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The Smallest Non-Autograph

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Abstract

Suppose that G is a simple, vertex-labeled graph and that S is a multiset. Then if there exists a one-to-one mapping between the elements of S and the vertices of G, such that edges in G exist if and only if the absolute difference of the corresponding vertex labels exist in S, then G is an *autograph*, and S is a *signature* for G. While it is known that many common families of graphs are autographs, and that infinitely many graphs are not autographs, a non-autograph has never been exhibited. In this paper, we identify the smallest non-autograph: a graph with 6 vertices and 11 edges. Furthermore, we demonstrate that the infinite family of graphs on n vertices consisting of the complement of two non-intersecting cycles contains only non-autographs for $n \geq 8$.

1. Introduction

Let $G = (V, E)$ be a simple, vertex-labeled graph, and let S be a multiset. If there exists a bijection $\pi : S \to V$ such that for every $s, t \in S$, the edge $(\pi(s), \pi(t)) \in E$ if and only if $|s - t| \in S$, then we say that G is an *autograph* and that S is a *signature* for G. For example, consider the path graph on three vertices P_3 shown in Figure 1. This is an autograph, since one possible signature for this graph is $S = \{1, 2, 4\}$.
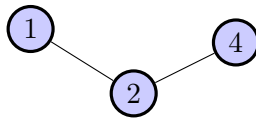

Figure 1. The path graph P_3 on three vertices. This is an autograph realized by the signature $\{1, 2, 4\}$. The edge between the vertices labeled 1 and 2 exists because $|1 - 2| = 1 \in S$, and similarly for the vertices labeled 2 and 4. However, the edge between 1 and 4 does not exist because $|1 - 4| = 3 \notin S$.

The use of a signature to define and categorize graphs is useful because it provides an exceptionally compact data structure for graphs — an autograph can be represented using only n numbers, whereas a non-sparse graph requires $O(n^2)$. The study of autographs — or undirected *difference graphs* — has led to the knowledge that many families of graphs are autographs, and indeed, signatures for a large number of common graph families have been discovered. For example, the set $S = \{1, 2, \ldots, n\}$ provides a signature for the complete graph K_n on n vertices. However, it is also known that infinitely many graphs on n vertices are not autographs, and further, it is conjectured that nearly all graphs are not autographs. Yet while [1] proved that infinitely many non-autographs exist, none has ever been demonstrated. In this paper, we prove that the graph M_6 on 6 vertices shown in Figure 2 is the smallest non-autograph. Furthermore, the infinite family of graphs $G_n = K_n \setminus 2C_{n/2}$, n even, consisting of a complete graph with two non-intersecting cycles deleted, contains only non-autographs for $n \geq 8$. An illustration of G_{10} is shown in Figure 3.

1.1. Motivation

The bijective nature of autographs becomes useful in a variety of practical contexts. The primary advantage is that since the adjacency matrix for an autograph can be computed from its signature, an autograph can be stored using $O(n)$ space, compared to $O(n^2)$ for a non-sparse graph. For large graphs, this order of magnitude savings can be of considerable practical value. For example, these savings could improve reliability if such information had to be transmitted over a noisy channel. Given its brevity, the representation of an autograph by its signature
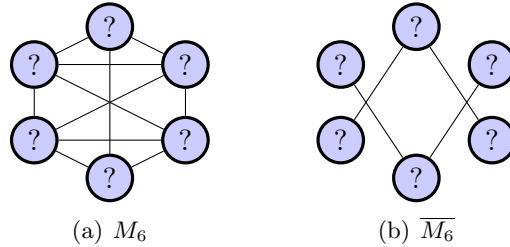
(a) M_6 (b) $\overline{M_6}$

Figure 2. The smallest non-autograph, M_6 (left), and its complement (right).

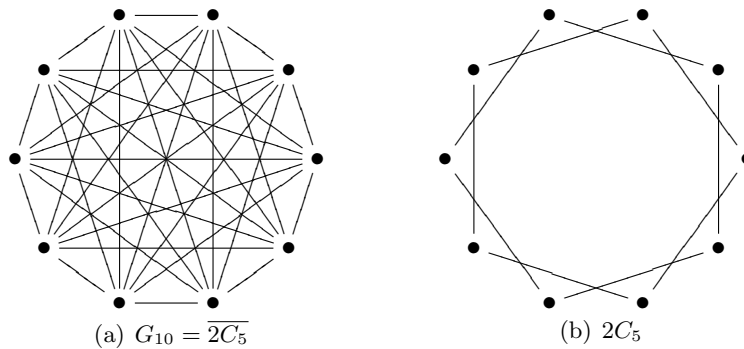(a) $G_{10} = \overline{2C_5}$ (b) $2C_5$

Figure 3. Illustration of $G_{10} = \overline{2C_5}$ and $2C_5$. We show that this graph is not an autograph.

provides some inherent increase in the probability of a successful transmission over a network.

Two problems are of interest:

Problem 1 (AUTOGRAPH). Given a simple graph G, determine if G is an autograph.

Problem 2 (SIGNATURE). Given an autograph G, find a signature S for G.

In Section 5, we discuss some computational complexity concerns of these two problems.

1.2. Related work

Within graph labeling, there is a substantial interest in *graceful labelings*, dating back to the work of [6] and [10]. A graph $G = (V, E)$ is considered *graceful* if the vertices can be numbered with integers chosen from $[0, |E|]$ such that every edge receives a distinct integer label defined by the absolute difference of the labels of its neighboring vertices. This theory led, in particular, to the Ringel-Kötzig graceful tree conjecture, which poses the question of whether all trees are graceful [2, 10].

However, while graphs may or may not admit a graceful labeling, the process of translating between vertex labels and graphs is not bijective. That is, if given a graph, one might be able to find a graceful labeling, but if given a graceful vertex labeling, one cannot recover the original graph. The desire to make this process bijective led to the notion of an *autograph*, as defined above. [3] found all signatures for graphs with at most four vertices, as well as trees, paths, cycles, complete graphs, pyramids and n-prisms. [5] investigated wheel graphs, and proved that they are *proper*[1] autographs for $n = 3, 4$ and 6 only. [7] used the *difference graph* terminology, wherein the edge labels may come from the difference (not necessarily absolute) of the vertex labels. In this realm [13] found conditions for building directed graphs from smaller difference graphs and also found that all cacti with girth at least 6 are autographs [12].

If the signature for a graph contains distinct elements, then the graph is known as a *monograph*. [14] studied the properties of monographs and discovered signatures for cycles, fan graphs, kite graphs and necklaces. [11] listed signatures for all graphs of order 5 and discovered signatures for gear graphs, triangular snakes, and dragons, among other things. The related construction of *mod difference digraphs*, in which $S = [n] = \{1, 2, \ldots, n\}$ and the edge labels are taken modulo n, were explored by [8].

1.3.   Our contributions

While most of the work cited above has focused on discovering signatures for families of graphs, thereby proving that they are autographs, comparatively less work has investigated the properties of *non-autographs*. [1] proved that there are infinitely many graphs on n vertices that are not autographs, but did not produce one. In that effort, monographs of low codegree — i.e., graphs in which each vertex is connected to most other vertices — were used to put constraints on possible signature values for monographs. We extend this work to demonstrate that a certain family of graphs of low codegree are not autographs. Moreover, while [3] found proper autograph signatures for all simple non-isomorphic graphs with at most 5 vertices, they exhibited three graphs with 6 vertices that were not proper autographs. We show that one of these graphs (Figure 2) is not an autograph, making it the smallest non-autograph.

Theorem 1. *M_6 is the smallest non-autograph.*

Theorem 2. *Let $G_n = K_n \backslash 2C_{n/2}$ be the complete graph on n vertices (n is even), with two non-intersecting cycles deleted. Then the infinite family of graphs G_n contains only non-autographs for $n \geq 8$.*

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[1]A proper autograph contains only positive signature values.

