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Rajan Amit Mehta  
_Smith College, rmehta@smith.edu_

Mathieu Stiénon  
_The Pennsylvania State University_

Ping Xu  
_The Pennsylvania State University_

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THE ATIYAH CLASS OF A DG-VECTOR BUNDLE

RAJAN AMIT MEHTA, MATHIEU STIÉNON, AND PING XU

En hommage à Charles-Michel Marle à l’occasion de son quatre-vingtième anniversaire

Abstract. We introduce the notions of Atiyah class and Todd class of a differential graded vector bundle with respect to a differential graded Lie algebroid. We prove that the space of vector fields $\mathfrak{X}(\mathcal{M})$ on a dg-manifold $\mathcal{M}$ with homological vector field $Q$ admits a structure of $L_\infty[1]$-algebra with the Lie derivative $L_Q$ as unary bracket $\lambda_1$, and the Atiyah cocycle $\mathrm{At}_\mathcal{M}$ corresponding to a torsion-free affine connection as binary bracket $\lambda_2$.

1. DG-MANIFOLDS AND DG-VECTOR BUNDLES

A $\mathbb{Z}$-graded manifold $\mathcal{M}$ with base manifold $M$ is a sheaf of $\mathbb{Z}$-graded, graded-commutative algebras $\{\mathcal{R}_U|U \subset M \text{ open}\}$ over $M$, locally isomorphic to $C^\infty(U) \otimes \hat{S}(V^\vee)$, where $U \subset M$ is an open submanifold, $V$ is a $\mathbb{Z}$-graded vector space, and $\hat{S}(V^\vee)$ denotes the graded algebra of formal polynomials on $V$. By $C^\infty(M)$, we denote the $\mathbb{Z}$-graded, graded-commutative algebra of global sections. By a dg-manifold, we mean a $\mathbb{Z}$-graded manifold endowed with a homological vector field, i.e. a vector field $Q$ of degree $+1$ satisfying $[Q, Q] = 0$.

Example 1.1. Let $A \to M$ be a Lie algebroid over $\mathbb{C}$. Then $A[1]$ is a dg-manifold with the Chevalley–Eilenberg differential $d_{CE}$ as homological vector field. In fact, according to Vaintrob [12], there is a bijection between the Lie algebroid structures on the vector bundle $A \to M$ and the homological vector fields on the $\mathbb{Z}$-graded manifold $A[1]$.

Example 1.2. Let $s$ be a smooth section of a vector bundle $E \to M$. Then $E[-1]$ is a dg-manifold with the contraction operator $i_s$ as homological vector field.

Example 1.3. Let $g = \sum_{i \in \mathbb{Z}} g_i$ be a $\mathbb{Z}$-graded vector space of finite type, i.e. each $g_i$ is a finite-dimensional vector space. Then $g[1]$ is a dg-manifold if and only if $g$ is an $L_\infty$-algebra.

A dg-vector bundle is a vector bundle in the category of dg-manifolds. We refer the reader to [10, 4] for details on dg-vector bundles. The following example is essentially due to Kotov–Strobl [4].


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The example above is a special case of a general fact [10], that LA-vector bundles [6, 7, 8] (also known as VB-algebroids [2]) give rise to dg-vector bundles.

Given a vector bundle \( E \xrightarrow{\pi} M \) of graded manifolds, its space of sections, denoted \( \Gamma(E) \), is defined to be \( \bigoplus_{j \in \mathbb{Z}} \Gamma_j(E) \), where \( \Gamma_j(E) \) consists of degree preserving maps \( s \in \text{Hom}(M, E[-j]) \) such that \( (\pi[-j]) \circ s = \text{id}_M \), where \( \pi[-j] : E[-j] \to M \) is the natural map induced from \( \pi \); see [10] for more details. When \( E \to M \) is a dg-vector bundle, the homological vector fields on \( E \) and \( M \) naturally induce a degree 1 operator \( Q \) on \( \Gamma(E) \), making \( \Gamma(E) \) a dg-module over \( \mathcal{C}^\infty(M) \).

