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On the Number of Embeddings of Minimally Rigid Graphs

Ciprian Borcea [1]   and Ileana Streinu [2]

Abstract

Rigid frameworks in some Euclidian space are embedded graphs having a unique local realization (up to Euclidian motions) for the given edge lengths, although globally they may have several. We study the number of distinct planar embeddings of minimally rigid graphs with n vertices. We show that, modulo planar rigid motions, this number is at most $\binom{2n-4}{n-2} \approx 4^n$. We also exhibit several families which realize lower bounds of the order of 2^n, 2.21^n and 2.88^n.

For the upper bound we use techniques from complex algebraic geometry, based on the (projective) Cayley-Menger variety $CM^{2,n}(C) \subset P_{\binom{n}{2}-1}(C)$ over the complex numbers C. In this context, point configurations are represented by coordinates given by squared distances between all pairs of points. Sectioning the variety with $2n - 4$ hyperplanes yields at most $deg(CM^{2,n})$ zero-dimensional components, and one finds this degree to be $D^{2,n} = \frac{1}{2}\binom{2n-4}{n-2}$. The lower bounds are related to inductive constructions of minimally rigid graphs via Henneberg sequences.

The same approach works in higher dimensions. In particular we show that it leads to an upper bound of $2D^{3,n} = \frac{2^{n-3}}{n-2}\binom{n-6}{n-3}$ for the number of spatial embeddings with generic edge lengths of the 1-skeleton of a simplicial polyhedron, up to rigid motions.

1   Introduction

In this paper we are concerned with *graph embeddings subject to edge lengths constraints*. We use *embedding* (or *realization*) in the extended sense, which allows some vertices to coincide and some edges to cross. For a given graph and for a given set of edge lengths, a natural question to ask is: *how many embeddings in R^d are there?*

Obviously, for a fixed dimension d, some graphs have a continuum of embeddings, or may have no embedding at all for particular choices of edge lengths.

We consider *minimally rigid graphs on n vertices in dimension d* (which have $dn - \binom{d+1}{2}$ edges and unique local realizations for generic choices of edge lengths), with particular regard to dimensions 2 and 3.

Our results give a general upper bound in arbitrary dimension, which is of the order of 2^{dn} for fixed d and n sufficiently large. We also exhibit a family of graphs inducing a lower bound of the order of 2.88^n in dimension 2.

Historical Perspective. *Distance geometry* relies on foundational work of Cayley on configurations of n points in Euclidean d-space. As elaborated by Menger (see [3], p. 237), this led to conditions characterizing systems of $\binom{n}{2}$ positive reals that arise as pairwise squared distances

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between n points in R^d. See also [4] and [10]. In dimension 3, distance geometry has been used for the study of molecular conformation in chemistry ([8], [9], [18]). Indeed, inter-atomic distance information can be obtained from nuclear magnetic resonance spectra of a molecule. Solving the graph embedding problem determines coordinates for the atoms, and hence the 3-dimensional shape of the molecule. Other applications include surveying and satellite ranging.

Our investigation is also related to classical studies in the Kinematics of mechanical linkages, in particular the problem of tracing algebraic curves. Wunderlich ([33]) gives an interesting family of generalized planar coupler curves with degree growing exponentially in the number of links. Related work was also done in combinatorial Rigidity Theory ([7], [31], [14]). Rigid frameworks are embedded graphs having a unique local realization for the given edge lengths. But globally there may be several realizations. Using special combinations of graphs and edge lengths, Saxe [27] has shown that it is NP-hard to solve the graph embedding problem in dimension two, as well as to determine whether it has a unique solution. Under the assumption of genericity, Hendrickson [17] studied conditions on rigid frameworks that guarantee a unique global realization.

Minimally Rigid Graphs. Hendrickson's graphs, as well as the complete graphs implicit in the Cayley-Menger conditions, are highly redundant in terms of distance dependencies. This makes the problem of reconstructing point sets from distances very challenging, as small perturbation errors in the input data will render the problem infeasible. In this paper we go to the lower end of the dependency spectrum and study the number of embeddings of *minimally rigid graphs in dimension d*.

In dimension two, these graphs possess a simple combinatorial characterization [22], and, for *generic* edge lengths, will have a *discrete* realization space. We'll refer to these graphs as *Laman graphs*. In dimension three, one does not have a comparable result. However, as observed by Gluck [13], a combination of arguments going back to Cauchy, Steinitz, Dehn, Weyl and Alexandrov ensures that 1-skeleta of simplicial polyhedra are minimally rigid graphs in dimension three.

Techniques and Results. Our approach emphasises the possibility of exploiting the *algebraic* character of the prevalent relations in distance geometry by rephrasing the problems in terms of associated *complex projective varieties*.

Indeed, our upper bounds are derived from *degree* computations for the complex projective varieties $CM^{d,n}$ induced by the Cayley-Menger determinant equalities for squared distances of n points in R^d. In [5] these *Cayley-Menger varieties* are identified with classical *determinantal varieties* of symmetric forms.

For planar realizations of Laman graphs we obtain:

Theorem 1.1 *Given a generic choice of edge lengths, a Laman graph with n vertices has at most $\binom{2n-4}{n-2}$ planar embeddings, up to rigid motions.*

In the direction of lower bounds, we exhibit several families which realize bounds of the order of 2^n, 2.21^n and 2.88^n.

