_{n-1} with the natural S^1-action defined by* (2.3)*. For* $\mathcal{A} \subseteq \{1, 2, \ldots, n-1\}$, *let $v_{\mathcal{A}} \in S_n$ be the permutation given in Definition 4.1, and let $p_{v_{\mathcal{A}}}$ be the corresponding Peterson Schubert class in $H^*_{S^1}(Y)$. Let $p_i := p_{s_i}$ denote the Peterson Schubert class corresponding to the singleton subset $\{i\}$. Then*

$$p_i \cdot p_{v_{\mathcal{A}}} = p_i(w_{\mathcal{A}}) \cdot p_{v_{\mathcal{A}}} + \sum_{\mathcal{A} \subsetneq \mathcal{B} \text{ and } |\mathcal{B}| = |\mathcal{A}| + 1} c_{i,\mathcal{A}}^{\mathcal{B}} \cdot p_{v_{\mathcal{B}}}, \tag{6.9}$$

where, for a subset $\mathcal{B} \subseteq \{1, 2, \ldots, n-1\}$ which is a disjoint union $\mathcal{B} = \mathcal{A} \cup \{k\}$,

- *if $i \notin \mathcal{B}$ then $c_{i,\mathcal{A}}^{\mathcal{B}} = 0$,*
- *if $i \in \mathcal{B}$ and $i \notin [\mathcal{T}_{\mathcal{B}}(k), \mathcal{H}_{\mathcal{B}}(k)]$, then $c_{i,\mathcal{A}}^{\mathcal{B}} = 0$,*
- *if $k \leq i \leq \mathcal{H}_{\mathcal{B}}(k)$, then*

$$c_{i,\mathcal{A}}^{\mathcal{B}} = (\mathcal{H}_{\mathcal{B}}(k) - i + 1) \cdot \begin{pmatrix} \mathcal{H}_{\mathcal{B}}(k) - \mathcal{T}_{\mathcal{B}}(k) + 1 \\ k - \mathcal{T}_{\mathcal{B}}(k) \end{pmatrix}, \tag{6.10}$$

- *if $\mathcal{T}_{\mathcal{B}}(k) \leq i \leq k-1$,*

$$c_{i,\mathcal{A}}^{\mathcal{B}} = (i - \mathcal{T}_{\mathcal{B}}(k) + 1) \cdot \binom{\mathcal{H}_{\mathcal{B}}(k) - \mathcal{T}_{\mathcal{B}}(k) + 1}{k - \mathcal{T}_{\mathcal{B}}(k) + 1}. \tag{6.11}$$

Moreover $p_i(w_{\mathcal{A}})$ as well as each $c_{i,\mathcal{A}}^{\mathcal{B}}$ is a non-negative integer.

Proof. The product $p_i \cdot p_{v_{\mathcal{A}}}$ in $H^*_{S^1}(Y)$ is a linear combination of the form (6.9) by Proposition 6.3. The first two claims about the vanishing of $c^{\mathcal{B}}_{i,\mathcal{A}}$ were shown in Lemma 6.6. The latter two claims (6.10) and (6.11) follow from straightforward computation using Lemma 6.9 and Lemma 6.11. Moreover, the assumptions on i imply that the first factor appearing in the product on the right hand side of (6.10) and (6.11), respectively, is a positive integer. Binomial coefficients are also positive integers, so we conclude that $c^{\mathcal{B}}_{i,\mathcal{A}}$ is always a non-negative integer. Finally, the fact that $p_i(w_{\mathcal{A}})$ is positive in the sense of Graham follows from Equation (6.4), or from Graham-positivity of Billey's formula. The result follows. □

We give two fully computed examples.