Lemma 1.5. Let \( E \to M \) be a vector bundle object in the category of graded manifolds and suppose \( M \) is a dg-manifold. If \( \Gamma(E) \) is a dg-module over \( \mathcal{C}^\infty(M) \), then \( E \) admits a natural dg-manifold structure such that \( E \to M \) is a dg-vector bundle. In fact, the categories of dg-vector bundles and of locally free dg-modules are equivalent.

In this case, the degree +1 operator \( Q \) on \( \Gamma(E) \) gives rise to a cochain complex

\[
\cdots \to \Gamma_i(E) \xrightarrow{Q} \Gamma_{i+1}(E) \to \cdots,
\]

whose cohomology group will be denoted by \( H^*(\Gamma(E), Q) \).

In particular, the space \( \mathfrak{X}(M) \) of vector fields on a dg-manifold \( (M, Q) \) (i.e. graded derivations of \( \mathcal{C}^\infty(M) \)), which can be regarded as the space of sections \( \Gamma(TM) \) of \( TM \), is naturally a dg-module over \( \mathcal{C}^\infty(M) \) with the Lie derivative \( L_Q : \mathfrak{X}(M) \to \mathfrak{X}(M) \) playing the role of the degree +1 operator \( Q \).

Thus we have the following

Corollary 1.6. For every dg-manifold \( (M, Q) \), the Lie derivative \( L_Q \) makes \( \Gamma(TM) \) into a dg-module over \( \mathcal{C}^\infty(M) \) and therefore \( TM \to M \) is naturally a dg-vector bundle.

Following the classical case, the corresponding homological vector field on \( TM \) is called the tangent lift of \( Q \).

Differential graded Lie algebroids are another useful notion. Roughly, a dg-Lie algebroid can be thought of as a Lie algebroid object in the category of dg-manifolds. For more details, we refer the reader to [10], where dg-Lie algebroids are called \( Q \)-algebroids.

Differential graded foliations constitute an important class of examples of dg-Lie algebroids.

Lemma 1.7. Let \( D \subset TM \) be an integrable distribution on a graded manifold \( M \). Suppose there exists a homological vector field \( Q \) on \( M \) such that \( \Gamma(D) \) is stable under \( L_Q \). Then \( D \to M \) is a dg-Lie algebroid with the inclusion \( \rho : D \to TM \) as its anchor map.

2. Atiyah class and Todd class of a dg-vector bundle

Let \( E \to M \) be a dg-vector bundle and let \( A \to M \) be a dg-Lie algebroid with anchor \( \rho : A \to TM \). An \( A \)-connection on \( E \to M \) is a degree 0 map \( \nabla : \Gamma(A) \otimes \Gamma(E) \to \Gamma(E) \) such that

\[
\nabla_{fX} = f\nabla_X
\]
and
\[ \nabla_X(f s) = \rho(X)(f)s + (-1)^{|X||f|}f \nabla_X s \]
for all \( f \in C^\infty(M) \), \( X \in \Gamma(A) \), and \( s \in \Gamma(E) \). Here we use the ‘absolute value’ notation to denote the degree of the argument. When we say that \( \nabla \) is of degree 0, we actually mean that \( |\nabla_X s| = |X| + |s| \) for every pair of homogeneous elements \( X \) and \( s \). Such connections always exist since the standard partition of unity argument holds in the context of graded manifolds. Given a dg-vector bundle \( E \to M \) and an \( A \)-connection \( \nabla \) on it, we can consider the bundle map \( \text{At}_E : A \otimes E \to E \) defined by
\[ \text{At}_E(X, s) := \mathcal{Q}(\nabla_X s) - \nabla_{Q(X)} s - (-1)^{|X|} \nabla_X (\mathcal{Q}(s)), \quad \forall X \in \Gamma(A), s \in \Gamma(E). \]

**Proposition 2.1.**
(1) \( \text{At}_E : A \otimes E \to E \) is a degree +1 bundle map and therefore can also be regarded as a degree +1 section of \( A^\vee \otimes \text{End} E \).
(2) \( \text{At}_E \) is a cocycle: \( \mathcal{Q}(\text{At}_E) = 0 \).
(3) The cohomology class of \( \text{At}_E \) is independent of the choice of the connection \( \nabla \).