For spatial realizations of 1-skeleta of simplicial polyhedra we obtain:

Theorem 1.2 *Given a generic choice of edge lengths, the 1-skeleton of a simplicial complex which has n vertices and is topologically a 2-dimensional sphere has at most $\frac{2^{n-3}}{n-2}\binom{n-6}{n-3}$ spatial embeddings, up to rigid motions.*

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In general, we have:

Theorem 1.3 *Let d be a given dimension. Let $\mathcal{G}$ be a class of connected graphs with $n \geq d+1$ vertices and $m = dn - \binom{d+1}{2}$ edges such that each graph allows an infinitesimally rigid realization in R^d.*

Then, for a generic choice of edge lengths, each graph in $\mathcal{G}$ has a finite number of embeddings in R^d and this number is bounded from above by $2D^{d,n}$. Here $D^{d,n}$ stands for the degree of the Cayley-Menger variety $CM^{d,n}(C) \subset P_{\binom{n}{2}-1}(C)$ and is given by the formula:

$$D^{d,n} = \prod_{k=0}^{n-d-2} \frac{\binom{n-1+k}{n-d-1-k}}{\binom{2k+1}{k}}$$

For d constant and n sufficiently large, $D^{d,n} \approx 2^{dn}$.

It is worth noticing that known (but general) techniques based on *real algebraic geometry* (Oleinik-Petrovskii-Milnor-Thom bounds) yield less good upper bounds.

Future Perspectives and Open Questions. Trying to bridge the still significant gap between the upper and lower bounds can be approached from both directions. The upper bound is presumably high, as it counts not only "realistic" solutions, but also real solutions which do not represent configurations and complex solutions in the Cayley-Menger variety.

On the other hand, in showing that Laman graphs which require Henneberg II constructions may have more than $c^n, c > 2$ planar embeddings, we give evidence that the lower bound may possibly be raised.

Obviously, an important problem for $d \geq 3$ is to identify and characterize classes of minimally rigid graphs.

2   Definitions and Preliminaries

In this section we define the necessary concepts: configuration space, rigid, infinitesimally rigid, d-minimally rigid graph. We also formulate the problem and, for the sake of comparison, use standard results from Real Algebraic Geometry to derive upper bounds that we'll later improve.

Rigidity Theoretic Definitions. Let G be a graph $G = (V, E)$, $V = \{1, \cdots, n\}$, $m = |E|$. A *framework* (G, L) is a graph G together with a set $L = \{l_{ij} | ij \in E\}$ of positive numbers $l_{ij} > 0$ interpreted as *edge lengths* associated to the edges. A *realization* or *embedding* $G(P)$ of (G, L) in some space R^d is given by a mapping of the vertices onto a set of points $P = \{p_1, \cdots, p_n\} \subset R^d$ such that l_{ij} equals the Euclidian distance between the two points p_i and p_j. Note that edges may cross and vertices may coincide.

Notation. The squared distance (resp. the distance) between two points labeled i and j will be denoted by d_{ij} (resp. l_{ij}) when given a priori, and by x_{ij} (squared distance only) when unknown.

The *realization* or *configuration space of (G, L) in R^d* is the set of all possible realizations, modulo the $\binom{d+1}{2}$ dimensional group of rigid motions (translations and orthogonal transformations).

Any graph has some embedding, for some edge lengths, but for given values of L the configuration space may be empty, finite or higher dimensional. A realization $G(P)$ is **rigid** when it cannot be deformed continuously into another (non-congruent) realization of the same framework. Otherwise, the realization is just one point of a higher dimensional component of the realization space, and the embedded graph is called in this case a *mechanism*, as it is *flexible*. The dimension of the local component to which the realization belongs is its *number of degrees of freedom*. A *one-degree-of-freedom mechanism* (abbreviated as *1DOF mechanism*) is an embedded framework whose component is one-dimensional (i.e. a curve).

The **rigidity matrix** A associated to a realization $G(P)$ is the following $m \times dn$ matrix. The rows are indexed by the m edges $ij \in E$ and the dn columns are grouped into d-tuples corresponding to the d coordinates of a point indexed by i, $i = 1, \cdots, n$. The ijth row has 0 entries everywhere, except in the ith and jth d-tuples of columns, where the entries are $p_i - p_j$, resp. $p_j - p_i$.

$$
ij \begin{array}{c} \begin{array}{cccccccc} 1 & \cdots & i & \cdots & j & \cdots & n \end{array} \\ \left(\begin{array}{ccccccc} & \cdots & & & & & \\ 0 & \cdots & p_i - p_j & \cdots & p_j - p_i & \cdots & 0 \\ \cdots & & & & & & \end{array} \right) \end{array}
$$

A solution $v \in R^{dn}$ of the linear system $Av = 0$ is called an *infinitesimal motion*. It is *trivial* if arising from an infinitesimal rigid transformation of R^d. The space of trivial infinitesimal motions is a $\binom{d+1}{2}$-dimensional linear subspace of the space of solutions of $Av = 0$. Hence the rank of the rigidity matrix is at most $dn - \binom{d+1}{2}$. It is known that it depends on both the combinatorial structure of the graph G and on the points of the embedding P.