This paper proceeds as follows: in Section 2 we develop some preliminary notions and lemmas that constrain possible signatures for graphs based on certain properties. In particular, we investigate the relationship between autographs and signatures composed of arithmetic progressions. In Section 3 we prove that M_6 is the smallest non-autograph. In Section 4 we prove that G_n is a non-autograph for $n \geq 8$. We conclude with open problems and thoughts on future work.

2. PRELIMINARIES

Recall that if the signature $S(G)$ for a graph G is a *set* (i.e., not a multiset), then that graph is a *monograph*. We show presently that both M_6 and G_n are monographs.

Observation 3. *If* $0 \in S(G)$, *then any vertex* $v \in V(G)$ *with* $\pi^{-1}(v) = 0$ *is adjacent to every vertex with a non-negative signature value.*

Proof. For any other vertex $w \in V(G)$ with $\pi^{-1}(w) > 0$, an edge between v and w exists, since $|\pi^{-1}(w) - \pi^{-1}(v)| = |\pi^{-1}(w)| = \pi^{-1}(w) \in S(G)$. ∎

Observation 4. *If two vertices* $v, w \in V(G)$ *have the same label, then* v *and* w *have the same neighborhood.*

Proof. Let $N_G(v), N_G(w)$ denote the set of vertices in G adjacent to vertices $v, w \in V(G)$, respectively, where $\pi^{-1}(v) = \pi^{-1}(w)$. For any $x \in N_G(v)$, $|\pi^{-1}(x) - \pi^{-1}(w)| = |\pi^{-1}(x) - \pi^{-1}(v)| \in S(G)$, since v, w have the same label. Thus, $x \in N_G(w)$. Similarly, for any $y \in N(w)$, $|\pi^{-1}(y) - \pi^{-1}(w)| = |\pi^{-1}(y) - \pi^{-1}(v)| \in S(G)$ and $y \in N_G(v)$. Therefore, $N_G(v) = N_G(w)$. ∎

It follows that if every vertex in a graph G has a different set of neighbors, then G is a monograph. Both M_6 and G_n have this property, and accordingly we restrict our attention to monographs throughout this paper.

2.1. Notation

We assume that G is a monograph with signature $S(G) = S = \{s_1, \ldots, s_n\}$, where $s_1 < s_2 < \cdots < s_n$. Let m be the number of negative signature values in $S(G)$, and set $S^- = \{s_1, \ldots, s_m\}$ and $S^+ = \{s_{m+1}, \ldots, s_n\}$. Thus, we have that

$$S(G) = S^- \cup S^+, \quad \text{with} \quad \underbrace{s_1 < s_2 < \cdots < s_m}_{negative} < \underbrace{s_{m+1} < \cdots < s_{n-1} < s_n}_{non\text{-}negatitive}.$$

If δ is the minimum degree of any vertex in G, then $k = n - 1 - \delta$ is the *codegree* of G — i.e., the maximum number of vertices to which each vertex in G is not adjacent.

Let $S = \{ts, (t+1)s, \ldots, (n-1+t)s\}$ be a signature for a proper monograph consisting of an arithmetic progression of n terms, beginning at ts and continuing in integer steps of size s. Note that S defines a monograph. We adopt the simpler notation $[t]_s^n$ for such a signature.

Definition. For any signature value $s \in S(G)$, where G is a monograph, the *difference set* $D(s) = \{|s - t| : t \in S, t \neq s\}$ is the set of all possible edge labels associated with the vertex $\pi(s)$ in G.

2.2. Basic results

Lemma 5. *In a monograph, the vertex labeled with the rth largest non-negative signature value can be adjacent to at most $r - 1$ vertices labeled with negative signature values.*

Proof. Note that s_{n-r+1} is the rth largest positive signature value, and $r - 1$ vertices have signature values greater than s_{n-r+1}. Assume by contradiction that s_{n-r+1} has at least r neighbors with negative signature values s_i, for $1 \leq i \leq m$. For all such s_i, $|s_{n-r+1} - s_i| = s_{n-r+1} + |s_i| \in S^+$. Since G is a monograph, we have identified at least r distinct values in S^+ that are greater than s_{n-r+1}, a contradiction. ∎

Corollary 6. *The number of negative signature values in any monograph with n vertices is at most $n - 1 - deg(s_n)$.*

Proof. Apply Lemma 5 to the maximum signature value s_n, and we get that s_n can only be adjacent to vertices with positive signature values, of which there are $n - m - 1$ other than s_n. Thus,

$$\deg(s_n) \leq n - m - 1 \Rightarrow m \leq n - 1 - \deg(s_n).$$ ∎

Lemma 7 (The Maximum Element Lemma for Monographs)**.** *In a monograph, if 0 is not in the signature and s_n is adjacent to all vertices with positive signature values, then*

$$s_n = s_{m+1} + s_{n-1} = s_{m+2} + s_{n-2} = \cdots = s_{\lfloor \frac{m+n}{2} \rfloor} + s_{\lceil \frac{m+n}{2} \rceil}.$$

Proof. Consider the difference set $D(s_n)$ of s_n. In decreasing order,

$$D(s_n) = \{s_n - s_{m+1}, s_n - s_{m+2}, \ldots, s_n - s_{n-1}\}.$$

By Lemma 5, s_n is only adjacent to vertices with non-negative signature values. We also know $0 \notin S$. Thus, the $n - m - 1$ terms of $D(s_n)$ are distinct and contained in $S^+ \setminus \{s_n\}$. Note that $S^+ \setminus \{s_n\}$ has $n - m - 1$ positive signature
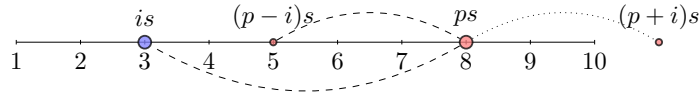
Figure 4. Example of one deletion from a complete graph signature. Illustrated is the case where $n = 10, i = 3$, and $p = 8$. For the vertex ps, if is is deleted, then the two dashed edges are lost, but the one dotted edge never existed. Thus, the degree of ps in the resulting graph is $n - 3$. This is Case 3 in the proof below $(p > \max(n - i, i).)$

values. Thus, there is a one-to-one correspondence between $D(s_n)$ and $S^+ \setminus \{s_n\}$, that is $D(s_n) = S^+ \setminus \{s_n\}$. In decreasing order,

$$S^+ \setminus \{s_n\} = \{s_{n-1}, s_{n-2}, \dots, s_{m+1}\}.$$

Since $D(s_n) = S^+ \setminus \{s_n\}$,

$$s_n - s_{m+1} = s_{n-1}, \quad s_n - s_{m+2} = s_{n-2} \dots .$$

The result now follows.   ■

2.3.   Arithmetic progressions

Recall that $S = [t]_s^n = \{ts, (t+1)s, \dots, (n-1+t)s\}$ is a signature for a proper monograph consisting of an arithmetic progression of n terms, beginning at ts and continuing in integer steps of size s. This section concerns the properties of graphs resulting from deleting members of $S = [1]_s^n$.

Lemma 8. *Let $S = [1]_s^n = \{s, 2s, \dots, ns\}$ be a signature for K_n. If the vertex of signature value is, for $1 \leq i \leq n$, is removed from K_n, the degree of ps, where $p \neq i$, $1 \leq p \leq n$ is*

$$\deg(ps) = \epsilon_i + \begin{cases} n - 4 & \text{if } p \in (i, n - i], \\ n - 2 & \text{if } p \in (n - i, i), \\ n - 3 & \text{otherwise}, \end{cases}$$

where

$$\epsilon_i = \begin{cases} 1 & \text{if } p = 2i, \ i \leq \lfloor \frac{n}{2} \rfloor, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. We will first consider the general case $p \neq 2i$. Before the deletion of is, $\deg(ps) = n - 1$ for all $1 \leq p \leq n$. The vertices whose degrees are potentially affected by the deletion of is have signature values $(p - i)s$ and $(p + i)s$. Note that $(p - i)s < ns$ and $(p + i)s > 0$ always hold. Thus, we will discuss the

relationship of $(p-i)s$ with 0, and $(p+i)s$ with ns to determine whether they are in the signature or affected by the deletion. We illustrate one case in Figure 4, but the reader is encouraged to envision how the other cases could be similarly illustrated.