Thus there is a natural cohomology class \( \alpha_E := [\text{At}_E] \) in \( H^1(\Gamma(A^\vee \otimes \text{End} E), Q) \). The class \( \alpha_E \) is called the Atiyah class of the dg-vector bundle \( E \to M \) relative to the dg-Lie algebroid \( A \to M \).

The Atiyah class of a dg-manifold, which is the obstruction to the existence of connections compatible with the differential, was first investigated by Shoikhet [11] in relation with Kontsevich’s formality theorem and Duflo formula. More recently, the Atiyah class of a dg-vector bundle appeared in Costello’s work [1].

We define the Todd class \( \text{Td}_E \) of a dg-vector bundle \( E \to M \) relative to a dg-Lie algebroid \( A \to M \) by
\[ \text{Td}_E := \text{Ber} \left( 1 - \frac{e^{-\alpha_E}}{\alpha_E} \right) \in \prod_{k \geq 0} H^k(\Gamma(\wedge^k A^\vee), Q), \]
where \( \text{Ber} \) denotes the Berezinian [9] and \( \wedge^k A^\vee \) denotes the dg vector bundle \( S^k(A^\vee[-1])[k] \to M \). One checks that \( \text{Td}_E \) can be expressed in terms of scalar Atiyah classes \( \alpha_k := \frac{1}{k!}(\text{Ber})^k \text{str} \alpha_k^E \in H^k(\Gamma(\wedge^k A^\vee), Q) \). Here \( \text{str} : \text{End} E \to C^\infty(M) \) denotes the supertrace. Note that \( \text{str} \alpha_k^E \in H^k(\wedge^k A^\vee) \) since \( \alpha_k^E \in \Gamma(\wedge^k A^\vee) \otimes_{C^\infty(M)} \text{End} E \). If \( A = TM \), we write \( \Omega^k(M) \) instead of \( \Gamma(\wedge^k T^m M) \).

3. **Atiyah class and Todd class of a dg-manifold**

Consider a dg-manifold \( (M, Q) \). According to Lemma [15], its tangent bundle \( TM \) is indeed a dg-Lie algebroid. By the Atiyah class of a dg-manifold \( (M, Q) \), denoted \( \alpha_M \), we mean the Atiyah class of the tangent dg-vector bundle \( TM \to M \) with respect to the dg-Lie algebroid \( TM \). Similarly, the Atiyah 1-cocycle of a dg-manifold \( M \) corresponding to an affine connection on \( M \) (i.e. a \( TM \)-connection on \( TM \to M \)) is the 1-cocycle defined as in Eq. (1).

**Lemma 3.1.** Suppose \( V \) is a vector space. The only connection on the graded manifold \( V[1] \) is the trivial connection.

**Proof.** Since the graded algebra of functions on \( V[1] \) is \( \wedge(V^\vee) \), every vector \( v \in V \) determines a degree \(-1\) vector field \( \iota_v \) on \( V[1] \), which acts as a contraction operator on \( \wedge(V^\vee) \). The \( C^\infty(V[1]) \)-module of all vector fields on \( V[1] \) is generated by its subset \( \{\iota_v\}_{v \in V} \). It follows that a connection \( \nabla \) on \( V[1] \) is completely determined
by the knowledge of $\nabla_{\alpha} t_{vw}$ for all $v, w \in V$. Since the degree of every vector field $\nabla_{\alpha} t_{vw}$ must be $-2$ and there are no nonzero vector fields of degree $-2$, it follows that $\nabla_{\alpha} t_{vw} = 0$. \qed

Given a finite-dimensional Lie algebra $g$, consider the dg-manifold $(\mathcal{M}, Q)$, where $\mathcal{M} = g[1]$ and $Q$ is the Chevalley-Eilenberg differential $d_{CE}$. The following result can be easily verified using the canonical trivialization $T\mathcal{M} \cong g[1] \times g[1]$.

