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# The pilot-wave perspective on spin

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The alternative pilot-wave theory of quantum phenomena—associated especially with Louis de Broglie, David Bohm, and John Bell—reproduces the statistical predictions of ordinary quantum mechanics but without recourse to special *ad hoc* axioms pertaining to measurement. That (and how) it does so is relatively straightforward to understand in the case of position measurements and, more generally, measurements, whose outcome is ultimately registered by the position of a pointer. Despite a widespread belief to the contrary among physicists, the theory can also account successfully for phenomena involving spin. The main goal of this paper is to explain how the pilot-wave theory’s account of spin works. Along the way, we provide illuminating comparisons between the orthodox and pilot-wave accounts of spin and address some puzzles about how the pilot-wave theory relates to the important theorems of Kochen and Specker and Bell. © 2014 American Association of Physics Teachers.

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## I. INTRODUCTION

In an earlier paper, which readers are encouraged to examine first and which I refer to hereafter as “the earlier paper,” I attempted to give a physicists’ introduction to the alternative, de Broglie-Bohm pilot-wave theory of quantum phenomena.<sup>1</sup> As a so-called “hidden variable theory,” the pilot-wave theory adds something to the state descriptions of ordinary quantum mechanics: in addition to the usual wave function  $\Psi$  obeying the usual Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi, \quad (1)$$

one also has *definite positions* for each particle in the system. For example, for a system of  $N$  spinless nonrelativistic particles, the position  $\vec{X}_n$  of the  $n$ th particle will evolve according to

$$\frac{d\vec{X}_n(t)}{dt} = \frac{\vec{j}_n(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N, t)}{\rho(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N, t)} \Big|_{\vec{x}_k = \vec{X}_k(t) \forall k}, \quad (2)$$

where  $\vec{j}_n = (\hbar/2mi)(\Psi^* \vec{\nabla}_n \Psi - \Psi \vec{\nabla}_n \Psi^*)$  is (what in ordinary QM is termed) the probability current associated with particle  $n$  and  $\rho = \Psi^* \Psi$  is (what in ordinary QM is termed) the probability density.

As explained in the earlier paper, the pilot-wave theory involves the quantum equilibrium hypothesis (QEH), according to which the particle positions are assumed to be *random*, with initial ( $t = 0$ ) probability distribution

$$P[\vec{X}_1 = \vec{x}_1, \dots, \vec{X}_N = \vec{x}_N] = |\Psi(\vec{x}_1, \dots, \vec{x}_N, 0)|^2. \quad (3)$$

It is then a purely mathematical consequence of Eqs. (1) and (2) that the probability distribution will remain  $|\Psi|^2$ -distributed for all times. The family of possible particle trajectories thus “flows along with”  $\rho$ , a property that has been dubbed the “equivariance” of the  $|\Psi|^2$  probability distribution.<sup>2</sup>

Critics of the theory often argue that the addition of these definite particle positions is pointless (or “metaphysical”) because at the end of the day the theory’s empirical predictions match those of ordinary quantum mechanics, which latter predictions are of course made using the wave function

alone. What the critics fail to appreciate, however, is that adding the particle positions allows something to be *subtracted* elsewhere in the system. In particular, the dynamical laws sketched above—namely, Eqs. (1) and (2)—constitute the *entirety* of the dynamical postulates of the theory. No additional axioms or special exceptions to the usual rules—such as the collapse postulate of ordinary QM—need to be introduced in order to understand measurement or, more generally, the emergence of the familiar everyday classical world.

In the earlier paper, this point was developed in the context of some simple scattering experiments involving single particles. In such situations, the fact that the scattered particle is found, whole, at some particular place at the end of the experiment is explained in the simplest imaginable way: there really is a particle following a definite trajectory and hence possessing a perfectly definite position at all times. The detector finds it at a particular place (even when its wave function is spread out across several different places) because it is already at a particular place before interacting with the detector. And equivariance, in light of the QEH, guarantees that the probability for the particle to *be at* a certain place at the end of the experiment perfectly matches what ordinary QM would instead describe as the probability that the measurement intervention triggers a collapse that makes the particle suddenly materialize at that place. It is thus clear, in pattern, how the pilot-wave theory reproduces the statistical predictions of ordinary QM for experiments that end with the measurement of the position of a particle.

That the pilot-wave theory makes the same predictions as ordinary QM for other kinds of measurements as well can be understood by including the particles constituting the measuring device in the system under study. Consider for example a simple schematic treatment of the measurement of, say, the energy of a particle with initial wave function  $\psi_0(x)$ .<sup>3</sup> The measurement apparatus is imagined to have a pointer with spatial coordinate  $y$  and initial wave function  $\phi(y) = \phi_0(y)$ , where  $\phi_0$  is a narrow packet centered at the origin, corresponding to a “ready” state for the apparatus.

An interaction between the particle and the pointer is regarded as a “measurement of the energy of the particle” if the Schrödinger equation (for the particle+pointer system) produces the following sort of time-evolution in the case

that the particle is initially in an “energy eigenstate”  $\psi_0(x) = \psi_i(x)$  with eigenvalue  $E_i$ :

$$\psi_i(x)\phi_0(y) \rightarrow \psi_i(x)\phi_0(y - \lambda E_i). \quad (4)$$

That is, at the end of the experiment, the wave function for the pointer is now sharply peaked around some new point  $y = \lambda E_i$  proportional to the initial energy  $E_i$  of the particle. The pointer, in short, indicates the energy of the particle.

But the linearity of the Schrödinger equation then immediately implies that, in the general case in which the particle is initially in an arbitrary superposition of energy eigenstates, the evolution goes as follows:

$$\left(\sum_i c_i \psi_i(x)\right)\phi_0(y) \rightarrow \sum_i c_i \psi_i(x)\phi_0(y - \lambda E_i). \quad (5)$$

The resulting “Schrödinger cat” type state is of course problematic from the point of view of ordinary QM: instead of registering some one definite outcome for the experiment, the pointer itself ends up in an entangled superposition. Enter the collapse postulate to save us from the troubling implications of universal Schrödinger evolution! But this is no problem at all in the pilot-wave theory, according to which the (directly perceivable) final disposition of the pointer is not to be found in the wave function, but instead in the actual position  $Y$  of the pointer particle at the end of the experiment. It follows from the equivariance property discussed earlier that, assuming the initial particle positions are random in accordance with the QEH, there will be a probability  $|c_i|^2$  that the actual value of  $Y$  at the end of the experiment is (near)  $\lambda E_i$ . That is, the pointer will point to a definite place, with probabilities given by the usual quantum rules, but, here, by *rigorously following* the basic dynamical rules, rather than making up special exceptions when they produce embarrassing results.

One can thus understand how the pilot-wave theory manages to reproduce the statistical predictions of ordinary QM, at least in cases where one measures position or measures some other quantity but by means of the position of a pointer. But ... what about spin? Of all the phenomena surveyed in undergraduate quantum mechanics courses, those involving intrinsic spin and its measurement—which Pauli famously described as indicating a mysterious, non-classical Zweideutigkeit or “two-valuedness”<sup>4</sup>—seem perhaps the least amenable to the pilot-wave type of analysis just sketched. Indeed, it is sometimes reported that the pilot-wave theory simply cannot deal with spin phenomena in a plausible way.<sup>5</sup> This view was expressed very eloquently by one of the anonymous referees of the earlier paper:

“It has indeed been shown that Bohmian mechanics is equivalent to non-relativistic quantum mechanics with respect to its predictions concerning particle positions. However, the scheme encounters major problems: in spite of a half-century of effort by its adherents, it has not been possible to incorporate spin into the theory in a convincing way ...”

This type of view is perhaps motivated by the various “no hidden variables” theorems, such as that due to Kochen and Specker, which show quite clearly that it is mathematically impossible to assign pre-existing values to all relevant spin components of an ensemble of particles such that the quantum mechanical predictions are reproduced.<sup>6,7</sup> That is, it is

known to be impossible to do, for all of the non-commuting components of spin, what the pilot-wave theory, as sketched above, does for position.

And yet, in fact, it is entirely *false* that the pilot-wave theory cannot deal with spin. Indeed, the truth is that the pilot-wave theory deals with spin in an almost shockingly natural—certainly a shockingly trivial—way. The goal of the rest of this paper is to resolve this paradox and explain how.

The main ideas of the paper are not new. The pilot-wave theory was applied to spin already in 1955 by Bohm *et al.*,<sup>8</sup> and numerical simulations of the theory’s account of spin measurements were carried out by Dewdney *et al.* in the 1980s.<sup>9</sup> Our approach here, following Bell’s more elegant treatment in Ref. 10, instead aims at simplicity and accessibility. In particular, the proposed delta-function model of a Stern-Gerlach experiment (the one genuine novelty in the paper) allows one to solve the Schrödinger equation and determine the pilot-wave particle trajectories (without any recourse to numerical simulations) using the “plane-wave packet” methods developed in the earlier paper.

The delta-function model is developed in the following section; the associated pilot-wave particle trajectories are then displayed and analyzed in Sec. III. Section IV explains how the pilot-wave theory deals with cases of repeated spin measurements, while Secs. V and VI develop examples to illustrate the “contextuality” and “non-locality” exhibited by the theory. Some concluding remarks about the relationship between the pilot-wave and orthodox points of view are then made in Sec. VII.

## II. THE STERN-GERLACH EXPERIMENT

The Stern-Gerlach experiment, first and most famously performed in 1922, involves subjecting a beam of particles to an inhomogeneous magnetic field so the particles experience a force proportional to a certain component of their intrinsic spin angular momenta.<sup>11</sup> Since, empirically, the incident beam gets split into two or more discrete sub-beams (as opposed to a continuous distribution), the experiment is usually understood as demonstrating the quantization of spin angular momentum.