Case 1. $p+i \leq n$ and $p-i > 0 \Rightarrow i < p \leq n-i$. Both $(p+i)s$ and $(p-i)s$ are in the signature. Thus, three edges adjacent to ps: $(is, ps), ((p-i)s, ps), ((p+i)s, ps)$ are deleted after the deletion of the vertex is and $\deg(ps) = n - 1 - 3 = n - 4$.

Case 2. $p+i \leq n$ and $p-i < 0 \Rightarrow p \leq \min(n-i, i)$. $(p+i)s$ is in the signature and $(p-i)s$ is not. Thus, two edges adjacent to ps: $(is, ps), ((p+i)s, ps)$ are deleted and $\deg(ps) = n - 1 - 2 = n - 3$.

Case 3. $p+i > n$ and $p-i > 0 \Rightarrow p > \max(n-i, i)$. $(p-i)s$ is in the signature and $(p+i)s$ is not. Two edges are deleted and $\deg(ps) = n - 1 - 2 = n - 3$. An example of this case is illustrated in Figure 4.

Case 4. $p+i > n$ and $p-i < 0 \Rightarrow n-i < p < i$. Neither $(p+i)s$ nor $(p-i)s$ is in the signature. Only the edge (is, ps) is deleted and $\deg(ps) = n-1-1 = n-2$.

If $p = 2i$, then $(p-i)s$ and is are the same vertex. That is, $(ps, (p-i)s)$ and (ps, is) are the same edge. Therefore, we overcounted the number of deleted edges by 1 and this defines ϵ_i. ∎

This result can be extended to the removal of two signature values, and we will use it in the proof of the main result.

Lemma 9. *Let* $S = [1]_s^n = \{s, 2s, \ldots, ns\}$ *be a signature for* K_n. *If vertices of signature value* is *and* js, *for* $1 \leq i < j \leq n$, *are removed from* K_n, *the degree of* ps, *where* $p \neq i, j$, $1 \leq p \leq n$ *is*

$$\deg(ps) = \epsilon_i + \epsilon_j + \gamma_{ij} + \beta_{ij}$$

$$+ \begin{cases} n-7 & \text{if } j < p \leq n-j, \\ n-6 & \text{if } \max(j, n-j) < p \leq n-i \text{ or } i < p \leq \min(n-j, j), \\ n-4 & \text{if } \max(i, n-i) < p < j \text{ or } n-j < p \leq \min(n-i, i), \\ n-3 & \text{if } n-i < p < i, \\ n-5 & \text{otherwise}, \end{cases}$$

where

$$\gamma_{ij} = \begin{cases} 2 & \text{if } p = i+j, \\ 0 & \text{otherwise}, \end{cases} \qquad \beta_{ij} = \begin{cases} 1 & \text{if } p = j-i, \\ 0 & \text{otherwise}, \end{cases}$$

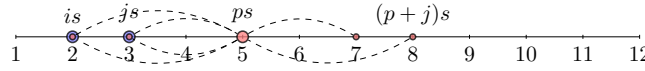*and* ϵ_i *and* ϵ_j *are defined as before.*

Figure 5. Example of two deletions from a complete graph signature. Illustrated is the case where $n = 12, i = 2, j = 3$, and $p = 5$. For the vertex ps, if is and js are deleted, then the six dashed edges are lost, but we have double-counted two of them. Thus, the degree of ps in the resulting graph is $n - 5$. This is Case 1 in the proof below, with $\gamma_{23} = 2$.
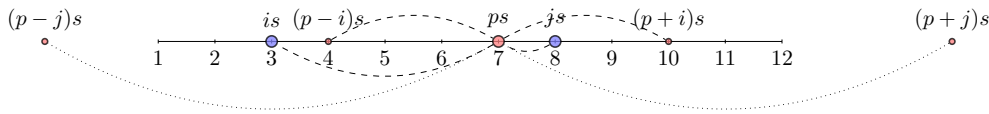

Figure 6. Example of two deletions from a complete graph signature. Illustrated is the case where $n = 12, i = 3, j = 8$, and $p = 7$. If is and js are deleted, then the four dashed edges are lost, but the two dotted edges never existed. Thus, the degree of ps in the resulting graph is $n - 5$. This is Case 9 in the proof below.

Proof. We will first consider the general case where $p \notin \{2i, 2j, j - i, j + i\}$. The vertices whose degree are potentially affected by the deletion of is, js have signature values of $(p - j)s, (p + j)s, (p - i)s$ and $(p + i)s$. Note that $(p - i)s < ns, (p - j)s < ns, (p + j)s > 0$ and $(p + i)s > 0$ always hold. Thus, we will discuss the relationship of $(p - i)s$ and $(p - j)s$ with 0, $(p + i)s$ and $(p + j)s$ with ns to determine whether these four values are in the signature or affected by the deletion. See Figures 5 and 6 for examples.

Case 1. $p - j > 0$ and $p + j \leq n \Rightarrow j < p \leq n - j$. Since $j > i$, we know $p - i > 0$ and $p + i \leq n$. Then $(p + j)s, (p - j)s, (p + i)s$ and $(p - i)s$ are all in the signature. Thus, six edges adjacent to ps: $(is, ps), ((p - i)s, ps), ((p + i)s, ps), (js, ps), ((p - j)s, ps), ((p + j)s, ps)$ are deleted and $\deg(ps) = n - 1 - 6 = n - 7$. An example is illustrated in Figure 5.

Case 2. $p - j > 0$, $p + j > n$ and $p + i \leq n \Rightarrow \max(j, \ n - j) < p \leq n - i$. Since $j > i$ and $p - j > 0$, we know $p - i > 0$. Then $(p - j)s, \ (p + i)s$ and $(p - i)s$ are in the signature. Thus, five edges adjacent to ps: $(is, ps), ((p - i)s, ps), ((p + i)s, ps), (js, ps), ((p - j)s, ps)$ are deleted and $\deg(ps) = n - 1 - 5 = n - 6$.

Case 3. $p - j > 0$, $p + j > n$ and $p + i > n \Rightarrow p > \max(j, \ n - i)$. Since $j > i$ and $p - j > 0$, we know $p - i > 0$. Then $(p - j)s$ and $(p - i)s$ are in the signature. Thus, four edges adjacent to ps: $(is, ps), ((p - i)s, ps), (js, ps), ((p - j)s, ps)$ are deleted and $\deg(ps) = n - 1 - 4 = n - 5$.

Case 4. $p + j \leq n$, $p - j \leq 0$ and $p - i > 0 \Rightarrow i < p \leq \min(j, \ n - j)$. Since $p + j \leq n$ and $j > i$, we know $p + i \leq n$. Then $(p - i)s, \ (p + j)s$ and $(p + i)s$ are

in the signature. Thus, five edges adjacent to ps: (is, ps), $((p-i)s, ps)$, (js, ps), $((p+j)s, ps)$, $((p-j)s, ps)$ are deleted and $\deg(ps) = n - 1 - 5 = n - 6$.

Case 5. $p + j \leq n$, $p - j \leq 0$ and $p - i \leq 0 \Rightarrow p < \min(i, \ n - j)$. Since $j > i$ and $p + j \leq n$, we know $p + i \leq n$. Then $(p+j)s$ and $(p+i)s$ are in the signature. Thus, four edges adjacent to ps: (is, ps), $((p+i)s, ps)$, (js, ps) and $((p+j)s, ps)$ are deleted and $\deg(ps) = n - 1 - 4 = n - 5$.

Case 6. $p - j < 0$, $p - i > 0$, $p + i > n \Rightarrow \max(i, n - i) < p < j$. Since $p + i > n$, we know $p + j > n$. Only $(p-i)s$ is in the signature. Thus, three edges adjacent to ps are deleted and $\deg(ps) = n - 1 - 3 = n - 4$.

Case 7. $p - i < 0$, $p + i > n \Rightarrow n - i < p < i$. Since $p + i > n$ and $p - i < 0$, we know $p + j > n$ and $p - j < 0$. Then, only two edges adjacent to ps: (is, ps), (js, ps) are deleted and $\deg(ps) = n - 1 - 2 = n - 3$.