**Lemma 3.2.** Let $(\mathcal{M}, Q) = (g[1], d_{CE})$ be the canonical dg-manifold corresponding to a finite-dimensional Lie algebra $g$. Then,

$$H^k(\Gamma(T^\vee \mathcal{M} \otimes \text{End} T\mathcal{M}), Q) \cong H^{k-1}_{CE}(g, g^\vee \otimes g^\vee \otimes g),$$

and

$$H^k(\Omega^k(\mathcal{M}), Q) \cong (S^k g^\vee)^g.$$

**Proposition 3.3.** Let $(\mathcal{M}, Q) = (g[1], d_{CE})$ be the canonical dg-manifold corresponding to a finite-dimensional Lie algebra $g$. Then the Atiyah class $\alpha_{g[1]}$ is precisely the Lie bracket of $g$ regarded as an element of $(g^\vee \otimes g^\vee \otimes g)^g \cong H^1(\Gamma(T^\vee \mathcal{M} \otimes \text{End} T\mathcal{M}), Q)$. Consequently, the isomorphism

$$\prod_k H^k(\Omega^k(\mathcal{M}), Q) \xrightarrow{\sim} (\mathcal{S}(g^\vee))^g$$

maps the Todd class $\text{Td}_{g[1]}$ onto the Duflo element of $g$.

**Example 3.4.** Consider a dg-manifold of the form $\mathcal{M} = (\mathbb{R}^{m|n}, Q)$. Let $(x_1, \cdots, x_m; x_{m+1} \cdots x_{m+n})$ be coordinate functions on $\mathbb{R}^{m|n}$, and write $Q = \sum_k Q_k(x) \frac{\partial}{\partial x_k}$. Then the Atiyah 1-cocycle associated to the trivial connection $\nabla \frac{\partial}{\partial x_k} = 0$ is given by

$$(3) \quad \text{At}_\mathcal{M} \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = (-1)^{|x_i|+|x_j|} \sum_k \frac{\partial^2 Q_k}{\partial x_i \partial x_j} \frac{\partial}{\partial x_k}$$

As we can see from (3), the Atiyah 1-cocycle $\text{At}_\mathcal{M}$ includes the information about the homological vector field of second-order and higher.

4. **Atiyah Class and Homotopy Lie Algebras**

Let $\mathcal{M}$ be a graded manifold. A $(1,2)$-tensor of degree $k$ on $\mathcal{M}$ is a $\mathbb{C}$-linear map $\alpha: \mathfrak{x}(\mathcal{M}) \otimes \mathfrak{x}(\mathcal{M}) \to \mathfrak{x}(\mathcal{M})$ such that $|\alpha(X, Y)| = |X| + |Y| + k$ and

$$\alpha(fX, Y) = (-1)^{|f||X|} f \alpha(X, Y) = (-1)^{|f||X|} \alpha(X, fY).$$

We denote the space of $(1,2)$-tensors of degree $k$ by $T_k^{1,2}(\mathcal{M})$, and the space of all $(1,2)$-tensors by $T^{1,2}(\mathcal{M}) = \bigoplus_k T_k^{1,2}(\mathcal{M})$.

The torsion of an affine connection $\nabla$ is given by

$$(4) \quad T(X, Y) = \nabla_X Y - (\nabla_Y X - [X, Y]).$$

The torsion is an element in $T^{1,2}_0(\mathcal{M})$. Given any affine connection, one can define its opposite affine connection $\nabla^{op}$, given by

$$(5) \quad \nabla^{op}_X Y = \nabla_X Y - T(X, Y) = [X, Y] + (-1)^{|X||Y|} \nabla_Y X.$$

The average $\frac{1}{2}(\nabla + \nabla^{op})$ is a torsion-free affine connection. This shows that torsion-free affine connections always exist on graded manifolds.
In this section, we show that, as in the classical situation considered by Kapranov in [3, 8], the Atiyah 1-cocycle of a dg-manifold gives rise to an interesting homotopy Lie algebra. As in the last section, let $(\mathcal{M}, Q)$ be a dg-manifold and let $\nabla$ be an affine connection on $\mathcal{M}$. The following can be easily verified by direct computation.