A realization is **infinitesimally rigid** if the rank of its associated rigidity matrix is precisely $dn - \binom{d+1}{2}$. This means that the only solutions $v \in R^{dn}$ of the system $Av = 0$ are the trivial infinitesimal motions. An infinitesimally rigid configuration is rigid, but the opposite is not true: there exist rigid embeddings which are not infinitesimally rigid.

Definition 2.1 *For a given dimension d, we say that a graph with n vertices is d-**minimally rigid** or **minimally rigid in dimension** d if it has $m = nd - \binom{d+1}{2}$ edges and allows an infinitesimally rigid realization in R^d. These graphs are also called d-**isostatic**, see [14].*

Remark. As a maximal rank condition, infinitesimal rigidity will hold true on the complement of an algebraic subvariety if holding true at one point. Consequently, a d-minimally rigid graph, as defined above, will have only infinitesimally rigid realizations in R^d, provided $d_{ij} = l_{ij}^2, ij \in E$ are chosen in the complement of a certain algebraic subvariety (i.e. *generically* as the term is used in algebraic geometry).

DIMENSION TWO. Laman [22] gave a complete characterization of 2-minimally rigid graphs: we will call them *Laman graphs*. They have $2n - 3$ edges, and each subset of k vertices spans at most $2k - 3$ edges.

Henneberg Sequences. Laman graphs have many elegant combinatorial properties. The so-called *Henneberg sequences* (see [19], [30]) are used to construct them in an inductive fashion. A Henneberg sequence for a graph G is a sequence $G_3, G_4, \cdots, G_n$ of Laman graphs on $3, 4, \cdots, n$ vertices, such that: G_3 is a triangle, $G_n = G$ and each graph G_{i+1} is obtained from the previous one G_i via one of two types of *steps*: type I and type II. A Henneberg step of type I adds a new

vertex and two new edges connecting this vertex to two arbitrary vertices of G_i. A Henneberg step of type II adds a new vertex and three new edges, and removes an old edge, more precisely: the three new edges must connect the new vertex to three old vertices such that at least two of them are joined via an edge; that edge will be removed. See Figure 1.
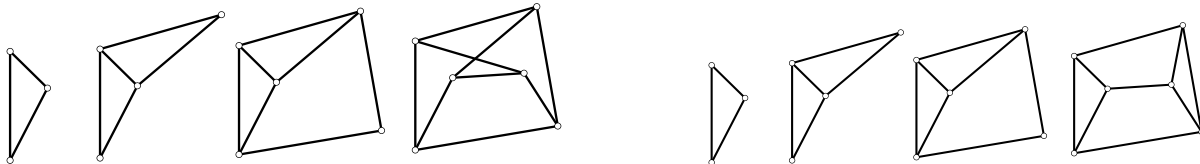

Figure 1: Henneberg constructions for two graphs on 6 vertices, using type I and type II steps. Left: $K_{3,3}$. Right: the Desargues graph.

Not all Laman graphs can be constructed using only type I steps. We call *Henneberg I graphs* those which can, and *Henneberg II* those which cannot. In particular, triangulations of planar polygons are Henneberg I graphs. The smallest examples of Henneberg II graphs are shown in Figure 1. The one on the right, the so-called *Desargues framework*, borrows its name from the classical Desargues configuration (two triangles in perspective) from plane projective geometry, which it induces for some special edge lengths. As an aside, we note that these special embeddings, illustrated in the last two cases of Fig. 2, are not infinitesimally rigid, or may even be flexible. Such embeddings are not *generic*, and therefore not treated in the general discussion of this paper. The first case in Fig. 2 is an example of a *generic* embedding.
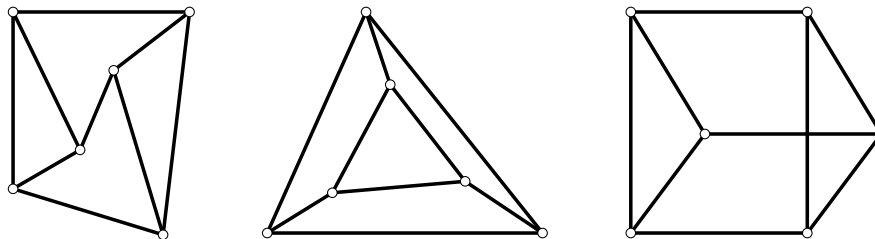

Figure 2: The Desargues framework and three types of its possible embeddings: infinitesimally rigid, rigid but not infinitesimally rigid and flexible.

DIMENSION THREE. A combinatorial characterization of minimally rigid graphs in dimension three or higher is not yet known. However, there is an important class of graphs which are minimally rigid in dimension 3, namely the 1-skeleta of simplicial convex polyhedra. By a theorem of Steinitz [28], these are the 3-connected maximally planar graphs (planar triangulations). The realization of such a graph as a 1-skeleton of a simplicial convex polyhedron is infinitesimally rigid, see Gluck [13].

HIGHER DIMENSIONS. Here is a simple argument showing that there exist generic minimally rigid graphs in any dimension d. Consider the class of graphs constructed inductively as follows. Start with the 1-skeleton of a d-simplex (the complete graph on d vertices). At each step afterwards, add a new vertex and d edges connecting it to d of the previously constructed vertices. It is straightforward to construct an infinitesimally rigid embedding of this graph, by

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embedding the vertices on a set of points in general position (an induction on the construction shows that the rank of the rigidity matrix is as expected).