Case 8. $p - i < 0$, $p + j > n$, $p + i \leq n \Rightarrow n - j < p \leq \min(i, n - i)$. Since $p - i \leq 0$, we know $p - j \leq 0$. Only $(p+i)s$ is in the signature. Thus, three edges adjacent to ps are deleted and $\deg(ps) = n - 1 - 3 = n - 4$.

Case 9. $p - j \leq 0$, $p - i > 0$, $p + j > n$, $p + i \leq n \Rightarrow \max(i, n - j) < p \leq \min(j, n - i)$. Two values $(p-i)s$ and $(p+i)s$ are in the signature. Thus, four edges adjacent to ps are deleted and $\deg(ps) = n - 1 - 4 = n - 5$. An example is illustrated in Figure 6.

Combining all the cases, we have the general result for $p \notin \{2i, 2j, j - i, j + i\}$. If $p = 2i, 2j$, we need to add ϵ_i, ϵ_j which are defined in the previous proof. If $p = i + j$, then $(p-i)s$ and js are the same vertex, and $(p-j)s$ and is are the same vertex. So we overcounted the number of deleted edges by 2 and this defines γ_{ij}. If $p = j - i$, then js and $(p-i)s$ are the same vertex, which defines β_{ij}. ∎

2.4.   Structure lemmas

Recall that s_1 is the smallest element in the proper monograph signature $S = S(G)$, and k is the codegree of G. Then [1] make the following observation:

Observation 10 (Fact 3.1). *The elements of S can be partitioned into at most $k + 1$ arithmetic progressions with common difference s_1. Furthermore, at least one such arithmetic progression has at least $\frac{n}{k+1}$ terms.*

Note that the arithmetic progressions can be as short as one element, and one of them begins with s_1. [1] continue by setting l equal to the number of terms in the longest such arithmetic progression.

Observation 11 (Fact 3.2). *Among the numbers $s_1, 2s_1, \ldots, (l - 1)s_1$, at most k are missing from S.*

While the previous observations hold for proper monographs only, it is natural to extend this logic to all monographs. The fact that for any signature values $a > 0$ and $b < 0$, an edge exists between them if and only if $|a - b| = a + |b| \in S$ leads to the following companion observation for negative signature values.

Observation 12. *For each negative signature value $s_i \in S$, where $\deg(s_i) \neq 0$, the elements of S^+ can be partitioned into at most k arithmetic progressions with common difference $|s_i|$. Furthermore, at least one such progression has at least $\frac{n}{k}$ terms.*

Proof. Consider any maximal arithmetic progression with common difference $|s_i|$, where $a \geq 0$ is the largest member of this arithmetic progression. Then $|a - s_i| = a + |s_i|$ is not in S, and hence a is not adjacent to s_i. Thus, the number of such arithmetic progressions is no more than the codegree k since k is defined as the maximum number of vertices which any vertex, including s_i, is not adjacent to. ∎

Observation 13. *If s_1 is the only negative signature value in S and $0 \notin S$, then among the numbers $|s_1|, 2|s_1|, \ldots, (l-1)|s_1|$, at most $k - 1$ are missing from S.*

Proof. Assume that s_1 is in the longest arithmetic progression with l terms and common difference $|s_1|$. Then, the l-term arithmetic progression is $\{s_1, 0, |s_1|, \ldots, (l-2)|s_1|\}$. This contradicts that $0 \notin S$, and so s_1 is not in the l-term arithmetic progression. Suppose a is the smallest term of an l-term arithmetic progression $a, a + |s_1|, \ldots, a + (l-1)|s_1|$. By definition of codegree k, vertex a fails to be adjacent to at most k vertices, not including a, but specifically including s_1 which is not in the l-term arithmetic progression. Each vertex in this sequence which is adjacent to a implies the existence of the label $a + t|s_1| - a = t|s_1|$. ∎

3. M_6 IS THE SMALLEST NON-AUTOGRAPH

Observe that vertices in M_6 with the same degree are neighbors and that all vertices of M_6 have degree 3 or 4. Furthermore, the largest clique contained in M_6 is of size 4. We first eliminate the possibility that zero is in a signature for M_6.

Lemma 14. *A set containing 0 cannot be the signature for M_6.*

Proof. Let $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$. Assume by contradiction that $0 \in S$ and S is a signature for M_6. By Corollary 6, M_6 has at most $m = 2$ negative signature values.

Case 1. $m = 1$. That is, $s_1 < 0$ and $s_2 = 0$. Note that by Observation 3, $\deg(s_2) = 4$, and in particular, s_2 is adjacent to s_3, s_4, s_5, and s_6. Thus, s_2 is not
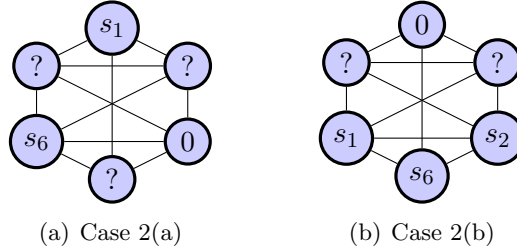
(a) Case 2(a)   (b) Case 2(b)

Figure 7. Possible configurations for M_6 discussed in Lemma 14.

adjacent to s_1, and so the only possible signature has the configuration depicted in Figure 7(a). Since s_2 and s_1 are not connected, it follows that $|s_2 - s_1| = s_2 + |s_1| = |s_1| \notin S$. But then, regardless of which vertex is labeled s_5, it must be adjacent to both s_1 and s_6. The first condition requires

$$|s_5 - s_1| = s_5 + |s_1| \in S \Rightarrow s_5 + |s_1| = s_6 \Rightarrow s_6 - s_5 = |s_1|.$$

But the second condition then implies $|s_6 - s_5| = |s_1| \in S$, contradicting the fact that $|s_1| \notin S$.

Case 2. $m = 2$. In this case $s_1 < s_2 < 0$ and $s_3 = 0$. It now follows from Observation 3 that $\deg(s_3) = 3$ and that in particular, s_3 is not adjacent to s_1 and s_2. However, it follows from Lemma 5 that $\deg(s_6) = 3$, and that s_6 is similarly not adjacent to s_1 and s_2. But this violates the structure of M_6, since then s_1 and s_2 are *both* non-adjacent to s_3 and s_6, and no such vertex exists in M_6. This impossible situation is depicted in Figure 7(b).   ∎

Lemma 15. *A set P that contains an arithmetic progression of 5 non-negative numbers and a negative number is not a signature for M_6.*

Proof. Assume by contradiction that $P = \{p, t, t + s, t + 2s, t + 3s, t + 4s\}$ is a signature for M_6, where $t > 0$, $s > 0$ and $p < 0$. We consider two cases based on whether s is in P or not.

Case 1. $s \in P$. Since $t + s \geq s > 0$, s can only equal t or $t + s$. If $s = t$, $P = \{p, s, 2s, 3s, 4s, 5s\}$. If $s = t + s$, $P = \{p, 0, s, 2s, 3s, 4s\}$. The vertices with signature values $s, 2s, 3s, 4s, 5s$ (if $s = t$) or $0, s, 2s, 3s, 4s$ (if $s = t + s$) form a clique of size 5 in M_6, which contradicts that the clique number of M_6 is 4.

Case 2. $s \notin P$. Consider the difference set of $t + 2s$,

$$D(t + 2s) = \{t + 2s - p, 2s, s, s, 2s\}.$$

Since $s \notin P$, we must have that $2s \in P$ since otherwise $\deg(t + 2s) \leq 1$, and M_6 does not contain such a vertex. Thus, $\deg(t + 2s) = 3$. Similarly, consider the

difference sets of $t + 3s$ and $t + s$,

$$D(t + 3s) = \{t + 3s - p, 3s, 2s, s, s\},$$

$$D(t + s) = \{t + s - p, s, s, 2s, 3s\}.$$

Since $s \notin P$, $\deg(t + 3s)$ and $\deg(t + s)$ cannot be 4. Then, three vertices would have degree 3: $t + 3s, t + 2s, t + s$. This contradicts that only two vertices of M_6 have degree 3.