1. The anti-symmetrization of the Atiyah 1-cocycle $\Lambda_{\mathcal{M}}$ is equal to $L_Q T$, so $\Lambda_{\mathcal{M}}$ is graded antisymmetric up to an exact term. In particular, if $\nabla$ is torsion-free, we have

$$\Lambda_{\mathcal{M}}(X, Y) = (-1)^{|X||Y|} \Lambda_{\mathcal{M}}(Y, X).$$

2. The degree $1 + |X|$ operator $\Lambda_{\mathcal{M}}(X, -)$ need not be a derivation of the degree $+1$ ‘product’ $\mathcal{X}(\mathcal{M}) \otimes_C \mathcal{X}(\mathcal{M}) \xrightarrow{\Lambda_{\mathcal{M}}} \mathcal{X}(\mathcal{M})$. However, the Jacobiator

$$(X, Y, Z) \mapsto \Lambda_{\mathcal{M}}(X, \Lambda_{\mathcal{M}}(Y, Z)) - \{(-1)^{|X|+1} \Lambda_{\mathcal{M}}(\Lambda_{\mathcal{M}}(X, Y), Z) + (-1)^{(|X|+1)(|Y|+1)} \Lambda_{\mathcal{M}}(Y, \Lambda_{\mathcal{M}}(X, Z))\},$$

of $\Lambda_{\mathcal{M}}$, which vanishes precisely when $\Lambda_{\mathcal{M}}(X, -)$ is a derivation of $\Lambda_{\mathcal{M}}$, is equal to $\pm L_Q(\nabla \Lambda_{\mathcal{M}})$. Hence $\Lambda_{\mathcal{M}}$ satisfies the graded Jacobi identity up to the exact term $L_Q(\nabla \Lambda_{\mathcal{M}})$.

Armed with this motivation, we can now state the main result of this note.

**Theorem 4.1.** Let $(\mathcal{M}, Q)$ be a dg-manifold and let $\nabla$ be a torsion-free affine connection on $\mathcal{M}$. There exists a sequence $(\lambda_k)_{k \geq 2}$ of maps $\lambda_k \in \text{Hom}(S^k(T\mathcal{M}), T\mathcal{M}[-1])$ starting with $\lambda_2 := \Lambda_{\mathcal{M}} \in \text{Hom}(S^2(T\mathcal{M}), T\mathcal{M}[-1])$ which, together with $\lambda_1 := L_Q : \mathcal{X}(\mathcal{M}) \rightarrow \mathcal{X}(\mathcal{M})$, satisfy the $L_\infty[1]$-algebra axioms. As a consequence, the space of vector fields $\mathcal{X}(\mathcal{M})$ on a dg-manifold $(\mathcal{M}, Q)$ admits an $L_\infty[1]$-algebra structure with the Lie derivative $L_Q$ as unary bracket $\lambda_1$ and the Atiyah cocycle $\Lambda_{\mathcal{M}}$ as binary bracket $\lambda_2$.

To prove Theorem 4.1, we introduce a Poincaré–Birkhoff–Witt map for graded manifolds.

It was shown in [5] that every torsion-free affine connection $\nabla$ on a smooth manifold $M$ determines an isomorphism of coalgebras (over $C^\infty(M)$)

$$\text{pbw}^\nabla : \Gamma(S(T\mathcal{M})) \xrightarrow{\cong} D(M),$$

(6)
called the Poincaré–Birkhoff–Witt (PBW) map. Here $D(M)$ denotes the space of differential operators on $M$.

Geometrically, an affine connection $\nabla$ induces an exponential map $TM \rightarrow M \times M$, which is a well-defined diffeomorphism from a neighborhood of the zero section of $TM$ to a neighborhood of the diagonal $\Delta(M)$ of $M \times M$. Sections of $S(T\mathcal{M})$ can be viewed as fiberwise distributions on $TM$ supported on the zero section, while $D(M)$ can be viewed as the space of source-fiberwise distributions on $M \times M$ supported on the diagonal $\Delta(M)$. The map $\text{pbw}^\nabla : \Gamma(S(T\mathcal{M})) \rightarrow D(M)$ is simply the push-forward of fiberwise distributions through the exponential map $\exp^\nabla : TM \rightarrow M \times M$ and is clearly an isomorphism of coalgebras over $C^\infty(M)$.