In a standard textbook treatment of the experiment, we assume a magnetic field

$$\vec{B} \approx \beta z \hat{z}, \quad (6)$$

as might be produced, for example, by the magnets indicated in Fig. 1.<sup>12</sup> For a neutral spin-1/2 particle (such as the silver atoms used in the original S-G experiment) the interaction Hamiltonian is

$$\hat{H} = -\mu \vec{\sigma} \cdot \vec{B} = -\mu \beta z \sigma_z, \quad (7)$$

where  $\mu$  is the magnitude of the particle’s magnetic moment and the components of  $\vec{\sigma}$  are, in the usual representation, the Pauli matrices. The eigenstates of this interaction Hamiltonian will obviously be (proportional to) spinors

$$\chi_{+z} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \chi_{-z} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ which satisfy}$$

$$\sigma_z \chi_{\pm z} = \pm \chi_{\pm z}. \quad (8)$$

Now let us consider the spatial degrees of freedom of the particle’s wave function, focusing in particular on the

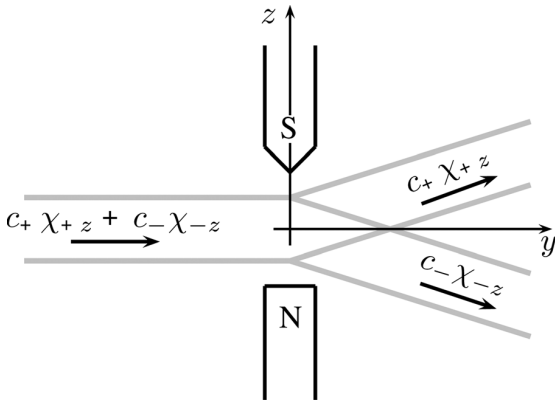


Fig. 1. Neutral spin-1/2 particles with initial spin state  $c_+\chi_{+z} + c_-\chi_{-z}$  move in the  $y$ -direction toward a Stern-Gerlach apparatus with magnetic field gradient along the  $z$ -direction. The field delivers an impulse to the  $\chi_{\pm z}$  components of the beam, so downstream there arise two spatially non-overlapping sub-beams with amplitudes  $c_+$  and  $c_-$ . In ordinary quantum mechanics, a measurement of the position of the particle downstream from the S-G device will find the particle in (or more accurately, cause it to materialize in) the upper/lower sub-beam with probability  $|c_{\pm}|^2$ .

dimension (here,  $z$ ) parallel to the magnetic field gradient. The beam is initially propagating, say, in the  $+y$ -direction, so let us assume that, in the  $z$ -direction, the wave function is initially constant (over the range where it is nonzero) and that the particles are in eigenstates of  $\sigma_z$ :

$$\Psi(z, 0) = A\chi_{\pm z}. \quad (9)$$

Now suppose the particles experience the magnetic field for a finite period of time  $T$  during which the interaction Hamiltonian above dominates all other terms. Then, the Schrödinger equation reads

$$i\hbar \frac{\partial \Psi}{\partial t} = -\mu\beta z \sigma_z \Psi, \quad (10)$$

which, for the assumed initial condition, has solution

$$\Psi(z, T) = A e^{\pm i\kappa z} \chi_{\pm z}, \quad (11)$$

where  $\hbar\kappa = \mu\beta T$  is the magnitude of the (upward or downward) momentum that the particle acquires as a result of traversing the field. For a beam of particles initially propagating in the  $y$ -direction, this impulse in the  $\pm z$ -direction deflects the beam either upward or downward depending on whether the particle was initially in the spin state  $\chi_{+z}$  or  $\chi_{-z}$ . Of course, in general, the initial spin might be an arbitrary linear combination of the two  $z$ -spin eigenstates:

$$\chi_0 = c_+\chi_{+z} + c_-\chi_{-z}. \quad (12)$$

But then, since Schrödinger's equation is linear, it follows that the S-G device will split the incoming wave function into two sub-beams—one with relative amplitude  $c_+$  that is deflected up, and one with relative amplitude  $c_-$  that is deflected down. The net effect is pictured in Fig. 1.

For purposes of discussing the pilot-wave account of these phenomena, it will be helpful to work with a simplified model that captures all the physics we've just reviewed but includes also an explicit treatment of the particle's other spatial degrees of freedom. This can be done, in the spirit of

Fig. 1, by imagining that the field  $\vec{B}$  is very strong, but is nonzero only in a vanishingly small region around  $y=0$ . That is, the interaction Hamiltonian of Eq. (7) is replaced with  $\hat{H} = -\mu b z \delta(y) \sigma_z$ , so that the total Hamiltonian governing the particle's motion in the  $yz$ -plane is

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} - \mu b z \delta(y) \sigma_z. \quad (13)$$

Now, the situation can be treated like an elementary 2D-scattering problem in wave mechanics.

To begin with, we imagine a "plane-wave packet" like those described in the earlier paper: the initial wave function should be a packet of length  $L$  along the  $y$ -direction and width  $w$  along the  $z$ -direction, with constant amplitude in the region where its amplitude doesn't vanish, propagating initially in the  $+y$ -direction with (reasonably sharply defined) energy  $E = \hbar^2 k^2 / 2m$ . Let us again start by assuming that the incident particle's spin degrees of freedom are described by one of the eigenspinors  $\chi_{\pm z}$ . Then, in the time period during which the incident packet is interacting with the field at  $y=0$ , the wave function can be taken to be (up to an overall time-dependent phase which we omit for simplicity)

$$\Psi(y, z) = \begin{cases} A e^{iky} \chi_{\pm z} + B z e^{-iky} \chi_{\pm z} & \text{for } y < 0, \\ C e^{ik'y} e^{\pm i\kappa z} \chi_{\pm z} & \text{for } y > 0, \end{cases} \quad (14)$$

in the appropriate regions of the  $yz$ -plane. (Note that the factor of  $z$ , in the term involving  $B$  that represents a reflected wave, is put in for future convenience.) This expression is a valid solution to the time-independent Schrödinger equation for  $y < 0$  and  $y > 0$  as long as

$$E = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 (k'^2 + \kappa^2)}{2m}. \quad (15)$$

In addition, the above expression should solve the Schrödinger equation at  $y=0$ . This will be the case if the wave function is continuous at  $y=0$ , requiring

$$A + Bz = C e^{\pm i\kappa z}, \quad (16)$$

and if the following condition on the  $y$ -derivatives of  $\psi$  (arrived at by integrating the time-independent Schrödinger equation from  $y = 0^-$  to  $y = 0^+$ ) is satisfied:

$$\left( \frac{\partial \psi}{\partial y} \Big|_{y=0^+} - \frac{\partial \psi}{\partial y} \Big|_{y=0^-} \right) = \mp \frac{2m\mu b}{\hbar^2} z \psi(0, z). \quad (17)$$

Plugging in our explicit expression for  $\psi(y, z)$  converts Eq. (17) into

$$ik' C e^{\pm i\kappa z} - ikA + ikBz = \mp \frac{2m\mu b}{\hbar^2} z C e^{\pm i\kappa z}. \quad (18)$$

These two conditions, Eqs. (16) and (18), cannot be satisfied exactly for all  $z$ . However, they can be approximately satisfied over the narrow range of  $z$  where the (width- $w$ ) wave packet has support. Thus, for example, Eq. (16) is valid to first order in  $\kappa w$  if  $A = C$  and

$$B = \pm i\kappa C. \quad (19)$$

Then Eq. (18) is satisfied to the same degree of approximation if  $k \approx k'$  (which implies, in light of Eq. (15), that  $\kappa \ll k$ ) and

$$\kappa = \frac{m\mu b}{\hbar^2 k}. \quad (20)$$

Notice that there are two distinct physical assumptions here. First, the impulse (of magnitude  $\hbar\kappa$ ) imparted to the particles by the Stern-Gerlach fields must be small compared to the initial momentum  $\hbar k$  of the particles, or equivalently  $\kappa/k = \mu b/2E$  should be small. (Notice that  $b$  has the same units as a magnetic field even though it is not one!) That is, the magnetic fields should be appropriately “gentle.” And second, the width  $w$  of the incident packet should be small compared to  $1/\kappa$ . Then Eq. (14) provides an acceptable description of the wave function in the vicinity of the Stern-Gerlach apparatus while the length- $L$  packet interacts with it. Note that, in these limits, the ( $z$ -dependent) amplitude of the reflected wave is (everywhere) small. Since it also plays no significant role in the physics to be discussed, we therefore drop it. (Those interested in the details should see Ref. 13 note.)

So far we have assumed that the spin degrees of freedom are given by one of the ( $z$ -direction) eigenspinors  $\chi_{\pm z}$ . But it is now straightforward to appeal to the linearity of the Schrödinger equation in order to write a solution appropriate for an arbitrary initial spinor like that of Eq. (12):

$$\Psi(y, z) = \begin{cases} A e^{iky} (c_+ \chi_{+z} + c_- \chi_{-z}) & y < 0, \\ A e^{ik'y} (e^{+ikz} c_+ \chi_{+z} + e^{-ikz} c_- \chi_{-z}) & y > 0, \end{cases} \quad (21)$$

where, of course, the expression for  $y < 0$  applies only within the range  $-w/2 < z < w/2$  where the plane-wave packet has support, and the expression for  $y > 0$  applies only within the triangular “overlap region” (see Fig. 1) where the two (separating) components of the wave coincide.