It would be interesting to find other classes of minimally rigid graphs with simple descriptions.

The Embedding Problem. Given a framework (G, L), one is interested in understanding its configuration space; in particular, topological invariants such as the number of connected components and their dimensions.

The configuration space is the orbit space of the group of rigid motions acting on the affine algebraic variety defined in R^{dn} by the equations:

$$|p_i - p_j|^2 = d_{ij}, \ ij \in E \tag{1}$$

Since the group is connected, the number of connected components can be counted on the algebraic variety itself.

Bounds from Real Semi-Algebraic Geometry. Consider an algebraic system of s equations of degree d in k variables $P_l[x_1, \cdots, x_k] = 0, l = 1, \cdots, s$, degree $P_l \leq d$. Its real solutions form a *real algebraic set*.

The following result from real algebraic geometry bounds the number of connected components or, more generally, the Betti numbers of an algebraic set.

Oleinik-Petrovskii-Milnor-Thom Theorem ([25], [23], [29], see also [2]): for equations of degree at most d, in k variables, the sum of the Betti numbers (and hence the number of connected components) is at most $d(2d - 1)^{k-1}$.

Applying the theorem to a minimally rigid graph in dimension d: the system has $m = dn - \binom{d+1}{2}$ equations of degree 2 in dn variables. The Oleinik-Petrovsky-Milnor-Thom bound implies at most $2 \times 3^{dn-1} \approx (3^d)^n$ components (i.e. realizations). In particular, in dimension 2 the bound is of the order 9^n.

We will obtain better upper bounds on the number of realizations by using a different algebraic object, the Cayley-Menger variety.

3 Cayley-Menger Varieties

A set of $\binom{n}{2}$ positive numbers $d_{ij} = l_{ij}^2$, $i, j = 1, \cdots, n, i < j$, has to satisfy certain algebraic conditions to be the set of squared Euclidian distances between n points in R^d. Some of these conditions are equalities (due to Cayley) and express the fact that the rank of the following matrix is at most $d + 2$:

$$\begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & d_{12} & \cdots & d_{1n} \\ 1 & & \cdots & & \\ 1 & d_{n1} & d_{n2} & \cdots & 0 \end{pmatrix}$$

The other conditions, due to Menger, are inequalities.

If we restrict our attention only to the equalities, we are led to consider a *complex projective variety* $CM^{d,n}(C) \subset P_{\binom{n}{2}-1}(C)$, which is called a *Cayley-Menger variety* in [5]. Here C stands for the field of complex numbers, to emphasize that we refer to the complex and not the real

variety. It is defined by setting to zero all the $(d+3) \times (d+3)$ minors of the following *symmetric* matrix:

$$
\begin{pmatrix}
0 & 1 & 1 & \cdots & 1 \\
1 & 0 & x_{12} & \cdots & x_{1n} \\
1 & & \cdots & & \\
1 & x_{n1} & x_{n2} & \cdots & 0
\end{pmatrix}
$$

where $(x_{ij})_{1 \le i < j \le n}$ are *complex homogeneous coordinates* in the projective space $P_{\binom{n}{2}-1}(C)$.

We arrive in this manner to a complex-projective formulation of our Embedding Problem: *Solve the system consisting of the Cayley determinantal equations, plus the linear conditions setting the graph edges to prescribed lengths.* The latter conditions have the form:

$$
\frac{x_{ij}}{x_{kl}} = \frac{d_{ij}}{d_{kl}}, \quad \text{that is: } d_{kl}x_{ij} = d_{ij}x_{kl}, \quad \text{for } ij, kl \in E
$$

which (for $m = |E| = dn - \binom{d+1}{2}$ edges) amounts to $dn - \binom{d+1}{2} - 1$ independent *hyperplane sections*.

As we are about to see, this is precisely the (complex) dimension of the Cayley-Menger variety $CM^{d,n}(C)$. Hence, by a general result in complex projective geometry, we have:

Proposition 3.1 *A codimension $dn - \binom{d+1}{2} - 1$ linear section of the Cayley-Menger variety $CM^{d,n}(C)$ has at most $deg(CM^{d,n}(C))$ isolated points.*

Here, $deg(CM^{d,n}(C))$ stands for the *degree* of our variety, which is defined as the number of (isolated) points in a *generic* codimension $dn - \binom{d+1}{2} - 1$ linear section.

Remark: Even with generic choices of edge lengths $(l_{ij})_{ij \in E}$, the linear sections corresponding to d-minimally rigid graphs remain very peculiar, and are never generic in the sense required in the definition of the degree. However, we know that a generic choice of edge lengths ensures that all configuration solutions, (which we also call "realistic", meaning both real *and* satisfying Menger's inequalities) are isolated points, because the resulting framework is infinitesimally rigid.

Corollary 3.2 *For a generic choice of edge lengths, a d-minimally rigid graph with n vertices has at most $2deg(CM^{d,n}(C))$ distinct planar embeddings, up to rigid motions.*

Indeed, there will be twice as many solutions to the original embedding problem than "realistic" solutions in the sectioning of the Cayley-Menger variety, because we count as distinct two realizations which are one the reflection of the other (modulo rigid transformation in the plane), while the Cayley-Menger approach automatically identifies congruent configurations. (The projective equivalence, which identifies similar configurations is no issue, since one can always consider a 'first' edge as the unit of length.)