Example 6.13. *Continuing Examples 5.8 and 5.10, suppose $n = 7, \mathcal{A} = \{1, 2, 3, 5, 6\}$ and $\mathcal{B} = \{1, 2, 3, 4, 5, 6\}$.*
Suppose first $i = 3$. Then from (6.4) *we immediately compute*

$$p_3(w_{\mathcal{A}}) = 3t.$$

In this case $\mathcal{B} = \mathcal{A} \cup \{4\}$, so $k = 4$ and $i = 3$, so we use (6.11). *We conclude that*

$$p_3 \cdot p_{v_{\mathcal{A}}} = 3t \cdot p_{v_{\mathcal{A}}} + 45 \cdot p_{v_{\mathcal{B}}},$$

which may also be checked directly using the computations given in Example 5.10 and (6.3).
Now suppose $i \notin \mathcal{A}$ but $i \in \mathcal{B}$, i.e. $i = 4$. In this case $k = i = 4$ and $i \notin \mathcal{A}$, so we immediately see $p_i(w_{\mathcal{A}}) = 0$. We also use (6.10) *to obtain the formula*

$$p_4 \cdot p_{v_{\mathcal{A}}} = 60 \cdot p_{v_{\mathcal{B}}},$$

which again may be checked explicitly using the computations in Example 5.10.

We conclude with some remarks about explicit presentations of $H^*_{S^1}(Y)$ and $H^*(Y)$ via generators and relations. By Proposition 6.2, the equivariant Chevalley-Monk formula above completely determines the ring structure of $H^*_{S^1}(Y)$. This leads to the following.

Corollary 6.14. *Let Y be the Peterson variety of type A_{n-1} with the natural S^1-action defined by* (2.3). *For $\mathcal{A} \subseteq \{1, 2, \ldots, n-1\}$, let $v_{\mathcal{A}} \in S_n$ be the permutation given in Definition 4.1, and let $p_{v_{\mathcal{A}}}$ be the corresponding Peterson Schubert class in $H^*_{S^1}(Y)$. Let $t \in H^*_{S^1}(Y)$ be the image of the generator $t \in H^*_{S^1}(\mathrm{pt}) \cong \mathbb{C}[t]$. Then the S^1-equivariant cohomology $H^*_{S^1}(Y)$ is given by*

$$H^*_{S^1}(Y) \cong \mathbb{C}[t, \{p_{v_{\mathcal{A}}}\}_{\mathcal{A} \subseteq \{1,2,\ldots,n-1\}}]/\mathcal{J}$$

where $\mathcal{J}$ is the ideal generated by the relations (6.9).

We next explain how the equivariant Chevalley-Monk formula of Theorem 6.12 yields a Chevalley-Monk formula for the *ordinary* cohomology of Peterson varieties, as well as an explicit ring presentation of $H^*(Y)$. For this discussion we denote by $\check{\sigma}_w \in H^*(G/B)$ and $\check{p}_w \in H^*(Y)$ the ordinary cohomology classes which are the images of the (equivariant) Schubert and Peterson Schubert classes under the forgetful maps $H^*_T(G/B) \to H^*(G/B)$ and $H^*_{S^1}(Y) \to H^*(Y)$, respectively. We have the following.

Lemma 6.15. *The classes $\{\check{p}_{v_{\mathcal{A}}}\}_{\mathcal{A} \subseteq \{1,2,\ldots,n-1\}}$ form a $\mathbb{C}$-basis of $H^*(Y)$ and the cohomology-degree-2 classes $\{\check{p}_i\}_{i=1}^{n-1}$ are a set of ring generators of $H^*(Y)$.*

Proof. It is well-known that the Schubert classes $\{\check{\sigma}_w\}_{w \in S_n}$ form a $\mathbb{C}$-basis of $H^*(G/B)$ and that the cohomology-degree-2 classes among the $\check{\sigma}_w$ generate the ring $H^*(G/B)$. Carrell and Kaveh show that the restriction map $H^*(G/B) \to H^*(Y)$ is surjective [6], so $H^*(Y)$ is generated in degree 2. Also, we have shown in previous sections that $H^*_T(G/B) \to H^*_{S^1}(Y)$ is surjective, and that the Peterson Schubert classes form a $H^*_{S^1}(\mathrm{pt})$-module basis for $H^*_{S^1}(Y)$. The compositions $H^*_T(G/B) \to H^*(G/B) \to H^*(Y)$ and $H^*_T(G/B) \to H^*_{S^1}(Y) \to H^*(Y)$ are equal, so we conclude that $H^*_{S^1}(Y) \to H^*(Y)$ is also surjective and hence the $\{\check{p}_{v_{\mathcal{A}}}\}$ are a $\mathbb{C}$-basis for $H^*(Y)$. □