Hence, P is not a signature for M_6. ■

Theorem 16. M_6 *is a non-autograph.*

Proof. Assume by contradiction that M_6 has a signature S. As noted in Section 2, M_6 is a monograph.

Since all vertices of M_6 have degree 3 or 4, M_6 has at most two negative signature values by Corollary 6. We consider three cases based on the number of negative signature values.

Case 1. $m = 2$. Let $s_1 < s_2 < 0 < s_3 < s_4 < s_5 < s_6$. Since s_6 is only adjacent to vertices with positive signature values, we know $\deg(s_6) = 3$ and $\deg(s_1) = \deg(s_2) = 4$, since s_1, s_2 are not adjacent to s_6 (see Figure 8). Then, s_1 and s_2 are both adjacent to s_5. Since $s_5 - s_1$ and $s_5 - s_2$ are both greater than s_5 and are both in the signature, we have that

$$s_5 - s_1 = s_6 \quad \text{and} \quad s_5 - s_2 = s_6.$$
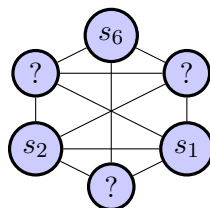
This contradicts that $s_1 < s_2$.

Figure 8. Possible configuration for M_6 with two negative signatures values (s_1 and s_2).

Case 2. $m = 1$. Let $s_1 < 0 < s_2 < s_3 < s_4 < s_5 < s_6$. By Lemma 5, s_6 is not adjacent to s_1. We consider two cases based on $\deg(s_1)$.

Case (a) $\deg(s_1) = 4$. Since s_1 is not adjacent to s_6, we know $\deg(s_6) = 3$. Note that s_1 is adjacent to s_2, s_3, s_4 and s_5. Thus, $s_5 - s_1 > s_5$ has to be s_6 and $s_6 - s_5 = |s_1|$ (see Figure 9(a)). Since $s_4 < s_4 - s_1 < s_5 - s_1 = s_6$, we have that $s_4 - s_1 = s_5$, and thus $s_5 - s_4 = |s_1|$. Since $s_3 < s_3 - s_1 < s_4 - s_1 = s_5$, we have
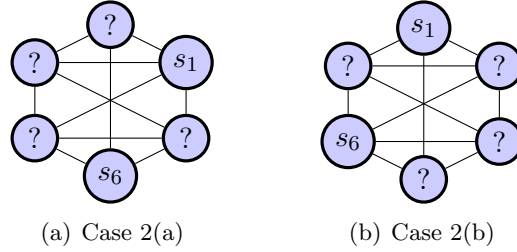
(a) Case 2(a)   (b) Case 2(b)

Figure 9. Possible configurations for M_6 in the case where s_1 is the only negative signature value. At left, $\deg(s_1) = 4$, while at right, $\deg(s_1) = 3$.

that $s_3 - s_1 = s_4$, hence $s_4 - s_3 = |s_1|$. Since $s_2 < s_2 - s_1 < s_3 - s_1 = s_4$, we have that $s_2 - s_1 = s_3$, and thus $s_3 - s_2 = |s_1|$. Therefore, s_2, s_3, s_4, s_5, s_6 form an arithmetic progression of length 5 with common difference $|s_1|$, that is,

$$s_3 - s_2 = s_4 - s_3 = s_5 - s_4 = s_5 - s_4 = -s_1.$$

By Lemma 15, S is not a signature for M_6.

Case (b) $\deg(s_1) = 3$. Since s_1 is not adjacent to s_6, we know $\deg(s_6) = 4$ and s_6 is adjacent to all vertices with positive signature values. By Lemma 7,

$$s_6 = s_2 + s_5 = s_3 + s_4 \Rightarrow s_6 - s_5 = s_2, \ \ s_5 - s_4 = s_3 - s_2.$$

We consider four cases based on the other vertex to which s_1 is not adjacent (see Figure 9(b)).

(i) s_5 is not adjacent to s_1. Then, $\deg(s_5) = 4$. Consider the intersection of the difference set of s_5 and S,

$$D(s_5) \cap S = \{s_6 - s_5, s_5 - s_2, s_5 - s_3, s_5 - s_4\}.$$

We already know that $s_6 - s_5 = s_2$. Note that all other elements of $D(s_5) \cap S$ are less than s_5 and greater than s_1, and thus can only take values of s_2, s_3 or s_4. Therefore, we can find a one-to-one correspondence between sets $D(s_5) \cap S$ and $\{s_2, s_3, s_4\}$. Namely,

$$s_5 - s_2 = s_4, \quad s_5 - s_3 = s_3, \quad \text{and} \ \ s_5 - s_4 = s_2.$$

Since $s_5 - s_4 = s_2$, we have that $s_3 - s_2 = s_2$, thus $s_3 = 2s_2$. Then $s_5 = 2s_3 = 4s_2$, $s_6 = s_5 + s_2 = 5s_2$ and $s_4 = s_5 - s_2 = 3s_2$. Therefore, s_2, s_3, s_4, s_5, s_6 form an arithmetic progression of length 5 with common difference s_2. By Lemma 15, S is not a signature for M_6.

(ii) s_4 is not adjacent to s_1. Then, $\deg(s_4) = 4$. Consider the intersection of the difference set of s_4 and S,

$$D(s_4) \cap S = \{s_4 - s_3, s_4 - s_2, s_6 - s_4, s_5 - s_4\}.$$

Note that $s_4 - s_3$ and $s_4 - s_2$ are both less than s_4 and greater than s_1. Thus, we must have $s_4 - s_3 = s_2$ and $s_4 - s_2 = s_3$. We already know that $s_6 - s_4 = s_3$. Then, $s_5 - s_4 < s_6 - s_4$ can only be s_2. Since $s_5 - s_4 = s_3 - s_2$, $s_3 = 2s_2$. Thus, $s_4 = 3s_2$, $s_5 = 4s_2$ and $s_6 = s_4 + s_3 = 5s_2$. Note that s_2, s_3, s_4, s_5, s_6 form an arithmetic progression with common common difference s_2. By Lemma 15, S is not a signature for M_6.

(iii) s_3 is not adjacent to s_1. Then, $\deg(s_3) = 4$. Consider the intersection of the difference set of s_3 and S,

$$D(s_3) \cap S = \{s_3 - s_2, s_4 - s_3, s_5 - s_3, s_6 - s_3\}.$$

Since $s_3 - s_2 < s_3$, $s_3 - s_2$ has to be s_2. We already know that $s_6 - s_3 = s_4$. Note that both $s_5 - s_3$ and $s_4 - s_3$ are less than $s_6 - s_3 = s_4$ and greater than $s_3 - s_2 = s_2$. Then $s_5 - s_3$ and $s_4 - s_3$ both are s_3, which contradicts that $s_4 \neq s_5$.

(iv) s_2 is not adjacent to s_1. Then, $\deg(s_2) = 4$. Consider the intersection of the difference set of s_2 and S,

$$D(s_2) \cap S = \{s_3 - s_2, s_4 - s_2, s_5 - s_2, s_6 - s_2\}.$$

We already know that $s_6 - s_2 = s_5$. All other elements of $D(s_2) \cap S$ are less than s_5 and greater than s_1, and thus can only take values of s_2, s_3 or s_4. Therefore, we can find a one-to-one correspondence between sets $D(s_2) \cap S$ and $\{s_2, s_3, s_4\}$. Namely,

$$s_3 - s_2 = s_2, \quad s_4 - s_2 = s_3, \quad \text{and} \quad s_5 - s_2 = s_4.$$

Thus, $s_3 = 2s_2$, $s_4 = 3s_2$, $s_5 = 4s_2$ and $s_6 = s_5 + s_2 = 5s_2$. Therefore, s_2, s_3, s_4, s_5, s_6 form an arithmetic progression of length 5 with common difference s_2. By Lemma 15, S is not a signature for M_6.