4   The Upper Bounds

We begin with a "naive" count for dimension: n points in R^d require dn parameters; equivalence under Euclidean motions and rescaling cuts down $\binom{d+1}{2}+1$ parameters, so that the *configuration*

space for n (ordered) points in the plane, modulo congruence and similarity, should be $(dn - \binom{d+1}{2} - 1)$-dimensional. The complex version $CM^{d,n}(C)$ (which is, technically, the Zariski-closure of the configuration space in $P_{\binom{n}{2}-1}(C)$), should have as many complex dimensions.

In order to be precise, we first relate *Cayley coordinates* and *Gram coordinates* for a configuration of n points $p_1, ..., p_n \in R^d$.

The *Cayley coordinates* are:

$$x_{ij} = x_{ij}(p) = |p_i - p_j|^2 = \langle p_i - p_j, p_i - p_j \rangle, \quad 1 \le i, j \le n$$

with $\langle \ , \ \rangle$ denoting the usual inner product in R^d.

For *Gram coordinates*, we have to **choose** one point as origin, and **we make the choice** $p_1 = 0$. Then we set:

$$y_{ij} = y_{ij}(p) = < p_i - p_1, p_j - p_1 > = < p_i, p_j >, \quad 2 \le i, j \le n$$

The relation is simply the *cosine theorem*:

$$y_{ij} = \frac{1}{2}(x_{1i} + x_{1j} - x_{ij}), \quad 2 \le i, j \le n$$

Normally, one looks at the Gram coordinates as arranged in a *symmetric $(n-1) \times (n-1)$ matrix* Y with entries $y_{ij} = y_{ji}$, while we've seen above the Cayley coordinates arranged in a *bordered symmetric $(n+1) \times (n+1)$ matrix*, which we denote by X.

It is a simple exercise (cf. [3] or [5]) to observe the rank relation:

$$rk(X) = 2 + rk(Y)$$

Since for configurations in R^d the Gram matrix has obviously rank at most d, we see the reason for defining the Cayler-Menger variety by the vanishing of all $(d+3) \times (d+3)$ minors in X. In fact, we have:

Proposition 4.1 *The passage from Cayley coordinates to Gram coordinates identifies the Cayley-Menger variety $CM^{d,n}(C)$ with the determinantal variety defined (projectively) by all complex (non-trivial) symmetric $(n-1) \times (n-1)$ matrices of rank at most d.*

This brings the matter into the territory of classical algebraic geometry, and we may simply refer to [15] and [11]. The degree computation goes back to Giambelli (cf. [12] [16] [21]).

The result converts our Corollary 3.2 into:

Theorem 4.2 *Let d be given and assume a generic choice of edge lengths. A d-minimally rigid graph with n vertices has at most*

$$2D^{d,n} = 2 \prod_{k=0}^{n-d-2} \frac{\binom{n-1+k}{n-d-1-k}}{\binom{2k+1}{k}}$$

embeddings in R^d, up to rigid motions.

This is our Theorem 1.3 in the Introduction. A direct calculation shows that this number is $O(2^{dn})$ for d fixed and n sufficiently large.

For dimension two we have:

Theorem 4.3 *Given a generic choice of edge lengths, a Laman graph with n vertices has at most $2deg(CM^{2,n}(C)) = \binom{2n-4}{n-2}$ planar embeddings, up to rigid motions.*

A direct computation of this particular degree is given in [5].

In dimension three, we obtain:

Theorem 4.4 *Given a generic choice of edge lengths, the 1-skeleton of a simplicial convex polyhedron with n vertices has at most $2deg(CM^{3,n}(C)) = \frac{2^{n-3}}{n-2}\binom{3n-6}{n-3}$ embeddings in R^3, up to rigid motions.*
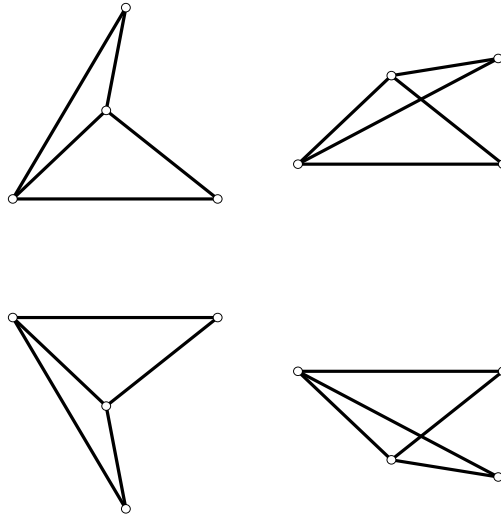

Figure 3: Triangulations have 2^{n-2} embeddings. An example for $n = 4$.

5 Lower Bounds

The simplest family of examples yielding an exponential lower bound is given by triangulations of simple polygons. Indeed, any triangle may be flipped over an adjacent internal diagonal, giving a total of exactly 2^{n-3} embeddings. Taking into account the mirror image of the whole graph doubles the number of possibilities to 2^{n-2}. See Fig. 3.

Our goal in this section is to improve this trivial bound. We do so by refining the analysis of minimally rigid graphs via Henneberg constructions. We show first that Henneberg I graphs have at most as many embeddings as triangulations have, and for particular choices of edge lengths this number is achieved. Then we improve the lower bound twice, using two types of special iterations of Desargues graphs.