### III. PARTICLE TRAJECTORIES IN THE PILOT-WAVE THEORY

Following the methods introduced in the earlier paper we can use the structure of the wave function in (especially) the “overlap region” to analyze the motion of the particles in the pilot-wave theory and to demonstrate explicitly that the theory reproduces the usual quantum-mechanical predictions for the outcome of a Stern-Gerlach spin measurement. The first question that must be addressed, though, is this: what is the analog of Eq. (2) when the wave function

$$\Psi = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} \quad (22)$$

is a multi-component spinor? The answer is simply that the equation is exactly the same, except we must sum over the spin index in both the numerator and denominator, or equivalently, writing  $\Psi^\dagger = (\Psi_+^* \Psi_-^*)$ , we say

$$\frac{d\vec{X}(t)}{dt} = \frac{\hbar}{2mi} \frac{\Psi^\dagger (\nabla \Psi) - (\nabla \Psi^\dagger) \Psi}{\Psi^\dagger \Psi} \Big|_{\vec{x}=\vec{X}(t)}. \quad (23)$$

Consider, for example, the wave function  $\Psi(y, z)$  in the  $y < 0$  region from Eq. (21). We have that

$$\Psi(y, z) = A e^{iky} \begin{pmatrix} c_+ \\ c_- \end{pmatrix} \quad (24)$$

so that

$$\Psi^\dagger(y, z) = A^* e^{-iky} (c_+^* \ c_-^*). \quad (25)$$

Plugging into Eq. (23) then gives

$$\frac{d\vec{X}}{dt} = \frac{\hbar k}{m} \hat{y}. \quad (26)$$

No matter its exact location within the wave, the particle will simply drift in the positive  $y$ -direction with the group velocity of the incident packet.<sup>13</sup>

Eventually, of course, the particle will encounter the fields at  $y=0$  and pass over into the overlap region on the  $y > 0$  side. Here, because the  $+$  and  $-$  components of  $\Psi$  have different dependencies on  $z$ , we end up with

$$\frac{d\vec{X}}{dt} = \frac{\hbar k'}{m} \hat{y} + \frac{\hbar \kappa}{m} (|c_+|^2 - |c_-|^2) \hat{z}, \quad (27)$$

which can be understood as a weighted average of the group velocities associated with the two (now differently directed) components of the wave.

The family of possible particle trajectories thus looks something like that illustrated in Fig. 2 for the particular case that  $|c_+|$  is a little bigger than  $|c_-|$ . It should be clear, for example, that if  $c_+ = 1$  and  $c_- = 0$  then, while in the overlap region, the particles will just move with the group velocity of the  $+$  component of the wave function and all incoming particles (no matter their lateral position within the

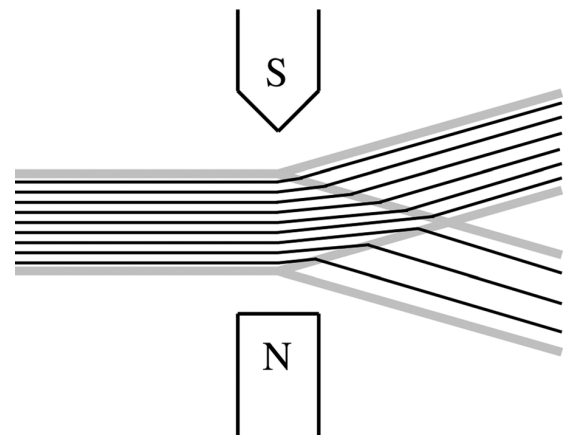


Fig. 2. Representative sample of possible particle trajectories through a  $z$ -oriented Stern-Gerlach apparatus for an initial wave function with spinor components  $c_+ = \sqrt{2/3}$  and  $c_- = \sqrt{1/3}$ . The particle will move along with the incident wave until crossing over into the overlap region where the  $+$  and  $-$  components of the wave are beginning to separate. In this region, the particle velocity is the weighted average, given in Eq. (27), of the group velocities associated with the two separating components of the wave. Depending on its initial lateral position within the incoming wave packet, the particle will either be shunted into the upper ( $+$ ) or lower ( $-$ ) fork. Subsequent detection of the particle in the upper/lower fork will result in its being identified as “spin up/down along  $z$ .”

wave) will be shunted into the upward sub-beam. Alternatively, if  $c_+ = 0$  and  $c_- = 1$  the particles in the overlap region will move with the group velocity of the—component and all incoming particles will be shunted downward.

The equivariance property mentioned in the introduction implies that if the position of the particle within the wave is random and  $|\Psi|^2$ -distributed at  $t = 0$ , it will remain  $|\Psi|^2$ -distributed for all subsequent times. This has the immediate consequence that the probability for the particle to be located (downstream of the Stern-Gerlach device) in the upper (lower) sub-beam, and hence counted as “spin-up” (“spin-down”) if its position there is detected, is  $|c_+|^2$  ( $|c_-|^2$ ). It is thus clear that the pilot-wave theory reproduces the usual quantum-mechanical predictions for the statistics of such spin measurements.

One of the nice things about the way we have set things up, though, is that this claim can be demonstrated explicitly by considering the “critical trajectory,”<sup>14</sup> which divides the trajectories emerging into the upper and lower beams. By definition, the critical trajectory arrives just at the vertex on the right of the triangular overlap region behind the magnets, as shown in Fig. 3. The slope of the critical trajectory is given by the ratio of the  $z$ - and  $y$ -components of the velocity in the overlap region, Eq. (27):

$$\text{slope} = \frac{|c_+|^2 - |c_-|^2 \kappa}{|c_+|^2 + |c_-|^2 k'} \quad (28)$$

On the other hand, as explained in the figure caption, the critical trajectory traverses a distance  $\Delta y = wk'/(2\kappa)$  in the  $y$ -direction and a distance  $\Delta z = wP_+ - w/2$  in the  $z$ -direction, where  $P_+$  is the fraction of the lateral width  $w$  of the beam with the property that, should the particle begin there, it will

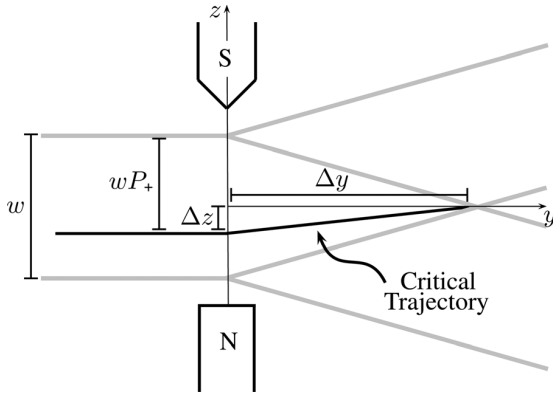


Fig. 3. Particles that begin above the critical trajectory will be shunted into the upper (“spin-up”) wave packet downstream of the Stern-Gerlach magnets, while those that begin below the critical trajectory will be shunted into the lower (“spin-down”) packet. For initial particle positions that are (in accordance with the QEH) uniformly distributed across the lateral extent of the incident packet, the probability for a particle to emerge with the “spin-up” packet will be the fraction of the lateral width ( $w$ ) of the packet that is above the critical trajectory, i.e., the distance from the critical trajectory to the top of the incident packet is  $wP_+$ . The vertical displacement  $\Delta z$  of the critical trajectory as it traverses the triangular overlap region is then  $wP_+ - w/2$ . By considering the right triangle that is the lower half of the overlap region (and whose hypotenuse is one edge of the “spin-up” packet), one can see that the horizontal extent  $\Delta y$  of the overlap region is  $(w/2)(\kappa/k')$ : the slope  $(w/2)/\Delta y$  of the hypotenuse should match the ratio  $\kappa/k'$  of the  $z$ - and  $y$ -components of the group velocity associated with the “spin-up” part of the wave.

end up going into the upper sub-beam. In short,  $P_+$  represents the probability that the S-G spin measurement will yield the outcome “spin-up.” Setting

$$\text{slope} = \frac{\Delta z}{\Delta y} \quad (29)$$

and solving for  $P_+$  one indeed recovers that

$$P_+ = |c_+|^2, \quad (30)$$

confirming explicitly that, indeed, the pilot-wave dynamics for the wave and particle are compatible in the way expressed by the formal equivariance property: the particle trajectories “bend just the right amount” to yield a final probability distribution for the particle position that is consistent with the usual QM predictions and, more importantly, what is seen in experiments.

#### IV. ADDITIONAL MEASUREMENTS

So far, we have concentrated on the measurement of the  $z$ -component of the spin using a Stern-Gerlach device whose magnetic field gradient is along the  $z$ -direction. By simply rotating the device, however, we can also measure the component of the spin along (say) an arbitrary direction  $\hat{n}$  in the  $xz$ -plane:

$$\hat{n} = \cos \theta \hat{z} + \sin \theta \hat{x}. \quad (31)$$

The whole measurement procedure can be analyzed in exactly the same way we’ve done above, *mutatis mutandis*: the eigenspinors  $\chi_{\pm n}$  will now be the eigenvectors of

$$\sigma_n = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}, \quad (32)$$

an arbitrary initial wave-packet (proportional to initial spinor  $\chi_0$ ) can be written as a linear combination of these eigenspinors

$$\chi_0 = c_{+n}\chi_{+n} + c_{-n}\chi_{-n}, \quad (33)$$

and the whole analysis of Secs. II and III goes through, with  $z$  being everywhere replaced by  $n$ .<sup>15</sup> Thus, it should be clear that the pilot wave theory reproduces the usual quantum statistical predictions for spin measurements, whether it is the  $z$ - or some other component of spin that is to be measured.