Even though, for a graded manifold $\mathcal{M}$ endowed with a torsion-free affine connection $\nabla$, we lack an exponential map $\exp^\nabla : T\mathcal{M} \rightarrow \mathcal{M} \times \mathcal{M}$, a PBW map can still be defined purely algebraically thanks to the iteration formula introduced in [5].
Lemma 4.2. Let $\mathcal{M}$ be a $\mathbb{Z}$-graded manifold and let $\nabla$ be a torsion-free affine connection on $\mathcal{M}$. The Poincaré-Birkhoff-Witt map inductively defined by the relations

$$\text{pbw}^\nabla(f) = f, \quad \forall f \in C^\infty(\mathcal{M});$$
$$\text{pbw}^\nabla(X) = X, \quad \forall X \in \mathfrak{X}(\mathcal{M});$$

and

$$\text{pbw}^\nabla(X_0 \odot \cdots \odot X_n) = \frac{1}{n+1} \sum_{k=0}^{n} (-1)^{|X_k|(|X_0|+\cdots+|X_{k-1}|)} \{ X_k \cdot \text{pbw}^\nabla(X_0 \odot \cdots \odot \hat{X}_k \odot \cdots \odot X_n)$$
$$- \text{pbw}^\nabla(\nabla X_k(X_0 \odot \cdots \odot \hat{X}_k \odot \cdots \odot X_n) \} ,$$

for all $n \in \mathbb{N}$ and $X_0, \ldots, X_n \in \mathfrak{X}(\mathcal{M})$, establishes an isomorphism

$$\text{pbw}^\nabla: \Gamma(S(T\mathcal{M})) \xrightarrow{\cong} D(\mathcal{M}).$$

of coalgebras over $C^\infty(\mathcal{M})$.

Now assume that $(\mathcal{M}, Q)$ is a dg-manifold. The homological vector field $Q$ induces a degree +1 coderivation of $D(\mathcal{M})$ defined by the Lie derivative:

$$L_Q(X_1 \cdots X_n) = \sum_{k=1}^{n} (-1)^{|X_1|+\cdots+|X_{k-1}|} X_1 \cdots X_{k-1}[Q, X_{k+1}]X_{k+1} \cdots X_n .$$

Now using the isomorphism of coalgebras $\text{pbw}^\nabla$ as in Eq. (7) to transfer $L_Q$ from $D(\mathcal{M})$ to $\Gamma(S(T\mathcal{M}))$, we obtain $\delta := (\text{pbw}^\nabla)^{-1} \circ L_Q \circ \text{pbw}^\nabla$, a degree 1 coderivation of $\Gamma(S(T\mathcal{M}))$. Finally, dualizing $\delta$, we obtain an operator

$$D : \Gamma(\hat{S}(T^\vee\mathcal{M})) \rightarrow \Gamma(\hat{S}(T^\vee\mathcal{M}))$$

as

$$\Gamma(\hat{S}(T^\vee\mathcal{M})) \cong \text{Hom}_{C^\infty(\mathcal{M})}(\Gamma(S(T\mathcal{M})), C^\infty(\mathcal{M})).$$

Theorem 4.3. Let $(\mathcal{M}, Q)$ be a dg-manifold and let $\nabla$ be a torsion-free affine connection on $\mathcal{M}$.

1. The operator $D$, dual to $(\text{pbw}^\nabla)^{-1} \circ L_Q \circ \text{pbw}^\nabla$, is a degree +1 derivation of the graded algebra $\Gamma(\hat{S}(T^\vee\mathcal{M}))$ satisfying $D^2 = 0$.

2. There exists a sequence $\{ R_k \}_{k \geq 2}$ of homomorphisms $R_k \in \text{Hom}(S^k T\mathcal{M}, T\mathcal{M}[-1])$, whose first term $R_2$ is precisely the Atiyah 1-cocycle $At_\mathcal{M}$, such that $D = L_Q + \sum_{k=2}^{\infty} R_k$, where $\hat{R}_k$ denotes the $C^\infty(\mathcal{M})$-linear operator on $\Gamma(\hat{S}(T^\vee\mathcal{M}))$ corresponding to $R_k$.

Finally we note that Theorem 4.1 is a consequence of Theorem 4.3.

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References


Department of Mathematics & Statistics, Smith College, 44 College Lane, Northampton, MA 01063
E-mail address: rmehta@smith.edu

Department of Mathematics, Pennsylvania State University, University Park, PA 16802
E-mail address: stienon@psu.edu
E-mail address: ping@math.psu.edu