Bounds on the number of embeddings for Henneberg I graphs. Let's first analyze the Henneberg constructions for triangulations of polygons. Starting from any triangle, at each step we add a new vertex and two new edges connecting it to two old vertices *which are already*

joined by an edge, hence at a fixed distance. When the edge lengths are given, this construction is easily carried out using ruler and compass. The newly added vertex can be chosen to be one of the two intersections of two circles with radii the edge lengths of the newly added edges (which are known to intersect, as the edge lengths are computed from an existing triangulation.) Any choice at step i is valid and does not affect the number of further options.
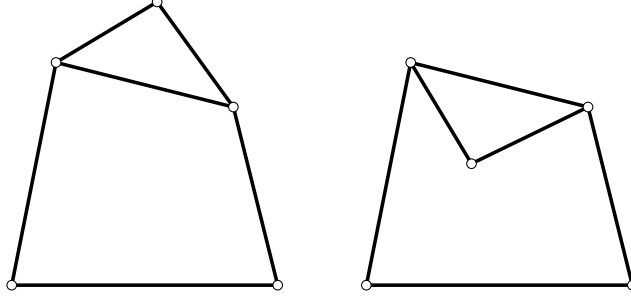

Figure 4: The 4-bar mechanism obtained from the Desargues graph, in two embeddings of the coupler triangle.

We now see that this analysis depends only on the fact that, at step i in the Henneberg I construction, the two vertices to be joined by the newly added edges to the new vertex are themselves connected by an edge, hence at fixed distance. This amounts to "adding a triangle" to the previously constructed graph. Therefore, for a realizable choice of edge lengths (which always exists) we get:

Lemma 5.1 *Any realizable Henneberg I graph obtained by "adding triangles" has exactly 2^{n-2} embeddings.*

In the general Henneberg I step, the two added edges may not form a triangle with an old edge. Therefore the distances between relevant points (such as those used to add two new diagonals at the next step) may vary between embeddings constructed up to step i, and this may render the next step infeasible (i.e. having two complex, instead of two real solutions). But we can carry out the Henneberg I construction and *choose* good edge lengths at each step, which will guarantee that all subsequent steps yield real solution. To do so, we will simply choose the edge lengths of the two new edges to be approximately equal, and sum up to a bit over the maximum distance (over all embeddings of the Henneberg I graph G to which this step is being applied) between the endpoints in G of the new edges. Hence:

Proposition 5.2 *Any realizable Henneberg I graph has at most 2^{n-2} embeddings. For any Henneberg I graph, there exists a choice of edge lengths such that the resulting framework has exactly 2^{n-2} embeddings.*

Number of Embeddings for the Desargues Framework. To beat this bound we must use Henneberg II graphs. For special edge lengths, the Desargues graph gives the smallest example with more than 2^{n-2} embeddings. Namely, on 6 vertices, it can have up to 24 ($> 2^{6-2} = 16$) embeddings.

Lemma 5.3 *There exist edge lengths for the Desargues graph which induce 24 embeddings.*

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First, notice that if you remove an edge from a Laman graph, the resulting graph is a one-degree-of-freedom mechanism. Once an edge is pinned down, the other vertices trace algebraic curves. The curve of the largest degree could then be intersected with a circle whose center is placed at another vertex to get intersection points allowing to place a new bar between the center of the circle and one of these intersection points. Notice also that such mechanisms contain several rigid subcomponents: the distances between vertices inside one such component stay fixed, no matter how the mechanism is moving.

Henneberg II graphs must contain at least one vertex of degree 3. If that vertex (and its adjacent edges) are removed, the resulting graph is a mechanism, whose vertices trace curves. The intersection of these curves with circles should (in general) be relatively easy to compute (even graphically, using for instance the Cinderella [26] software). To be able to easily put the three edges back in a way that would give as many embeddings as possible for the given edge lengths, it would help if two of these edges would stay at fixed distance (i.e. have an edge between them, or belong to a rigid component of the mechanism, which we will assume is grounded when the curve is traced). To achieve this goal, we use a *floating center* idea: place a circle anywhere, of any radius, as long as it gives the desired number of crossings with the curve traced by the mechanism. Then join the center via two bars to the grounded bar of the mechanism: this yields the grounded triangle of the Desargues configuration. To complete the construction, add a bar between the center of the circle and any of the intersection points. This can be done in as many ways as we had crossings. An additional set of embeddings is obtained by flipping (about the grounded edge) the two edges that were used to ground the center of the floating circle.

It turns out that the Desargues graph has all the properties to make this construction work.
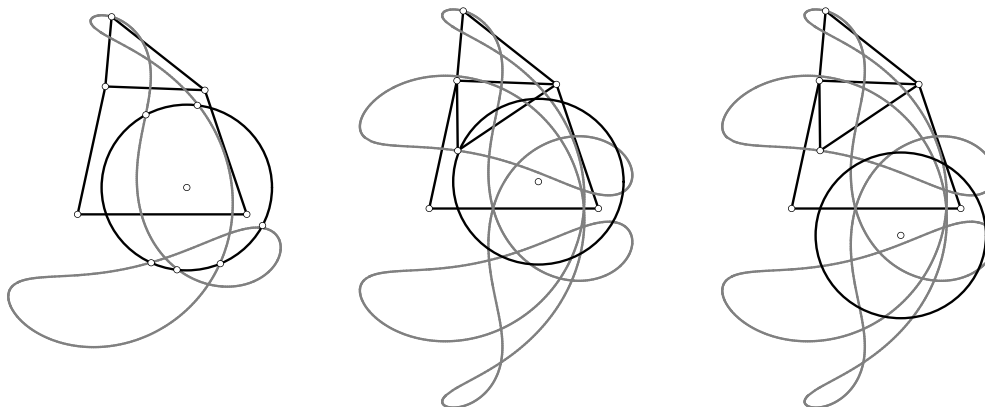