Perhaps the theory will fail, however, when we consider the possibility of *subsequent* and/or *repeated* measurements? The pilot-wave theory is, after all, a deterministic hidden variable theory: for a given experimental setup, the outcome of a spin measurement is determined by the initial state of the “particle” (meaning here the particle + wave combination). Standard textbook discussions might perhaps suggest that there would be a problem.

For example, suppose that particles emerging from the “spin-up” port of a  $z$ -oriented Stern-Gerlach device ( $SG_z$ ) are subsequently sent through an  $SG_x$  device; those particles emerging as “spin-up” along  $x$  are then subjected to a further measurement of their spin’s  $z$ -component (see Fig. 4). The result, of course, is that fully half of the particles entering it will emerge from the final  $SG_z$  device as “spin-down.” The usual interpretation is that spin-along- $z$  and spin-along- $x$  are

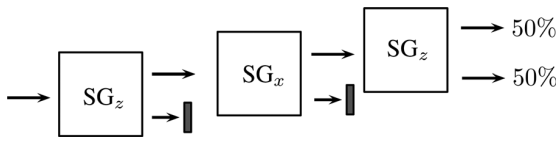


Fig. 4. Schematic representation of a series of Stern-Gerlach spin measurements, in which the devices are replaced by “black boxes” with two output ports, one for “spin-up” along the direction  $n$  indicated by the box’s label “SG $_n$ ,” and one for “spin-down.” Here, particles that emerge as “spin-up” along the  $z$ -direction and then “spin-up” along the  $x$ -direction are subjected to a further measurement of  $z$ -spin. Half of the particles exiting this final SG $_z$  device are “spin-up,” with the other half being “spin-down.” As discussed in the text, both ordinary QM and the pilot-wave theory are able to account for the observed statistics, because in both theories the state of a particle is in general *affected* by subjecting it to a spin measurement. In particular, the state of a particle exiting the first SG $_z$  device is not the same as the state of that same particle when it exits the intermediate SG $_x$  device.

incompatible properties, corresponding to non-commuting quantum mechanical operators, so the determination of a definite value for spin-along- $x$  completely erases the previously-definite value for spin-along- $z$ . Surely a non-quantum, deterministic hidden-variable theory could not reproduce this paradigmatically quantum result?

But a little thought reveals that, yes, the pilot-wave theory reproduces this prediction quite easily. The crucial point is that, although the particles *entering* the SG $_x$  device were being guided by waves proportional to  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ , the wave guiding those particles that successfully *emerge* from the “spin-up” port of the SG $_x$  device is proportional to  $\chi_{+x} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ . The  $\chi_{+x}$  and  $\chi_{-x}$  components of the wave get separated by the SG $_x$  device, whereas the particle goes one way or the other. And, by definition, if the particle ends up in the “spin-up-along- $x$ ” beam, downstream of the SG $_x$  device, the component of the wave now surrounding and guiding it is the “spin-up-along- $x$ ” part. So it is as if the particle’s quantum state has “collapsed,” as per the usual quantum mechanical rules, even though, in fact, no collapse—no violation of the usual Schrödinger evolution of the wave—has occurred. The other, “spin-down-along- $x$ ,” part of the wave is still present—it’s just “over there,” far away from the actual location of the particle, and hence dynamically irrelevant to the present or future motion of the particle.<sup>16</sup> This separation of the different spin components of the wave packet of course happens in ordinary quantum mechanics as well. But in ordinary QM there is nothing like the *actual* position of the particle, in addition to the wave function, to warrant talk of the particle *actually* being up there (i.e., talk of the spin measurement having actually had the outcome “spin-up”), and so additional dynamical postulates are required. This is an illustration of the point made in the introduction, that although the pilot-wave theory adds something to the physical state descriptions, it is not really more (let alone pointlessly more) complicated than ordinary QM because this addition allows for the subtraction of the otherwise rather dubious measurement axioms.

What is actually shown by this kind of example is not that one cannot explain the observed statistics of spin measurements with a non-quantum, deterministic, hidden-variable theory, but rather only that the physical state of a particle (known, say, to be “spin-up-along- $z$ ”) must be *affected* by a measurement of (for example) its spin-along- $x$ . Ordinary

quantum theory of course exhibits this behavior: in quantum mechanics, the state of the particle, after a measurement, is postulated to “collapse” to the eigenstate of the appropriate operator corresponding to the actually-observed outcome. (It is worth noting here that the “actually-observed outcome” is yet *another* additional postulate of the theory—for Bohr, for example, this was realized in some classical macroscopic pointer somewhere, whose *physical* relationship to the particle in question is, at best, obscure.) The point here is that the pilot-wave theory also exhibits this behavior: the physical state of (in particular) the wave guiding the particle is different before, and after, the intermediate SG $_x$  device. But in the pilot-wave theory, this difference—the physical influence of the measurement on the properties of the system in question—is a natural and straightforward consequence of the *usual* dynamical laws.

## V. CONTEXTUALITY

In the previous section I stressed that the pilot-wave theory is not the “naive” sort of hidden-variable theory that is sometimes discussed (and, when discussed, always refuted with great pomp!) in textbooks. In these “naive” theories, measurements simply passively reveal some pre-existing value of the property being measured, without affecting the state of the particle. In the pilot-wave theory, by contrast, there is a natural mechanism (which, quite literally, is nothing but Eqs. (1) and (2) that define the theory’s dynamics) whereby the physical measurement intervention affects the state of the measured system. There is thus a sense in which, for the pilot-wave theory just as for ordinary QM, the measurement cannot be thought of as passively revealing some pre-existing quantity, but should instead be thought of as an active intervention which brings about the new, final state of the particle corresponding to the measurement outcome.

But there is an even deeper sense in which, for the pilot-wave theory, the measurement cannot be thought of as passively revealing a pre-existing value. Let us discuss this in terms of a concrete example. Recall the SG $_z$  device sketched in Fig. 1: the magnets produce a magnetic field near the origin that can be approximated by the expression in Eq. (6). The field is such that a *classical* particle, with magnetic dipole moment in the  $+z$  direction, will feel a force in the  $+z$  direction and hence be deflected up upon passing through the SG device. Likewise, a classical particle with magnetic dipole moment in the  $-z$  direction will feel a force in the  $-z$  direction and will hence be deflected down. This is, in essence, the basis for saying that, when a particle emerges from the SG device having been deflected up, it is “spin-up along  $z$ ,” etc.

But other ways of measuring the  $z$ -component of a particle’s spin can also be contemplated. For example, imagine a device—let us call it here an SG $_z'$  device—like that indicated in Fig. 5. The structure is identical to the original SG $_z$  device except that the polarity of the magnets has been reversed and hence Eq. (6) is replaced with

$$\vec{B}' \approx -\beta z \hat{z}. \quad (34)$$

This reversal of the magnetic field gradient reverses the effect on particles passing through the device: now, a classical particle with a magnetic dipole moment in the  $+z$  direction will feel a force in the  $-z$  direction and hence be deflected *down* upon passing through the device, while a

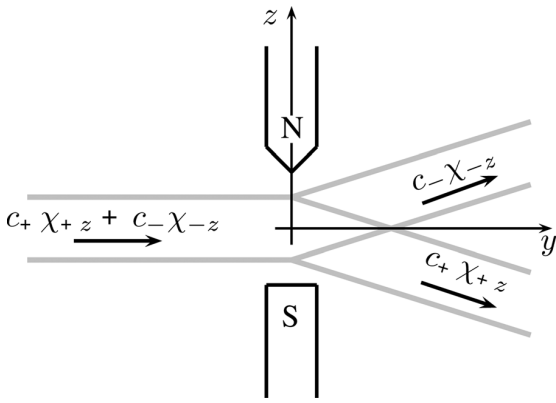


Fig. 5. An alternative way to “measure the  $z$ -component of the spin of a particle” is to pass the particle through the  $SG'_z$  device shown here. This is the same as an  $SG_z$  device, but with the polarity of the magnets—and hence the calibration of the device—reversed: particles that deflect *up* are now counted as “spin-down along  $z$ ,” while particles that deflect *down* are now counted as “spin-up along  $z$ .”

classical particle with magnetic dipole moment in the  $-z$  direction will feel a force in the  $+z$  direction and will hence be deflected *up*. The  $SG'_z$  device is a perfectly valid one for measuring the  $z$ -component of the spin of a particle; it merely has a different “calibration” (one might say) than the original device. Whereas, with the original device, particles deflected *up* are declared to be “spin-up along  $z$ ,” particles deflected *up* by the modified device are instead declared to be “spin-down along  $z$ ,” and vice versa. And indeed, a full quantum mechanical analysis, parallel to that undertaken for the original  $SG_z$  device in Sec. III, leads to the conclusion that an incident wave packet proportional to  $c_+\chi_{+z} + c_-\chi_{-z}$  will be split, upon passage through the  $SG'_z$  device, into two sub-beams, one proportional to  $c_+\chi_{+z}$  that has been deflected down and the other proportional to  $c_-\chi_{-z}$  that has been deflected up.