In contrast, the element $t \in H^*_{S^1}(Y)$ given by the image of the cohomology-degree-2 generator of $\mathbb{C}[t] \cong H^*_{S^1}(\mathrm{pt})$ lies in the kernel of the forgetful map $H^*_{S^1}(Y) \to H^*(Y)$. This can be seen from the fact that Y is the fiber of the bundle $Y \to Y \times_{S^1} ES^1 \to BS^1$.

From this discussion we immediately obtain the following consequence of Theorem 6.12.

Corollary 6.16. *(**"The (ordinary) Chevalley-Monk formula for Peterson varieties."**) Let Y be the Peterson variety of type A_{n-1}. For $\mathcal{A} \subseteq \{1, 2, \ldots, n-1\}$, let $v_{\mathcal{A}} \in S_n$ be the permutation given in Definition 4.1, and let $\check{p}_{v_{\mathcal{A}}}$ be the image under the forgetful map $H^*_{S^1}(Y) \to H^*(Y)$ of the corresponding Peterson Schubert class $p_{v_{\mathcal{A}}}$ in $H^*_{S^1}(Y)$. Let $\check{p}_i := \check{p}_{s_i}$ denote the class corresponding to the singleton subset $\{i\}$. Then*

$$\check{p}_i \cdot \check{p}_{v_{\mathcal{A}}} = \sum_{\mathcal{A} \subsetneq \mathcal{B} \text{ and } |\mathcal{B}|=|\mathcal{A}|+1} \check{c}^{\mathcal{B}}_{i,\mathcal{A}} \cdot \check{p}_{v_{\mathcal{B}}}, \tag{6.12}$$

where, for a subset $\mathcal{B} \subseteq \{1, 2, \ldots, n-1\}$ which is a disjoint union $\mathcal{B} = \mathcal{A} \cup \{k\}$, the structure constant $\check{c}^{\mathcal{B}}_{i,\mathcal{A}}$ is equal to the structure constant given in Theorem 6.12, i.e.

$$\check{c}^{\mathcal{B}}_{i,\mathcal{A}} = c^{\mathcal{B}}_{i,\mathcal{A}} \in \mathbb{Z}_{\geq 0}.$$

In particular, each $\check{c}^{\mathcal{B}}_{i,\mathcal{A}}$ is a non-negative integer.

Proof. The statement is immediate from Theorem 6.12 and the observation that $p_i(w_{\mathcal{A}})$, being a multiple of t, goes to zero under the forgetful map $H^*_{S^1}(Y) \to H^*(Y)$. □

Since the cohomology-degree-2 elements $\{\check{p}_i\}_{i=1}^{n-1}$ generate the ring, Corollary 6.16 completely determines the ring structure of $H^*(Y)$. In particular, in analogy to Corollary 6.14, we obtain the following.

Corollary 6.17. *Let Y be the Peterson variety of type A_{n-1}. For $\mathcal{A} \subseteq \{1, 2, \ldots, n-1\}$, let $v_{\mathcal{A}} \in S_n$ be the permutation given in Definition 4.1, and let $\check{p}_{v_{\mathcal{A}}}$ be the image under $H^*_{S^1}(Y) \to H^*(Y)$ of the corresponding Peterson Schubert class in $H^*_{S^1}(Y)$. Then the ordinary cohomology $H^*(Y)$ is given by*

$$H^*_{S^1}(Y) \cong \mathbb{C}[\{\check{p}_{v_{\mathcal{A}}}\}_{\mathcal{A} \subseteq \{1,2,\ldots,n-1\}}]/\check{\mathfrak{J}}$$

where $\check{\mathfrak{J}}$ is the ideal generated by the relations (6.12).