Figure 5: Left: one of the coupler curves, the "floating" circle and their 6 crossings. The bottom bar is grounded to the plane, the upper three vertices are mobile and the top vertex traces the curve. Middle: The 2 coupler curves and a placement of a circle crossing each of them in 6 points. Right: The symmetric placement of the circle from the middle picture induces another 12 crossings.

The construction is illustrated in Figures 4 and 5. Fig. 4 shows, in two embeddings, the mechanism obtained by removing a vertex of degree 3. It is assumed that the bottom edge is grounded and the motion of the mechanism is guided by the rotation of one of the adjacent edges around a grounded vertex. The curves traced by the degree 2 vertex of the triangle attached to this 4-bar mechanism have received a lot of attention in mechanical engineering (kinematics):

indeed, in the '50's a whole atlas of such "coupler curves" has been published ([20], see also [24]). It is known that they are curves of degree 6.

We have used Cinderella ([26]) to get a favorable arrangement for the coupler curves. Fig. 5(left) shows one curve and a position of a "floating" circle intersecting it in 6 points. The static black-and-white pictures shown here do not capture the kinematic intricacy of the arrangement. A Cinderella applet allowing the user to vary the edge lengths and experiment with various shapes of the curves is available from:http://cs.smith.edu~streinu/Research/Embed/coupler.html.

The two flipped triangles on the coupler edge give two distinct and symmetric curves of degree 6 each. We can find a position of a circle intersecting each of them in 6 points, see Figure 5(middle). The symmetric arrangement gives another 12 crossings, for a total of 24. See Fig. 5(right). This proves Lemma 5.3.
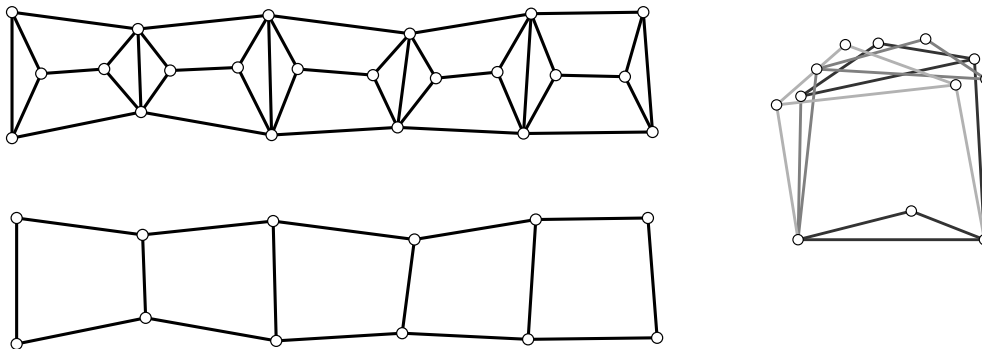

Figure 6: Left: The "caterpillar" construction and its underlying structure. Right: The structure of the "fan" construction.

Iterating the Desargues framework. A simple observation for an iterative construction is that if we glue together, along any edge, two Desargues frameworks, we get a Laman graph. For instance, we could glue them in a "caterpillar" fashion, as in Fig. 6(left). The structure of this construction is illustrated in Fig. 6(middle): it is a planar graph with quadrilateral faces (obtained from the Desargues graphs by retaining just one of the underlying 4-bar mechanisms used in obtaining many embeddings) and whose dual graph is a path. Any such glueing of quadrilaterals whose dual is a tree would give the same bound. Intuitively, this is the equivalent of glueing together triangles to get a polygon triangulation (the dual of a polygon triangulation being a tree).

Lemma 5.4 *There exist edge length for which the number of embeddings of the iterated "caterpillar" Desargues frameworks is of the order of $24^{\frac{n}{4}} \approx (2.21)^n$.*

Proof: Each Desargues graph adds 4 new vertices and multiplies by 24 the total number of possible embeddings. □

To get a better bound we use the following observation.

Lemma 5.5 *A small perturbation of the lengths of the moving edges of the 4-bar mechanism from Lemma 5.3 yields slightly perturbed coupler curves, which still intersect the two symmetric circles in 24 points.*

We now fix a bar in the plane, place on it a 4-bar mechanism satisfying Lemma 5.3, fix the positions of the symmetric triangles and the corresponding circles that give the 24 embeddings. Then perturb the 4-bar mechanism (but not the base triangle) several times as in Lemma 5.5, to obtain a *fan*-like glueing of Desargues configurations. See Figure 6(right). Then the same two symmetric circles (or circles of slightly perturbed radii, but with the same center) cross each pair of coupler curves (one pair for each of these mechanisms) in 24 points, giving the desired number of embeddings.