The significance of these two distinct pieces of experimental apparatus—both equally qualified to “measure the  $z$ -component of the spin of a particle”—becomes clear when we consider what happens when, for example, a particle with initial wave function proportional to  $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  is incident on the two devices. For this case, the particle velocity in the overlap region is proportional to  $\hat{y}$ , i.e., the particle trajectories simply continue in a straight line all the way through the overlap region until they finally emerge into one of the now spatially separated sub-beams corresponding to their initial lateral positions within the incident packet. That part is identical whether the original  $SG_z$  or the alternative  $SG'_z$  device is involved. But there is a crucial difference between the two devices: if (say) the particle happens to start in the upper half of the incident packet, it is destined to eventually find itself in the upward-deflected sub-beam and hence be counted as “spin-up along  $z$ ”—if the measurement is carried out using the original  $SG_z$  device. But if instead the measurement is carried out using the alternative  $SG'_z$  device, the exact same “particle”—that is, the exact same wave function and the exact same particle location in the upper half of the wave packet—is instead destined to eventually find itself in the upward-deflected sub-beam and hence be counted as “spin-down along  $z$ .”<sup>17</sup>

In short, for the (deterministic, hidden variable) pilot-wave theory, the outcome of “measuring the  $z$ -component of the

spin of the particle” is *not* simply a function of the initial state of the “particle” (i.e., the particle + wave complex). The exact same initial state can yield either of the two possible measurement outcomes, depending on which of two possible experimental devices is chosen for performing the experiment.

This is a concrete illustration of the fact that is usually put as follows: for the pilot-wave theory, spin is “contextual.” That is, the outcome of a measurement of a certain component of a particle’s spin depends, in the pilot-wave theory, not just on the initial state of the particle but also on the overall experimental context, that is, on the particular “way” that the measurement is implemented.

This “contextuality” should be contrasted to the “non-contextual” type of hidden variable theory in which for each (Hermitian) quantum mechanical operator (corresponding to some “observable” property), each particle in an ensemble possesses a definite value that is simply revealed by the appropriate kind of experiment, independent of which specific experimental implementation is used to measure the observable in question. In particular, a non-contextual hidden-variable theory would (by definition) assign a definite value to the  $z$ -component of each particle’s spin, independent of whether the spin is measured using an  $SG_z$  or an  $SG'_z$  device. More generally, non-contextuality can be understood as the requirement that each observable should possess a particular definite value, that will be revealed by a measurement of that observable, *no matter what compatible observables are perhaps being measured simultaneously*. (The following section provides a concrete example of this more general sort of contextuality.)

That the “no hidden variables” theorems of von Neumann, Jauch-Piron, and Gleason tacitly assume non-contextuality (or an even stronger and less innocent requirement) was first clearly pointed out in Bell’s landmark 1966 paper, “On the Problem of Hidden Variables in Quantum Mechanics.”<sup>18</sup> (This paper, incidentally, was written in 1964, prior to Bell’s more famous 1964 paper proving what is now called “Bell’s Theorem,” but remained unpublished until 1966 due to an editorial accident.) The somewhat more famous Kochen-Specker “no hidden variables” theorem, which appeared the year after Bell’s paper, also assumes non-contextuality, as discussed in the beautiful review paper by Mermin.<sup>6,7</sup>

As Bell explains, these proofs all rely on relating “in a nontrivial way the results of experiments that cannot be performed simultaneously.” (For example, they assume that the results of an  $SG_z$ -based and an  $SG'_z$ -based measurement should be the same.) Bell elaborates:

“It was tacitly assumed that measurement of an observable must yield the same value independently of what other measurements may be made simultaneously. Thus as well as [some observable with corresponding QM operator  $\hat{A}$ ] say, one might measure *either*  $[\hat{B}]$  *or*  $[\hat{C}]$ , where  $[\hat{B}]$  and  $[\hat{C}]$  commute with  $\hat{A}$ , but not necessarily with each other]. These different possibilities require different experimental arrangements; there is *no a priori* reason to believe that the results for  $[\hat{A}]$  should be the same. The result of an observation may reasonably depend not only on the state of the system (including hidden variables) but also on the complete disposition of the apparatus.”<sup>18</sup>

Bell then references Bohr’s insistence on remembering “the impossibility of any sharp distinction between the behavior of atomic objects and the interaction with the



measuring instruments which serve to define the conditions under which the phenomena appear.”<sup>19</sup> Abner Shimony has aptly described Bell’s invocation of Bohr (the arch-opponent of hidden variables) in Bell’s *defense* of hidden variables (against the theorems supposedly, but not actually, showing them to be impossible) as a “judo-like manoeuvre.”<sup>20</sup>

For our purposes, the upshot of all this is the following. In the literature on hidden variables one often finds the implication that the requirement of non-contextuality is quite reasonable in the sense that contextuality would supposedly involve putting in “by hand” an obviously implausible *ad hoc* kind of apparatus-dependence. But as the simple example of the contextuality of the pilot-wave theory (involving the distinct outcomes for  $SG_z$ - and  $SG'_z$ -based measurements of  $z$ -spin) makes clear, this is not the case at all. The “context-dependence” arises in a perfectly straightforward and natural way, again as a direct consequence of the fundamental dynamical postulates of the theory given by Eqs. (1) and (2).

## VI. NON-LOCALITY

Bell’s 1966 paper closes by noting that Bohm’s 1952 pilot-wave theory eludes the various “no hidden variables” theorems by means of its contextuality—but that the context-dependence implies a non-local action at a distance in situations involving entangled but spatially-separated particles: “in this theory an explicit causal mechanism exists whereby the disposition of one piece of apparatus affects the results obtained with a distant piece.”<sup>18</sup> Let us explore this type of situation, in which the pilot-wave theory’s contextuality implies a non-local dependence on a remote context.

Consider then a pair of spin-1/2 particles, propagating in opposite directions along (and already widely separated along) the  $y$ -axis. The initial spatial part of the 2-particle wave function can be taken to be

$$\psi(\vec{x}_1, \vec{x}_2) = \Phi_k^{(0,d,0)}(\vec{x}_1)\Phi_{-k}^{(0,-d,0)}(\vec{x}_2), \quad (35)$$

where  $\Phi_k^{(0,d,0)}(\vec{x})$  represents a plane-wave packet<sup>1</sup> centered at  $\vec{x} = (0, d, 0)$  with central wave vector  $\vec{k} = k\hat{y}$ . Consider first the case in which the spin part of the initial state also factorizes, so that the total state can be written

$$\Psi(\vec{x}_1, \vec{x}_2) = \left(\Phi_k^{(0,d,0)}(\vec{x}_1)\chi_0^1\right)\left(\Phi_{-k}^{(0,-d,0)}(\vec{x}_2)\chi_0^2\right). \quad (36)$$

We may introduce the so-called “conditional wave function” (CWF) for particle 1 as follows:

$$\Psi_1(\vec{x}) = \Psi(\vec{x}, \vec{x}_2)|_{\vec{x}_2=\vec{X}_2}, \quad (37)$$

where  $\vec{X}_2$  is, of course, the actual position of particle 2. The CWF for particle 2 is defined in a parallel way.

These CWFs are convenient because the dynamical law for the particle positions, Eq. (2), can be reformulated to give, for each particle, an expression for the particle velocity in terms of its own CWF:

$$\frac{d\vec{X}_n}{dt} = \frac{\hbar}{2mi} \frac{\Psi_n^\dagger(\vec{\nabla}\Psi_n) - (\vec{\nabla}\Psi_n^\dagger)\Psi_n}{\Psi_n^\dagger\Psi_n} \Big|_{\vec{x}=\vec{X}_n}. \quad (38)$$

The CWFs can thus be thought of as “one-particle wave functions,” each of which guides its corresponding particle,

in exactly the way we have discussed previously for the pilot-wave theory of single particles.

The CWFs have, however, several perhaps-unexpected properties that should be acknowledged. First, it should be appreciated that, in general, the CWFs do not obey some simple one-particle Schrödinger equation.<sup>22</sup> Evaluating Eq. (36) at  $\vec{x}_2 = \vec{X}_2$  gives an overall (time-dependent) multiplicative factor, which means that (even in the case of non-entangled particles) the overall phase and normalization of the CWF can change erratically. This, however, is irrelevant to the motion of the associated particle because any multiplicative factor simply cancels out in Eq. (38). In addition, the CWF of *each* particle carries spin indices for *both* particles. In the case where the two-particle wave function is factorizable, as in Eq. (36), the particle-2 spinor ( $\chi_0^2$ ) is irrelevant to the motion of particle 1, and vice versa. The “extra” spinor thus acts just like the overall multiplicative constant. It is thus clear that when the two-particle wave function factorizes—when the two particles are not entangled—the motions of the two particles will be fully independent: if the particles subsequently encounter Stern-Gerlach devices, each will (independently and separately) act exactly as described in the previous sections.

If, however, the spins of the two particles are initially entangled, as for example in

$$\Psi(\vec{x}_1, \vec{x}_2) = \psi(\vec{x}_1, \vec{x}_2) \frac{1}{\sqrt{2}} \left( \chi_{+z}^1 \chi_{-z}^2 - \chi_{-z}^1 \chi_{+z}^2 \right), \quad (39)$$

then the non-locality appears. Suppose for example that both particles are to be subjected to  $SG_z$ -based measurements of their  $z$ -spins. And suppose that particle 1 encounters its  $SG_z$  device first. A simple calculation shows that, in the overlap region behind the magnets, the velocity of particle 1 will be proportional to  $\hat{y}$  (i.e., it will simply continue in a straight horizontal line through the overlap region). It will then exit the overlap region into one or the other of the two downstream sub-beams, depending on its initial  $z$ -coordinate: if the particle happens to have started in the upper half of the packet it will end up deflecting up and being counted as “spin-up,” whereas if it happens to have started in the lower half of the packet it will end up deflecting down and being counted as “spin-down.” Supposing, for notational simplicity, that the two now spatially-separated sub-beams are bent (say, by additional appropriate  $SG$  type magnets) so that they again propagate in the  $+y$ -direction but with displacements  $\pm\Delta$  in the  $z$ -direction,<sup>21</sup> the two-particle wave function after particle 1 has passed through its  $SG_z$  device will take the form

$$\Psi = \frac{1}{\sqrt{2}} \left[ \Phi_k^{(0,d',\Delta)}(\vec{x}_1) \Phi_{-k}^{(0,-d',0)}(\vec{x}_2) \chi_{+z}^1 \chi_{-z}^2 - \Phi_k^{(0,d',-\Delta)}(\vec{x}_1) \Phi_{-k}^{(0,-d',0)}(\vec{x}_2) \chi_{-z}^1 \chi_{+z}^2 \right]. \quad (40)$$

That is, the overall wave function is a superposition of two terms: one, proportional to  $\chi_{+z}^1 \chi_{-z}^2$ , in which the particle 1 wave packet has been displaced a distance  $\Delta$  in the *positive*  $z$ -direction, and the other, proportional to  $\chi_{-z}^1 \chi_{+z}^2$ , in which the particle 1 wave packet has been displaced a distance  $\Delta$  in the *negative*  $z$ -direction.