Case 3. $m = 0$. Let s denote the smallest signature value in S. By Observation 10, S can be partitioned into at most three maximal arithmetic progressions of common difference s. S cannot consist of one arithmetic progression with common difference s since $[1]_s^n$ is a signature for the complete graph. Thus, we consider two cases.

Case (a) Three arithmetic progressions. Let the three progressions be $[1]_s^{l_0}$, $[a_1]_s^{l_1}$, and $[a_2]_s^{l_2}$, where we assume without loss of generality that $s < a_1 < a_2$. Note that s is not adjacent to either a_1 or a_2 since $[a_1]_s^{l_1}$ and $[a_2]_s^{l_2}$ are maximal progressions. Thus, $\deg(s) = 3$ and $\deg(a_1) = \deg(a_2) = 4$. Note that s is not in

the complete subgraph on 4 vertices. Then, the length of $[1]_s^{l_0}$, l_0, must be less than 4. Moreover, l_1 and l_2 are less than 4. Assume by contradiction that $l_i = 4$, where $i = 1, 2$. Then, the other two arithmetic progressions have length 1, and there exists an arithmetic progression of four terms: $\{a_i, a_i + s, a_i + 2s, a_i + 3s\}$. Since a_i is not adjacent to s only, $a_i + s - a_i, a_i + 2s - a_i, a_i + 3s - a_i \in S$, thus $s, 2s, 3s \in S$, which contradicts that $l_0 = 1$. Therefore, l_0, l_1, l_2 are all less than 4. We discuss three cases of l_0.

(i) $l_0 = 1$. Since l_1, l_2 are both less than 4, $S = \{s, a_1, a_1 + s, a_2, a_2 + s, a_i + 2s\}$, where i is 1 or 2. However, $|a_i + 2s - a_i| = 2s \notin S$, contradicting that a_i is adjacent to every vertex besides s.

(ii) $l_0 = 3$. Then $S = \{s, 2s, 3s, a_1, a_2, a_i + s\}$, where $i = 1$ or 2. Since $2s, 3s$ are adjacent, $2s$ and $3s$ both have degree 4, and $\deg(a_i + s) = 3$. Note that $a_i + s$ is not adjacent to $2s, 3s$. Since $a_1 - 2s < a_1$ and a_1 is adjacent to $2s$, then $a_1 - 2s \in \{s, 2s, 3s\}$. Since $\{s, 2s, 3s\} \in S$, $a_1 = 5s$. If $a_i + s = a_1 + s = 6s$, then $a_i + s$ is adjacent to $3s$, a contradiction. Thus, $l_1 = 1$, $l_2 = 2$ and $S = \{s, 2s, 3s, 5s, a_2, a_2 + s\}$. Since $a_2 - 2s < a_2 \in S$, $a_2 - 2s = a_1 = 5s$, hence $a_2 = 7s$. However, $a_2 - 3s = 4s \notin S$, which contradicts that $3s$ is adjacent to every vertex besides $a_2 + s$.

(iii) $l_0 = 2$. Note that a_1 is adjacent to $2s$. We discuss three cases based on l_1, l_2.

A. $l_1 = 2$, $l_2 = 2$. Then $S = \{s, 2s, a_1, a_2, a_1 + s, a_2 + s\}$. Since a_2 is adjacent to every vertex besides s, $a_2 - a_1, a_2 + s - a_1, a_2 - (s + a_1) \in S$. They form an arithmetic progression of length 3 with common difference s, contradicting that $l_0 = l_1 = l_2 = 2$.

B. $l_1 = 1$, $l_2 = 3$. Then $S = \{s, 2s, a_1, a_2, a_2 + s, a_2 + 2s\}$, and $a_2 - a_1, a_2 + s - a_1, a_2 + 2s - a_1 \in S$. They form an arithmetic progression of length 3 with common difference s, and S only contains one arithmetic progression of three terms: $\{a_2, a_2 + s, a_2 + 2s\}$. Thus, $a_2 - a_1 = a_2$, a contradiction.

C. $l_1 = 3$, $l_2 = 1$. Then $S = \{s, 2s, a_1, a_2, a_1 + s, a_1 + 2s\}$ and $a_1 + s - a_2, a_1 + 2s - a_2 \in S$. We know $\{s, 2s\}$ is the only arithmetic progression of two terms in S. Then $a_1 + s - a_2 = s$, a contradiction.

Case (b) Two arithmetic progressions. Let the two maximal progressions be $[1]_s^{l_0}, [a_1]_s^{l_1}$, that is,

$$S = \{s, 2s, \ldots, l_0 s\} \cup \{a_1, a_1 + s, \ldots, a_1 + (l_1 - 1)s\}.$$

Consider the difference set of s,

$$D(s) = \{a_1 - s, a_1, a_1 + s, \ldots, a_1 + (l_1 - 2)s\} \cup \{s, 2s, \ldots, (l_0 - 1)s\}.$$

Since $[a_1]_s^{l_1}$ is a maximal arithmetic progression, $a_1 - s$ cannot be in the signature. Note that all terms in $D(s)$ besides $a_1 - s$ are in the signature, and $\deg(s) = 4$. Since s is only non-adjacent to a_1, $\deg(a_1) = 3$. We discuss five cases based on l_0 and l_1.

(i) $l_0 = 1$ $l_1 = 5$. Then, $S = \{s, a_1, a_1 + s, a_1 + 2s, a_1 + 3s, a_1 + 4s\}$. Consider the difference set of $a_1 + 2s$,

$$D(a_1 + 2s) = \{s, s, 2s, 2s, a_1 + s\}.$$

Thus, $\deg(a_1 + 2s) = 3$. However, $a_1 + 2s$ is not adjacent to a_1, which contradicts that a_1 and $a_1 + 2s$ both have degree 3.

(ii) $l_0 = 2$, $l_1 = 4$. Then, $S = \{s, 2s, a_1, a_1 + s, a_1 + 2s, a_1 + 3s\}$. Consider the difference set of $a_1 + 2s$,

$$D(a_1 + 2s) = \{s, s, 2s, a_1, a_1 + s\}.$$

All elements of $D(a_1 + 2s) \in S$, which contradicts that all vertices have degree 3 or 4.

(iii) $l_0 = 3$, $l_1 = 3$. Then, $S = \{s, 2s, 3s, a_1, a_1 + s, a_1 + 2s\}$. Consider the difference set of $2s$,

$$D(2s) = \{s, s, a_1 - 2s, a_1 - s, a_1\}.$$

We know that $a_1 - s \notin S$. If $a_1 - 2s \notin S$, then $\deg(2s) = 3$. However, $2s$ is not adjacent to a_1, the other vertex with degree 3, a contradiction. Thus, $a_1 - 2s \in S$. Since $a_1 - 2s < a_1$ and $a_1 > 4s$, $a_1 - 2s$ can only be $3s$. Consider the difference set of $a_1 = 5s$,

$$D(a_1) = \{4s, 3s, 2s, s, 2s\}.$$

Only $4s$ in $D(a_1)$ is not in S, contradicting that $\deg(a_1) = 3$.

(iv) $l_0 = 4$, $l_1 = 2$. Then, $S = \{s, 2s, 3s, 4s, a_1, a_1 + s\}$. Consider the difference set of $2s$. Following the same argument, we deduce that $a_1 - 2s$ can only be $4s$, thus $a_1 = 6s$. Consider the difference set of $a_1 = 6s$,

$$D(a_1) = \{5s, 4s, 3s, 2s, s\}.$$

Only $5s$ in $D(a_1)$ is not in S, contradicting that $\deg(a_1) = 3$.

(v) $l_0 = 5$, $l_1 = 1$. Then, $S = \{s, 2s, 3s, 4s, 5s, a_1\}$. Consider the different sets of $2s, 3s, 4s, 5s$:

$$D(2s) = \{s, s, 2s, 3s, a_1 - s\},$$
$$D(3s) = \{2s, s, s, 2s, a_1 - 3s\},$$
$$D(4s) = \{3s, 2s, s, s, a_1 - 4s\},$$
$$D(5s) = \{4s, 3s, 2s, s, a_1 - 5s\}.$$

Thus, $s, 2s, 3s, 4s, 5s$ all have degree 4, which contradicts that two vertices have degree 4.