Proposition 5.6 *There exist edge lengths for which the number of embeddings of the iterated "fan" Desargues frameworks is of the order of $24^{\frac{n}{3}} \approx (2.88)^n$.*

Proof: Each Desargues graph adds now 3 new vertices and 24 possible embeddings. □

As a final comment, we note that we can combine the perturbation idea from the previous construction with the "caterpillar structure" glueing to get another family of examples achieving the same bound.

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References

[1] S. Basu. Different Bounds on the Different Betti Numbers of Semi-Algebraic Sets, *Proc. 17th Symp. on Comput. Geometry (SoCG)*, Medford, Massachusetts, 2001, pp. 288-292.

[2] S. Basu and R. Pollack and M.-F. Roy. Algorithms in Real Algebraic Geometry, book, in preparation.

[3] M. Berger. Geometry I and II, Springer Verlag, 1987.

[4] L. Blumenthal. Theory and Applications of Distance Geometry, Chelsea, Bronx NY, 1970.

[5] C. Borcea. Point Configurations and Cayley-Menger Varieties, manuscript, http://arxiv.org/abs/math/0207110, 2002.

[6] J. Canny. Complexity of Robot Motion Planning, MIT Press, Cambridge, MA, 1988.

[7] R. Connelly. Rigidity, in P. M. Gruber and J. M. Wills (eds.) *Handbook of Convex Geometry*, vol. A, North Holland, 1993, pp. 223-272.

[8] G. M. Crippen. Distance Geometry and Conformational Calculations. Research Studies Press, John Wiley, 1981.

[9] G. M. Crippen and T. F. Havel. Distance Geometry and Molecular Conformation. Research Studies Press, John Wiley, 1988.

[10] M. M. Deza and M. Laurent. Geometry of Cuts and Metrics. Springer Verlag, 1997.

[11] W. Fulton. Intersection theory, Springer-Verlag, 1984.

[12] G. Z. Giambelli. Sulle varietá rappresentate coll'annullare determinanti minori contenuti in un determinante simmetrico od emisimmetrico generico di forme. *Atti della R. Acc. Sci. di Torino* 44 (1905/1906), 102-125.

[13] H. Gluck. Almost all simply connected closed surfaces are rigid, in *Geometric Topology*, Lecture Notes in Mathematics No. 438, Springer Verlag, Berlin, 1975, 225-239.

[14] J. Graver, B. Servatius and H. Servatius. Combinatorial Rigidity, Amer. Math. Soc., Graduate Studies in Mathematics vol.2, 1993.

[15] J. Harris. Algebraic Geometry, Graduate Texts in Math. 133, Springer-Verlag, 1993.

[16] J. Harris and L. W. Tu. On symmetric and skew-symmetric determinantal varieties, *Topology* 23 (1984), 71-84.

[17] B. Hendrickson. Conditions for unique graph realizations, *SIAM J. Comput.*, 21(1) (1992) 65-84.

[18] B. Hendrickson. The Molecule Problem: Determining Conformation from Pairwise Distances, PhD Thesis, Computer Science Department, Cornell University. Technical Report 90-1159, 1990.

[19] L. Henneberg. Die graphische Statik der starren Systeme, Leipzig, 1911.

[20] J. A. Hrones and G. I. Nelson. Analysis of the Fourbar Linkage, MIT Technology Press, Cambridge, MA, 1951.

[21] T. Józefiak, A. Lascoux and P. Pragacz. Classes of determinantal varieties associated with symmetric and skew-symmetric matrices. *Math. USSR Izvestija* 18 (1982), 575-586.

[22] G. Laman. On graphs and rigidity of plane skeletal structures. *Eng. Math.*, 4 (1970), 331-340.

[23] J. Milnor. On the Betti numbers of real varieties, *Proc. AMS* 15 (1964), 275-280.

[24] R. L. Norton. Design of Machinery. McGraw Hill, 2nd ed. 2001.

[25] O. A. Oleinik and I. B. Petrovskii. On the topology of real algebraic surfaces, *Izv. Akad. Nauk SSSR* 13 (1949), 389-402.

[26] J. Richter-Gebert and U. Kortenkamp. Cinderella, an interactive software package for Geometry, Springer Verlag, 1999.

[27] J. B. Saxe. Embedability of weighted graphs in *k*-space is strongly NP-hard, in *Proc. Allerton Conf. on Communications, Control and Computing* (1979), 480-489.

[28] E. Steinitz and H. Rademacher. Vorlesungen über die Theorie der Polyedern, Springer, Berlin, 1934.

[29] R. Thom. Sur l'homologie des variétés réelles, *Differential and Combinatorial Topology*, Princeton University Press, Princeton, (1965), 255-265.

[30] W. Whiteley. Matroids from Discrete Applied Geometry, *Contemporary Mathematics*, vol. 197, pp. 171-311, Amer. Math. Soc. 1996.

[31] W. Whiteley. Rigidity and Scene Analysis, in *Handbook of Discrete and Computational Geometry*, (Jacob E. Goodman, Joseph O'Rourke, eds.) (1997), 893-916.

[32] W. Whiteley. Rigidity of Molecular Structures, in *Rigidity Theory and Applications* (Thorpe and Duxbury, eds.), Kluwer, 1999.

[33] W. Wunderlich. Hohere Koppelkurven, *Österiechisches Ingenieur Archiv*, XVII(3) (1963), 162-165.