The actual location  $\vec{X}_1$  of particle 1 will be either near  $z = \Delta$  or near  $z = -\Delta$  (again, depending on its random initial position). It is thus meaningful already at this stage to speak of the actual outcome of the  $SG_z$ -based measurement of the  $z$ -spin of particle 1. The crucial point is now that the CWF

for particle 2—the thing that will determine how particle 2 behaves when it subsequently encounters its  $SG_z$  device—depends on where particle 1 ended up. If particle 1 went up (that is, if  $\vec{X}_1$  is now in the support of  $\Phi_k^{(0,d,\Delta)}$ ) then the CWF of particle 2 is (proportional to)

$$\Psi_2(\vec{x}) = \Phi_{-k}^{(0,-d',0)}(\vec{x}) \chi_{-z} \quad (41)$$

and its subsequent interaction with an  $SG_z$  device will, independent of the exact position  $\vec{X}_2$ , result in its being deflected *down*. Whereas if instead particle 1 ended up going down (that is, if  $\vec{X}_2$  is now in the support of  $\Phi_k^{(0,d',-\Delta)}$ ) then the CWF of particle 2 is instead (proportional to)

$$\Psi_2(\vec{x}) = \Phi_{-k}^{(0,-d',0)}(\vec{x}) \chi_{+z} \quad (42)$$

and its subsequent interaction with an  $SG_z$  device will, independent of the exact position  $\vec{X}_2$ , instead result in its being deflected *up*. That is, the CWF for particle 2 collapses, as a result of the measurement on particle 1, to a state of definite  $z$ -spin. This happens, however, despite the fact that the two-particle wave function obeys the (unitary) Schrödinger equation without exception! And the usual quantum mechanical statistics are reproduced: the two possible joint outcomes (particle 1 is spin-up and particle 2 is spin-down, or particle 1 is spin-down and particle 2 is spin-up) each occur with 50% probability. Of course, the pilot-wave theory being after all deterministic, these probabilities are quite reducible. In particular, which of the two joint outcomes occurs depends on the random initial position of particle 1 (specifically, its initial  $z$ -coordinate).

In order to make the non-local character of the theory absolutely clear, it is helpful to now consider an alternative scenario in which particle 1 is not subjected to any measurement. For example, perhaps Alice, who is stationed there next to it, decides (at the last possible second before particle 1 arrives) to yank the  $SG_z$  device out of the way. Then, the two-particle wave function remains in a state described by Eq. (39) until particle 2 arrives at its  $SG_z$  device. But then particle 2 will behave in just exactly the way we previously described for particle 1 (when it was the first to encounter an  $SG_z$  device): its velocity in the overlap region behind the magnets will be proportional to  $\hat{y}$  and the outcome of the spin measurement will depend on the initial position (in particular, the  $z$ -coordinate) of particle 2.

The non-locality is thus clear: even for the same initial state—two-particle wave function given by Eq. (39) and, say, both particle positions,  $\vec{X}_1$  and  $\vec{X}_2$ , in the upper halves of their respective packets—the outcome of the measurement on particle 2 depends on what Alice chooses to do in the vicinity of particle 1. If Alice subjects particle 1 to an  $SG_z$ -based measurement, that measurement will have outcome “spin-up,” the CWF of particle 2 will collapse to a state proportional to  $\chi_{-z}$ , and the subsequent measurement of particle 2’s  $z$ -spin will have outcome “spin-down.” On the other hand, if Alice removes the  $SG_z$  device, the measurement of particle 2’s  $z$ -spin will instead have the outcome spin-up.<sup>23</sup> It is clear that the non-locality is a form of contextuality in which the part of the experimental “context” affecting the realized outcome of the experiment is *remote* (Fig. 6).

Although we have focused here on the concrete example in which the  $z$ -components of the spins of both particles are (perhaps) measured, it should now be clear that, and how, the

pilot-wave theory accounts for the empirically observed correlations in the more general sort of case in which arbitrary components of spin are measured. One of the particles will encounter its measuring device *first*; the outcome of this first measurement will be determined by the random initial position of this measured particle within its wave packet; the completion of this first measurement induces a collapse in the distant particle’s CWF; and this in turn determines the statistics for a subsequent measurement on the distant particle. (Showing that, in addition, the expected statistical results are reproduced even if one or both of the measuring devices are replaced with alternative  $SG'_n$  devices is left as an exercise for the reader.)

That so much hangs on which measurement happens *first* (or more mathematically, that the CWF of each particle is the two-particle wave function evaluated at the actual *current* position of the other particle, no matter how distant) makes it clear that the non-locality of the pilot-wave theory will be difficult to reconcile with relativity. This is, however, in principle no different from the non-locality implied by the collapse postulate of ordinary QM. (Indeed, the collapse of the pilot-wave theory’s CWFs can and should be understood as a mathematically precise *derivation* of the usual textbook approach to measurement, in a theory where imprecise notions like “measurement” play no fundamental role.) And of course, the “grossly non-local structure” of the pilot-wave theory and ordinary QM turns out to be “characteristic... of any such theory which reproduces exactly the quantum mechanical predictions”—as shown in Bell’s other landmark paper, of 1964.<sup>24</sup>

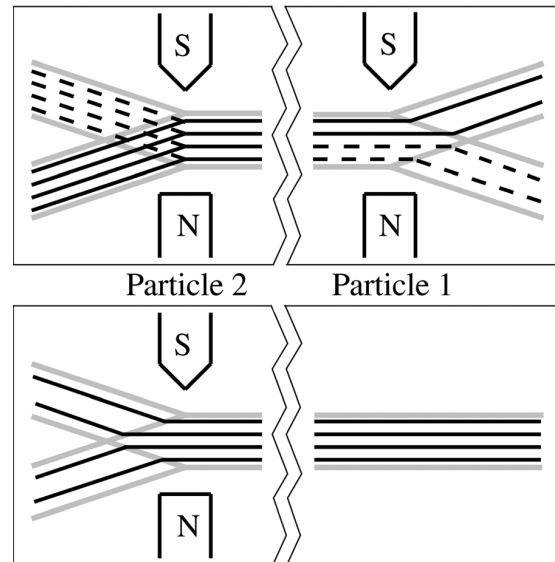


Fig. 6. Illustration of the two scenarios discussed in the text. In the top frame, particle 1 (on the right) encounters its  $SG_z$  device first; the particle is found to be spin-up (solid trajectories) or spin-down (dashed trajectories) depending on the initial  $z$ -coordinate of the particle. If particle 1 goes up, the collapse suffered by the CWF of particle 2 (on the left) causes it to go down (solid trajectories) regardless of its initial  $z$ -coordinate. On the other hand, if particle 1 goes down, the collapse causes particle 2 instead to go up (dashed trajectories) regardless of its initial  $z$ -coordinate. In the lower frame, particle 1 is not subjected to any measurement. The result of measuring the  $z$ -spin of particle 2 is then determined by the initial  $z$ -coordinate of particle 2. The difference between the two scenarios exemplifies the contextuality of the pilot-wave theory (since the result of measuring the spin of particle 2 depends not only on the initial state but on whether or not the spin of particle 1 is measured jointly) and also its non-locality (since the choice of whether or not to measure particle 1 could be made at space-like relativistic separation from the measurement of particle 2).

## VII. CONCLUSIONS

Contrary to an apparently widespread belief, the pilot-wave theory has no trouble accounting for phenomena involving spin. It does so, actually, in the most straightforward imaginable way: the wave function (a scalar-valued field on the configuration space in the case of spinless particles) is replaced with the appropriate spinor-valued function, evolving according to the appropriate generalization of the Schrödinger equation, just exactly as in ordinary QM. The dynamical law for the motion of the particles, Eq. (2), doesn't change at all—one need simply recall, again from ordinary QM, that for particles with spin the definitions of both  $\vec{j}_n$  and  $\rho$  involve summing over the spin indices. Thus understood, the pilot-wave theory reproduces the usual quantum statistical predictions for spin measurements (including sequences of measurements performed on a single particle, and joint measurements performed on pairs of perhaps-entangled particles) but without any additional postulates or *ad hoc* exceptions to the usual dynamical laws. And it does this, incidentally, in a way that could have been anticipated: spin measurements *too* (like position measurements and measurements whose results are registered by the position of a pointer) ultimately come down to the position of a particle (downstream from an appropriate Stern-Gerlach device).