We have exhausted all possible signatures for M_6, without finding one. This completes the proof. ∎

4. G_n Are Non-Autographs

Let $G = \overline{2C_{\frac{n}{2}}}$, for some even $n \geq 8$. That is, let G_n consist of the graph formed by taking the complement of two disjoint cycles of length $n/2$. A symmetric depiction of G_n for $n = 10$ is shown in Figure 3, with which we also include a drawing of its complement for clarity.

We begin by making three observations about the structure of G_n that will prove useful later on.

Observation 17. *The clique number of G_n is at most $\frac{n}{2}$, where n is even.*

Proof. Consider cycles of length $\frac{n}{2}$ in $\overline{G_n}$. If $4 \mid n$, the maximum clique includes every other vertex in each cycle. If $4 \nmid n$, each cycle in $\overline{G_n}$ has an odd length, and so the maximum clique has $\frac{n}{2} - 1$ vertices. In both cases, the clique number is no more than $\frac{n}{2}$. ∎

Observation 18. *The size of the largest independent set in G_n is 2, where n is even.*

Proof. Note that $\overline{G_n}$ contains two cycles of length greater than or equal to 5. Every vertex is non-adjacent to exactly two vertices. Those two vertices are always adjacent to avoid forming a 3-cycle in $\overline{G_n}$. ∎

Observation 19. *The length of the longest path in $\overline{G_n}$ is $\frac{n}{2} - 1$, where n is even.*

Proof. Take one of the cycles in $\overline{G_n}$ and remove one edge. ∎

We first prove a helpful lemma based on our previous results on arithmetic progressions.

Lemma 20. *A set containing 0 cannot be the signature for G_n for some even $n \geq 8$.*

Proof. Note that $\delta(G_n) = n - 3$, and the codegree of G_n is $k = 2$. By Corollary 6 we have that there are at most two negative signature values. Assume by contradiction that $0 \in S = \{s_1, s_2, \ldots, s_n\}$ and S is the signature for G_n. By Observation 3, it is clear that if S contains 0, it must contain exactly two negative signature values, since otherwise $\deg(0) > n - 2$. Thus, since the vertex labeled

with 0 is adjacent to all $n - 3$ vertices with positive signature values, it is not adjacent to s_1 and s_2. Then $s_1 < s_2 < 0$, $s_3 = 0$, and $0 < s_4 < s_5 < \cdots < s_n$. By Lemma 5, s_n is also not adjacent to s_1 and s_2. Thus, s_1, s_n, s_2 and s_3 form a 4-cycle in $\overline{G_n}$. This is only possible when $n = 8$, as shown in Figure 10. In this case, s_7 is adjacent to both s_1 and s_2. Hence,

$$s_7 - s_1 > s_7 \in S, \qquad s_7 - s_2 > s_7 \in S \Rightarrow s_7 - s_1 = s_7 - s_2 = s_8.$$
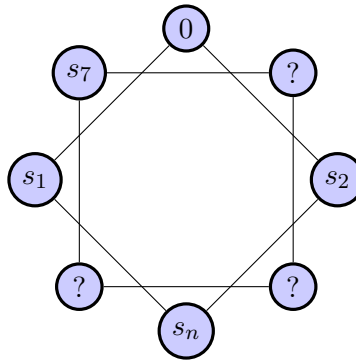
This contradicts the fact that G_n is a monograph.   ∎

Figure 10. $\overline{G_8}$: In G_8, s_7 is adjacent to both s_1 and s_2, creating the impossible situation in which there are two distinct signature values greater than s_7.

Lemma 21. *The set $S = \{s, 2s, \ldots, ns\} \setminus \{is\}$ is not a signature for G_{n-1}, where $n \geq 9$ and n is odd.*

Proof. We can immediately apply Lemma 8 to obtain the degree of vertices in the monograph generated by S. Note that every vertex of G_{n-1} has degree $n - 4$. Only for $p \in (i, n - i]$ and $p \neq 2i$, $\deg(ps) = n - 4$. Clearly, $(i, n - i] \setminus \{2i\}$ containing $n - 2i - 1$ terms does not include all of $[1, n] \setminus \{i\}$ containing $n - 1$ terms. Therefore, some vertices in S do not have degree $n - 4$, and S cannot be a signature for G_{n-1}.   ∎

Lemma 22. *The signature $S = \{s, 2s, \ldots, ns\} \setminus \{is, js\}$ for some $i < j$ is not a signature for G_{n-2}, where $n \geq 10$ and n is even.*

Proof. Assume by contradiction that S is a signature for G_{n-2}. We apply Lemma 9 to obtain the degree of vertices in the monograph generated by S. Since all vertices of G_{n-2} have degree $n - 5$, there does not exist p such that $\max(i, n - i) < p < j$ or $n - j < p \leq \min(n - i, i)$ or $n - i < p < i$, since this would already imply too large of a degree for ps. Thus, $n - i \geq j \Rightarrow n \geq i + j$ if all vertices have degree $n - 5$.
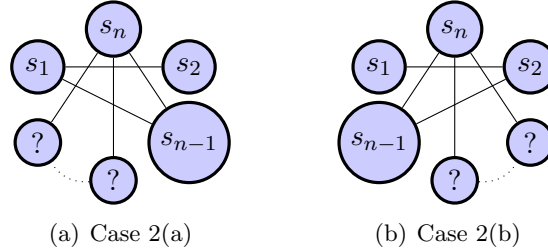
(a) Case 2(a)   (b) Case 2(b)

Figure 11. Possible configurations for G_n with $m = 2$. At left, s_{n-1} is adjacent to s_1 but not s_2. At right, the opposite.

If there exists p such that $j < p \leq n - j$, then there must exist only one such value p in $(j, n - j]$ and $\gamma_{ij} = 2$ for the vertex with signature value ps so that $\deg(ps) = n - 5$. Therefore, it must be the case that j and $n - j$ are consecutive integers, and in particular, $j + 1 = n - j$. Thus $p = i + j = j + 1 = n - j$, and hence $n = 2j + 1$, which contradicts that n is even. Therefore, there does not exist p such that $j < p \leq n - j$, and $\gamma_{ij} = 0$ for all vertices. Thus, $i + j > n$, which contradicts the previous conclusion that $n \geq i + j$.   ■

Our main result follows.

Theorem 23. *Let* $G_n = \overline{2C_{\frac{n}{2}}}$, *for some even* $n \geq 8$. *Then* G_n *is not an autograph.*

Proof. First, note that no two vertices in G_n have the same set of neighbors, and so by Observation 4, G_n is a monograph. Next, note that $\delta(G_n) = n - 3$, and the codegree of G_n is $k = 2$. It follows from Corollary 6 that $m \leq 2$. We thus consider three cases.

Case 1. $m = 2$. Let $s_1 < s_2 < 0$. Note that the maximum signature value s_n is adjacent to all vertices with positive signature values, namely $s_3, s_4, \ldots, s_{n-1}$. By Lemma 7,

$$s_n = s_3 + s_{n-1} \quad \Rightarrow \quad s_n - s_{n-1} = s_3.$$

Since s_n is not adjacent to either s_1 or s_2 and s_1, s_2 have different neighborhoods, it follows that s_1 and s_2 are adjacent. Likewise, s_{n-1} is adjacent to either s_1 or s_2, but not both. Consider two cases where s_{n-1} is adjacent to s_2 or s_1, as depicted in Figure 11.

Case (a). s_{n-1} is adjacent to s_1. Then $s_{n-1} - s_1 > s_{n-1}$ must be s_n. This implies that $s_{n-1} - s_1 = s_n$, and so $s_n - s_{n-1} = -s_1$. But now,

$$s_2 - s_1 = s_2 + s_n - s_{n-1} < s_n - s_{n-1} = s_3.$$

Since s_3 is the smallest positive signature value, $s_2 - s_1 < s_3$ indicates that $s_2 - s_1 \notin S$, contradicting that s_2 and s_1 are adjacent.