Appendix A. Module bases for Borel-equivariant cohomology with field coefficients

In this appendix, we state a fact (with proof) about bases for modules over graded rings, which in particular applies to our setting of Borel-equivariant cohomology with field coefficients. The statement is well-known, perhaps obvious, to the experts. However, we were unable to find a clear reference in the literature, and include it here for completeness, convenience, and future use.

It is known [3, 18] that if $H^*(X; \mathbb{F})$ for a field $\mathbb{F}$ is finite-dimensional and concentrated in even degree, then the Borel-equivariant cohomology of $H^*_T(X; \mathbb{F})$ is a free $H^*_T(\mathrm{pt}; \mathbb{F})$-module, with a non-canonical module isomorphism to the tensor product

$$H^*_T(X; \mathbb{F}) \cong H^*_T(\mathrm{pt}; \mathbb{F}) \otimes_{\mathbb{F}} H^*(X; \mathbb{F}).$$

Suppose X is a T-space such that the above holds. In such a situation, it is natural to ask for a $H^*_T(\mathrm{pt}; \mathbb{F})$-module basis for the equivariant cohomology $H^*_T(X; \mathbb{F})$, such that the basis elements correspond in some way to elements of the ordinary cohomology $H^*(X; \mathbb{F})$.

Motivated by this question, we prove below a general theorem about graded rings and modules over graded rings. One consequence is that in many common situations in the toric topology of algebraic varieties, the Betti numbers of the ordinary cohomology $H^*(X; \mathbb{F})$ of a T-space determine the number of elements of a given degree in a module basis for $H^*_T(X; \mathbb{F})$.

Let R be a graded ring and M an R-module. Suppose M is graded compatibly with the R-module structure in the sense that $M \cong \bigoplus_{k \geq 0} M_k$ as additive groups and the R-module structure takes $R_i \times M_k$ to M_{i+k}. We assume $R_0 \cong \mathbb{F}$. Hence, since M is an R_0-module, it also has the structure of an $\mathbb{F}$-vector space, with each M_k an $\mathbb{F}$-subspace. Let $M_{\leq k} = \bigoplus_{j \leq k} M_j$ denote the subspace of M consisting of graded pieces of degree at most k.

Proposition A.1. *Let $\mathbb{F}$ be a field. Let $R = \bigoplus_{i \geq 0} R_i$ be a graded $\mathbb{F}$-algebra such that R_k is finite-dimensional for all $k \geq 0$, and $R_0 \cong \mathbb{F}$. Let M be a free finitely-generated R-module of the form*

$$M = R \otimes_{\mathbb{F}} V$$

for a finite-dimensional graded $\mathbb{F}$-vector space V, where the R-module structure on the right hand side is given by ordinary multiplication on the first factor and the grading on M is given by

$$M_k = \bigoplus_{i+j=k} R_i \otimes_{\mathbb{F}} V_j.$$

Suppose $\{m_{\mu,k}\}$ is a subset of M satisfying

- $\deg(m_{\mu,k}) = k$,
- *the number of $m_{\mu,k}$ of degree k is precisely $\dim_{\mathbb{F}}(V_k)$, and*
- *the $\{m_{\mu,k}\}$ are R-linearly independent in M.*

Then the $\{m_{\mu,k}\}$ are an R-module basis of M.

Proof. Since the $\{m_{\mu,k}\}$ are assumed R-linearly independent, it suffices to show that they R-span M. Let N denote the R-submodule of M generated by the $\{m_{\mu,k}\}$. We will show that $N = M$ by proving inductively that for each $k \geq 0$ we have

- $N_{\leq k} = M_{\leq k}$ and moreover,
- $M_{\leq k}$ is R-generated by the subset $\{m_{\mu,j} : j \leq k\}$ of elements $m_{\mu,j}$ of degree less than or equal to k.