The theory exhibits “contextuality,” as illustrated in the two examples discussed: the result of a measurement depends not only on the state of the particle prior to measurement but also on the measurement's specific experimental implementation. In particular, the outcome of a measurement of the  $z$ -component of the spin of a particle can depend on whether the spin is measured using an  $SG_z$  or an  $SG'_z$  device, and/or can depend on which (commuting) observable is also being jointly measured. Considered in the abstract, the idea of a contextual hidden variable theory perhaps sounds contrived and implausible. Such, at least, has apparently been the thinking behind the suggestions that the various “no hidden variables” theorems (due to von Neumann, Kochen-Specker, and so on) rule out the possibility of (or even an important class of) hidden variable theories. But the example of the pilot-wave theory shows, quite simply and conclusively, that they don't. And furthermore, like it or lump it, the theory (backed up by Bell's theorem) shows that there is nothing the least bit contrived or intolerable in the sort of contextuality that is required to eliminate the “unprofessionally vague and ambiguous” aspects of ordinary QM in favor of something clear and mathematically precise.<sup>28</sup>

The idea that there should be something contrived or intolerable about contextuality undoubtedly arises from the idea that, if a property really exists, measurement of it should—by definition—simply reveal its value. It would be hard, actually, to disagree with this sentiment. The key question, though, is precisely whether any such property exists. As has been discussed in illuminating detail in Ref. 25, the real lesson to be taken away from examining the pilot-wave perspective on spin is that so-called “contextual properties” (like the individual spin components in the pilot-wave theory) are not properties at all.<sup>26</sup> They simply do not exist and there is nothing mysterious about this at all, just as there is nothing mysterious in the fact that the eventual flavor of a loaf of bread (which depends not just on the ingredients but also on how it is later baked!) is not a pre-existing property of the raw dough:

“Note that one can completely understand what's going on in [a] Stern-Gerlach experiment without invoking any putative property of the electron such

as its actual  $z$ -component of spin that is supposed to be revealed in the experiment. For a general initial wave function there is no such property. What is more, the transparency of the [pilot-wave] analysis of this experiment makes it clear that there is nothing the least bit remarkable (or for that matter “nonclassical”) about the *nonexistence* of this property.”<sup>27</sup>

As explained by Bell, the appearance to the contrary—that is, the tacit assumption that there *must* be some real “ $z$ -component of spin” property that the measurements unveil—seems to arise from the unfortunate and inappropriate connotations of the word “measurement”:

“the word comes loaded with meaning from everyday life, meaning which is entirely inappropriate in the quantum context. When it is said that something is “measured” it is difficult not to think of the result as referring to some pre-existing property of the object in question. This is to disregard Bohr's insistence that in quantum phenomena the apparatus as well as the system is essentially involved. If it were not so, how could we understand, for example, that measurement of a component of “angular momentum”—in an arbitrarily chosen direction—yields one of a discrete set of values? When one forgets the role of the apparatus, as the word measurement makes all too likely, one despairs of ordinary logic—hence “quantum logic.” When one remembers the role of the apparatus, ordinary logic is just fine.”<sup>29</sup>

Unfortunately, even some of the people in the best possible position to understand this important point—namely, proponents of the pilot-wave theory—have missed it and have thus pointlessly laden the theory with additional variables corresponding to such actual spin components.<sup>30</sup> One goal of the present paper is thus to present, in a clear and accessible way, the *simplest possible* pilot-wave account of spin, in order to rectify misconceptions like that held by the anonymous referee quoted in the introduction.

Finally, note that it is by no means only champions of hidden variable theories that are sometimes seduced into thinking of spin components as real properties. Although the official party line of the orthodox quantum theory, as expressed, for example, in standard textbooks, is of course that there are no such things, one often finds that the authors, perhaps swept away by all the talk of “measuring” “observables,” are somewhat conflicted about this. Townsend, for example, whose beautiful textbook begins with five full chapters on spin, stresses early on that, because the operators corresponding to distinct spin components fail to commute, “the angular momentum never really ‘points’ in any definite direction.”<sup>31</sup> This already implies the usual, orthodox view that there is simply no such thing as a spin angular momentum vector, pointing in some particular direction, at all.

And yet—even for Townsend, who is *far more careful* about this issue than most authors<sup>32</sup>—there is a tendency to occasionally slip into suggesting that there is such a thing, or at least that (even though there isn't) it is sometimes helpful to pretend that there is. Thus, for example, Townsend later explains that “placing [a spin-1/2] particle in a magnetic field in the  $z$  direction rotates the spin of the particle about the  $z$ -axis as time progresses...” We also find statements about

how, for example, in the weak decay of the muon, “the positron is preferentially emitted in a direction opposite to the spin direction of the muon” and even a description of  $\langle +x|\psi(t)\rangle$  and  $\langle +y|\psi(t)\rangle$ , the amplitudes for a particle in spin state  $|\psi(t)\rangle$  to be found, respectively, spin-up along the  $x$ - and  $y$ -directions, as “the components of the intrinsic spin in the  $x$ -  $y$  plane.” So perhaps after all particles *do* have definite spin vectors? No, Townsend reminds us that this language and the associated physical picture are not to be taken too seriously:

“However, we should be careful not to carry over *too* completely the classical picture of a magnetic moment precessing in a magnetic field since in the quantum system the angular momentum ... of the particle cannot actually be pointing in a specific direction because of the uncertainty relations ...” [Emphasis added.]

It is not my intention here to criticize Townsend’s text, which I really do think is wonderful. Surely it would be made much worse if all the linguistic shortcuts criticized above were “fixed.” (Just imagine the dreariness and impenetrability of a textbook that explained, for example, that “the positron is preferentially emitted in a direction opposite to the direction along which measurement of the muon’s spin, should such a measurement have been performed prior to its decay, would have given the highest probability of yielding the outcome ‘spin-up.’”) Nevertheless, there really is a sense—highlighted especially by Townsend’s use of the word “too” in the passage just quoted—in which the orthodox view insists on *retaining* the classical picture (of a little spinning ball of charge with definite spin angular momentum vector) but simultaneously *apologizing* for this, by demanding that the picture not be carried over *too* completely, not be taken *too* seriously.

The reason for this schizophrenia, I suspect, is that orthodox quantum physicists are, after all, physicists. They cannot just “shut up and calculate”—not completely. They need *some* sort of visualizable picture of what, physically, the mathematical formalism describes, or they simply cannot keep track of what in the world they are talking about.<sup>33</sup> So they retain the classical picture while simultaneously, out of the other sides of their mouths, rejecting it.

Perhaps then, at the end of the day, the most important thing about the pilot-wave perspective on spin is simply that it provides a picture of what might actually be going on physically in phenomena involving spin—a picture that does not involve any spinning balls of charge, but which is nevertheless completely and absolutely *clear* and *precise* and which *can be taken seriously*, without apologies or double-speak.

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<sup>1</sup>T. Norsen, “The pilot-wave perspective on scattering and tunneling,” *Am. J. Phys.* **81**(4), 258–266 (2013).

<sup>2</sup>S. Goldstein, D. Dürr, and N. Zanghí, “Quantum equilibrium and the origin of absolute uncertainty,” *J. Stat. Phys.* **67**, 843–907 (1992).

<sup>3</sup>Note that here, in discussing the measurement from the point of view of orthodox QM, the word “particle” is used in an extended sense—meaning,

for example, “an electron.” Of course, according to orthodox QM, an electron is not literally a particle in the sense of being a point-like object following a definite trajectory. Whereas, according to the pilot-wave theory, a single electron comprises both a quantum wave and a (literal) particle. Readers are hereafter assumed to be able to judge, from the context, in which sense the word “particle” is used.

<sup>4</sup>W. Pauli, “Über den Zusammenhang des Abschlusses der Elektronengruppen im Atom mit der Komplexstruktur der Spektren,” *Z. Phys.* **31**, 765–783 (1925).

<sup>5</sup>See, for example, Lubos Motl, “David Bohm born 90 years ago,” <<http://motls.blogspot.com/2007/12/david-bohm-born-90-years-ago.html>>.

<sup>6</sup>S. Kochen and E. Specker, “The problem of hidden variables in quantum mechanics,” *J. Math. Mech.* **17**, 59–87 (1967).

<sup>7</sup>N. D. Mermin, “Hidden variables and the two theorems of John Bell,” *Rev. Mod. Phys.* **65**(3), 803–815 (1993).

<sup>8</sup>D. Bohm, R. Schiller, and J. Tiomno, “A causal interpretation of the Pauli equation (A),” *Suppl. Nuovo Cimento* **1**, 48–66 (1955); see also T. Takabayasi, “The vector representation of spinning particle in the quantum theory, I,” *Prog. Theor. Phys.* **14**, 283–302 (1955).

<sup>9</sup>See, e.g., C. Dewdney, P. R. Holland, and A. Kyprianidis, “What happens in a spin measurement,” *Phys. Lett. A* **119**, 259–267 (1986); C. Dewdney, P. R. Holland, and A. Kyprianidis, “A causal account of non-local Einstein-Podolsky-Rosen spin correlations,” *J. Phys. A* **20**, 4717–4732 (1987).

<sup>10</sup>J. S. Bell, “Introduction to the hidden variable question,” *Proceedings of the International School of Physics ‘Enrico Fermi,’ Course II: Foundations of Quantum Mechanics* (Academic, New York, 1971), pp. 171–181. A somewhat more detailed elaboration can be found in J. S. Bell, “Quantum mechanics for cosmologists,” in *Quantum Gravity 2*, edited by C. Isham, R. Penrose, and D. Sciama (Clarendon Press, Oxford, 1981), pp. 611–637. Both essays are reprinted in J. S. Bell, *Speakable and Unsayable in Quantum Mechanics*, 2nd ed. (Cambridge U.P., Cambridge, 2004).

<sup>11</sup>W. Gerlach and O. Stern, “Der experimentelle Nachweis der Richtungsquantelung im Magnetfeld,” *Z. Phys.* **9**, 349–352 (1922).

<sup>12</sup>Note that this expression for the field is simplified and idealized: A more precise expression would involve field components perpendicular to  $\hat{z}$ —which would allow the divergence of  $\vec{B}$  to vanish, as it of course should—as well as a large constant field in the  $z$ -direction. As it is usually explained, this large constant field causes rapid precession about the  $z$ -axis: This makes any transverse forces (that would otherwise arise from the transverse magnetic field gradients) average to zero. It is much simpler, though, to simply look the other way in regard to the violation of Maxwell’s equations, and thus avoid needing to talk about any of these subtleties, by simply pretending that Eq. (6) accurately describes the field to which the particles are subjected.

<sup>13</sup>If the reflected wave term from Eq. (14) is retained, the detailed particle trajectories are a little more complicated. During the time period when the incident and reflected packets overlap, the particle velocity in that ( $y < 0$ ) overlap region is slightly oscillatory, with an average velocity—in precisely, the sense explained and used in the earlier paper—that is slightly less than  $\hbar k/m$  by an amount proportional to  $(\kappa z)^2$ . Thus, particles very near the trailing edge of the incident packet will in fact be overtaken by the trailing edge of the packet before reaching  $y = 0$  and will hence be swept away, back out to the left, by the reflected packet. In short, particles that happen to begin in a small (parabolic) slice near the trailing edge of the incident packet will not behave as described in the main text, but will instead reflect. The behavior of the particles that do transmit through to the  $y > 0$  region, however, is (in the applicable  $\kappa w \rightarrow 0$  limit) unaffected. So we will continue to suppress discussion at this level of detail in the main text.

<sup>14</sup>D. Bohm and B. Hiley, *The Undivided Universe* (Routledge, London, 1993), pp. 73–78.

<sup>15</sup>Actually there is one subtle point that deserves to be addressed. In analyzing the measurement of  $z$ -spin, we have assumed that the incident wave packet has an essentially rectangular cross-sectional profile, with (as mentioned explicitly) some width  $w$  along the  $z$ -direction and (as only really assumed tacitly) some width  $w'$  along the  $x$ -direction. This makes the problem “effectively 2D” in the sense that (for the width- $w'$  range of  $x$ -values where the wave function has support and where the particle might therefore be) nothing depends on  $x$ . If, however, the incident wave function is the same but the SG apparatus is rotated through some arbitrary angle (not an integer multiple of  $90^\circ$ ), this nice symmetry is spoiled. Everything still works as claimed; it’s just that one has to think a little bit (about, e.g., slicing the 3D wave packet up into a stack of planes perpendicular to  $\hat{n}$ ) to

convince oneself of this. Alternatively, one can recognize that the statistics of the measurement outcomes will not depend on the exact cross-sectional profile of the incident packet and hence imagine that the cross-sectional profile of the incident packet is rotated in tandem with the SG device so that the analysis remains transparent.

<sup>16</sup>Of course, the mere fact that the two sub-beams are spatially separated is not sufficient to prove that the empty packet is forevermore dynamically irrelevant to the motion of the particle. The “empty” spin-down sub-beam could be made to become dynamically relevant, for example, by arranging (by use, say, of appropriate magnetic fields) to deflect the spin-down beam back up toward the spin-up beam, at which point interference effects—including perhaps the re-constitution of a guiding wave proportional to  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ —could occur. But if, as suggested in Fig. 4, the spin-down sub-beam is blocked by a brick wall, then the usual process of decoherence will render it impossible, at least for all practical purposes, to arrange the appropriate overlapping of the two components of the wave function, and treating the wave as having “effectively collapsed” is entirely warranted.

<sup>17</sup>This particular illustration of the “contextuality” of spin (for the pilot-wave theory) is due, in essence, to David Z. Albert: see *Quantum Mechanics and Experience* (Harvard U.P., Cambridge, MA, 1994), pp. 153–155.

<sup>18</sup>J. S. Bell, “On the problem of hidden variables in quantum mechanics,” *Rev. Mod. Phys.* **38**, 447–452 (1966), reprinted in Bell (2004), Ref. 10.

<sup>19</sup>N. Bohr, “Discussions with Einstein on epistemological problems in atomic physics,” in *Albert Einstein: Philosopher Scientist*, edited by P. A. Schilpp (Library of Living Philosophers, Evanston, Illinois, 1949).

<sup>20</sup>A. Shimony, “Contextual hidden variables theories and Bell’s inequalities,” *Brit. J. Phil. Sci.* **35**(1), 25–45 (1984).

<sup>21</sup>We assume here that  $\Delta > w$  so that the two sub-beams downstream of the SG device do not overlap spatially.

<sup>22</sup>T. Norsen, “The theory of (Exclusively) local beables,” *Found. Phys.* **40**, 1858–1884 (2010).

<sup>23</sup>It is perhaps worth noting explicitly that the non-locality cannot be used for signaling: if a large ensemble of pairs are prepared in the state (39), and Alice chooses at the last minute whether to subject all of the particles on her side to an SG<sub>2</sub> measurement, or some other spin measurement, or no measurement, Bob will see the same 50/50 statistics on his side regardless of her choice.

<sup>24</sup>J. S. Bell, “On the Einstein Podolsky Rosen Paradox,” *Physics* **1**(3), 195–200 (1964), reprinted in Bell (2004), Ref. 10. For a detailed analysis, see S. Goldstein, T. Norsen, D. Tausk, and N. Zanghí, “Bell’s theorem,” *Scholarpedia* **6**(10), 8378-200, <[http://www.scholarpedia.org/article/Bell's\\_theorem](http://www.scholarpedia.org/article/Bell's_theorem)>.

<sup>25</sup>M. Daumer, D. Dürr, S. Goldstein, and N. Zanghí, “Naive Realism about Operators” *Erkenntnis* **45**, 379–397 (1996).

<sup>26</sup>The authors of Ref. 25 describe themselves as following Bell in concluding that, for the pilot-wave theory, “spin is not real.” Bell, for example, writes (in the first essay of Ref. 10): “We have here a picture in which although the wave has two components, the particle has only position. . . . The particle does not ‘spin,’ although the experimental phenomena

associated with spin are reproduced. Thus the picture resulting from a hidden-variable account of quantum mechanics need not very much resemble the traditional classical picture that the researcher may, secretly, have been keeping in mind. The electron need not turn out to be a small spinning yellow sphere.” Gestures in the direction of this same point were made even earlier by M. Renninger, “Zum Wellen-Korpuskel-Dualismus,” *Z. Phys.* **136**, 251–261 (1953), translated by W. De Baere as “On Wave-Particle Duality,” <<http://arxiv.org/abs/physics/0504043>>; and still earlier by Lorentz, who discussed in 1922 a pilot-wave theory of photons, previously suggested by Einstein; see H. A. Lorentz, *Problems of Modern Physics* (Dover, New York, 1967), p. 157.

<sup>27</sup>D. Dürr, S. Goldstein, and N. Zanghí, “Quantum equilibrium and the role of operators as observables in quantum theory,” *J. Stat. Phys.* **116**, 959–1055 (2004). Reprinted as Chapter 3 of D. Dürr, S. Goldstein, and N. Zanghí, *Quantum Physics Without Quantum Philosophy* (Springer-Verlag, Berlin, 2013).

<sup>28</sup>J. S. Bell, “Beables for quantum field theory,” preprint CERN-TH 4035/84 (1984), reprinted in Bell (2004), Ref. 10.

<sup>29</sup>J. S. Bell, “Against ‘Measurement,’” in *62 Years of Uncertainty*, edited by A. I. Miller (Plenum, New York, 1990), reprinted in Bell (2004), Ref. 10.

<sup>30</sup>See, for example, P. Holland, *The Quantum Theory of Motion* (Cambridge U.P., Cambridge, 1993), Chap. 9; C. Dewdney, P. Holland, and A. Kyprianidis, op cit. (1987); D. Bohm and B. Hiley, *The Undivided Universe* (Routledge, London, 1993), Chap. 10; C. Dewdney, “Constraints on quantum hidden-variables and the Bohm theory,” *J. Phys. A* **25**, 3615–3626 (1992).

<sup>31</sup>J. S. Townsend, *A Modern Approach to Quantum Mechanics*, 2nd ed. (University Science Books, Mill Valley, CA, 2012). All quotes are from Chaps. 3 and 4.

<sup>32</sup>For example, in D. Griffiths, *Introduction to Electrodynamics*, 2nd ed. (Prentice-Hall, Englewood Cliffs, NJ, 1989), the author writes: “all magnetic fields are due to electric charges in motion, and in fact, if you could examine a piece of magnetic material on an atomic scale you would find tiny currents: electrons orbiting around nuclei and electrons spinning on their axes.” And later: “Since every electron constitutes a magnetic dipole (picture it, if you wish, as a tiny spinning sphere of charge), you might expect paramagnetism to be a universal phenomenon.” Interestingly, Griffiths is more carefully orthodox about spin in his *Introduction to Quantum Mechanics*, 2nd ed. (Pearson Prentice Hall, Upper Saddle River, NJ, 2004).

<sup>33</sup>In this respect it is somewhat telling that the worst offenses against the official orthodox view tend to occur in figures and illustrations. Almost all quantum mechanics textbooks, for example, include diagrams depicting the precession, induced by the presence of a magnetic field, of a particle’s spin vector about some axis. Townsend’s text (Ref. 31) avoids that particular misleading suggestion of a classical picture. But his front cover art—depicting a Stern-Gerlach spin measurement much like our Fig. 1—includes little circles with arrows (pointing, respectively, up and down) in the downstream sub-beams. The circles are even yellow, inviting us to recall Bell’s warning, quoted earlier in Ref. 26 note.