We begin with the base case $k = 0$. In this case

$$M_0 = R_0 \otimes_{\mathbb{F}} V_0.$$

By assumption R_0 is a one-dimensional $\mathbb{F}$-vector space so $\dim_{\mathbb{F}}(M_0) = \dim_{\mathbb{F}}(V_0)$. By hypothesis there exist $\dim_{\mathbb{F}}(V_0)$ many elements $m_{\mu,0}$. These elements are assumed R-linearly independent, so in particular they are $\mathbb{F}$-linearly independent. Hence they $\mathbb{F}$-span an $\mathbb{F}$-subspace of M_0 of dimension $\dim_{\mathbb{F}}(M_0)$, so they are a basis; we conclude $N_0 = M_0$. We also see that M_0 is R-generated by the $\{m_{\mu,0}\}$, as required.

Now suppose by induction that $N_{\leq k} = M_{\leq k}$ and that $M_{\leq k}$ is R-generated by the elements $\{m_{\mu,j}\}$ with $j \leq k$. We wish to show that $N_{\leq k+1} = M_{\leq k+1}$ for which it would suffice to show $N_{k+1} = M_{k+1}$. By definition $N_{k+1} \subseteq M_{k+1}$, so it suffices to show $\dim_{\mathbb{F}} N_{k+1} \geq \dim_{\mathbb{F}} M_{k+1}$. We first observe that M_{k+1} may be decomposed as

$$M_{k+1} = (R_0 \otimes V_{k+1}) \bigoplus \left(\bigoplus_{\substack{i+j=k+1 \\ i>0}} R_i \otimes V_j \right). \tag{A.1}$$

We first claim that any element in the second factor of this direct sum decomposition is an R-linear combination of elements $m_{\mu,j}$ for $j \leq k$. Indeed, any element in $R_i \otimes V_j$ with $i > 0$ can be written as an R-multiple of an element $1 \otimes V_j \in M_j$ for $j \leq k$. By the inductive hypothesis $M_j = N_j$ for $j \leq k$, and by definition the $\{m_{\mu,j}\}$ for $j \leq k$ are an R-basis for $N_{\leq k}$. Multiplying an R-linear combination of $\{m_{\mu,j}\}$ for $j \leq k$ by an element of R is still an R-linear combination of $\{m_{\mu,j}\}$ for $j \leq k$; in particular the result is still in N.

We now claim the $\mathbb{F}$-span of the degree-$(k+1)$ elements $\{m_{\mu,k+1}\}$ and the second factor in (A.1) is all of M_{k+1}. Note that

$$\mathrm{span}_{\mathbb{F}}\langle m_{\mu,k+1}\rangle \cap \left(\bigoplus_{\substack{i+j=k+1\\ i>0}} R_i \otimes V_j \right) = \{0\}$$

since the $\{m_{\mu,j}\}_{j\leq k+1}$ are R-linearly independent and in particular $\mathbb{F}$-linearly independent. Since $|\{m_{\mu,k+1}\}| = \dim_{\mathbb{F}}(V_{k+1})$ and $\mathrm{span}_{\mathbb{F}}\langle m_{\mu,k+1}\rangle \subseteq N_{k+1}$, we conclude $\dim_{\mathbb{F}} N_{k+1} \geq \dim_{\mathbb{F}} M_{k+1}$, as desired.

□

Remark A.2. *We emphasize that it is crucial in this proof, as well as in the applications to T-spaces mentioned above, that we are working with vector spaces over a field $\mathbb{F}$. In particular, the analogous conclusion does* not *hold for arbitrary generalized equivariant cohomology theories. For instance, for Borel-equivariant cohomology with $\mathbb{Z}$ coefficients, Darius Bayegan has shown via explicit calculation that the Peterson Schubert classes $\{p_{v_A}\}$ in this manuscript are* not *an $H^*_{S^1}(\mathrm{pt};\mathbb{Z})$-module basis of $H^*_{S^1}(Y;\mathbb{Z})$, although they are a $H^*_{S^1}(\mathrm{pt};\mathbb{C})$-module basis of $H^*_{S^1}(Y;\mathbb{C})$ by Theorem 4.12.*

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