Case (b). s_{n-1} is adjacent to s_2. Then,

$$s_{n-1} - s_2 = s_n.$$

Since $s_n = s_3 + s_{n-1}$ by Lemma 7, it follows that $s_2 = -s_3$.

By Observation 12, $\{s_3, s_4, \ldots, s_n\}$ can be partitioned into at most two arithmetic progressions with common difference $|s_2| = s_3$. If $\{s_3, s_4, \ldots, s_n\}$ is an arithmetic progression with common difference s_3, the graph would contain a complete subgraph of $n - 2$ vertices, contradicting that the clique number of G_n cannot exceed $n/2$ (Observation 17). Thus, $\{s_3, s_4, \ldots, s_n\}$ consists of two arithmetic progressions with common difference s_3. Let the two arithmetic progressions be

$$\{s_3, 2s_3, \ldots, l_0 s_3\} \text{ and } \{a_1, a_1 + s_3, \ldots, s_n\},$$

where $a_1 \neq (l_0 + 1)s_3$. Note that s_1 is adjacent to all vertices other than s_n and s_{n-1}. In particular, s_1 is adjacent to $l_0 s_3$. Then, $l_0 s_3 - s_1 = (l_0 + 2)s_3$ is in the signature and $(l_0 + 2)s_3 \in \{a_1, \ldots, s_n\}$. Thus, $\{a_1, \ldots, s_n\}$ consists of multiples of s_3 and $a_1 > (l_0 + 1)s_3$.

Since $|s_1| > |s_2| = s_3$, $|s_1| > s_n - s_{n-1}$, and hence $s_{n-1} + |s_1| > s_n$. Thus, $|s_{n-2} - s_1| = s_{n-2} + |s_1| = s_{n-1} - s_3 + |s_1| > s_n - s_3 = s_{n-1}$. Since s_{n-2} is adjacent to s_1, $s_{n-2} + |s_1|$ must be s_n. Therefore,

$$s_n - s_{n-2} = |s_1| = 2s_3.$$

Since $a_1 > (l_0 + 1)s_3$, $a_1 + s_3 > (l_0 + 2)s_3$ and all terms in $\{a_1, a_1 + s_3, \ldots, s_n\}$ except for a_1 are greater than $(l_0 + 2)s_3$. It follows that $a_1 = (l_0 + 2)s_3$. Therefore, the signature S is

$$S = \{-2s_3, -s_3\} \cup \{s_3, 2s_3, \ldots, l_0 s_3\} \cup \{(l_0 + 2)s_3, (l_0 + 3)s_3, \ldots, s_n\}.$$

Consider the difference set of s_3,

$$D(s_3) = \{3s_3, 2s_3\} \cup \{s_3, 2s_3, \ldots, (l_0 - 1)s_3\}$$

$$\cup \{(l_0 + 1)s_3, (l_0 + 2)s_3, \ldots, s_n - s_3\}.$$

Observe that in $D(s_3)$ only $(l_0 + 1)s_3$ is not in the signature. Thus, $\deg(s_3) = n - 2$, which contradicts that every vertex in G_n has degree $n - 3$.

Case 2. $m = 1$. Let $s_1 < 0$. Observation 12 implies that S consists of s_1 and at most two arithmetic progressions with common difference $|s_1|$. Two cases are discussed based on the number of arithmetic progressions in S. Recall that l denotes the length of the longest arithmetic progression.

Case (a) One arithmetic progression. Then,

$$S = \{s_1\} \cup \{s_2, s_2 + |s_1|, \ldots, s_2 + (l - 1)|s_1|\},$$

$$D(s_1) = \{s_2 + |s_1|, s_2 + 2|s_1|, \ldots, s_2 + (l-1)|s_1|, s_2 + l|s_1|\}.$$

In $D(s_1)$, only $s_2 + l|s_1| \notin S$ and s_1 is only non-adjacent to $s_2 + (l-1)|s_1|$, contradicting that every vertex of G_n has degree $n-3$.

Case (b) Two arithmetic progressions. By Observation 13, at most one of $\{|s_1|, 2|s_1|, \ldots, (l-1)|s_1|\}$ is not in S. If $|s_1| \notin S$, we suppose the two arithmetic progressions of S are $\{2|s_1|, 3|s_1|, \ldots, (l_1+1)|s_1|\}$ and $\{s_n - (l_2-1)|s_1|, \ldots, s_n - |s_1|, s_n\}$. Since we assume $|s_1| \notin S$, $\{2|s_1|, 3|s_1|, \ldots, (l_1+1)|s_1|\}$ and $\{s_1, s_n - (l_2-1)|s_1|, \ldots, s_n - |s_1|, s_n\}$ form two paths in $\overline{G_n}$, starting with $2|s_1|, s_1$ and ending with $(l_1+1)|s_1|, s_n - (l_2-1)|s_1|$, respectively. This contradicts that $\overline{G_n}$ consists of two cycles. Therefore, $|s_1| \in S$ and $|s_1|$ is the smallest term in one arithmetic progression.

Suppose the smallest term of the other arithmetic progression is s_j. If no element from $\{|s_1|, 2|s_1|, \ldots, (l_1-1)|s_1|\}$ is missing, we suppose the length of the other arithmetic progressions is l_2. Thus,

$$S = \{s_1\} \cup \{|s_1|, 2|s_1|, \ldots, (l-1)|s_1|\}$$
$$\cup \{s_j, s_j + |s_1|, \ldots, s_j + (l_2-1)|s_1|\},$$
$$D(|s_1|) = \{2|s_1|\} \cup \{|s_1|, 2|s_1|, \ldots, (l-2)|s_1|\}$$
$$\cup \{|s_j - |s_1||, s_j, s_j + |s_1|, \ldots, (l_2-2)|s_1|\}.$$

In $D(|s_1|)$, only $|s_j - |s_1||$ may not be in S and $\deg(|s_1|) \geq n-2$. This contradicts every vertex of G_n has degree $n-3$.

Thus, exactly one member of $\{|s_1|, 2|s_1|, \ldots, (l-1)|s_1|\}$ is not in S. Let $t|s_1|$ be that value. Since we know $|s_1| \in S$, it remains to check two possibilities.

Case (a) $2 \leq t \leq l-2$. In this case

$$S = \{s_1\} \cup \{|s_1|, 2|s_1|, \ldots, (t-1)|s_1|\} \cup \{(t+1)|s_1|, (t+2)|s_1|, \ldots, n|s_1|\}.$$

This is also not a signature for G_n, since the vertex labeled $(n-1)|s_1|$ is adjacent to every other vertex except for $(n-1-t)|s_1|$. This contradicts that every vertex in G_n has degree $n-3$.

Case (b) $t = l-1$. In this case there exists a maximal arithmetic progression of length exactly $l-2$, namely $\{|s_1|, \ldots, (l-2)|s_1|\}$. By definition of l, there also exists a maximal arithmetic progression of length l with common difference $|s_1|$. By Observation 3, $0 \notin S$, and so s_1 is not in any arithmetic progression with common difference $|s_1|$. The first term of the arithmetic progression of length l is greater than s_1, and its last term is greater than $(l-2)|s_1|$. Note that the two arithmetic progressions are disjoint. If they have a common term, then all terms following the common term are the same since both arithmetic progressions have common difference $|s_1|$. Then $\{|s_1|, \ldots, (l-2)|s_1|\}$ can be extended to the last

term of the l-term arithmetic progression, and would not be a maximal arithmetic progression. Thus, we must have that $l+l-2 = n-1$, thus $l = \frac{n+1}{2}$, contradicting that n is even.

Case 3. $m = 0$. Let s denote the smallest signature value in S. By Observation 10, S can be partitioned into at most three maximal arithmetic progressions of common difference s, one of which begins with s. S cannot consist of one arithmetic progression with common difference s since $[1]_s^n$ is a signature for the complete graph K_n.

Case (a) Two arithmetic progressions. Let the two maximal progressions be $[1]_s^{l_0}, [a_1]_s^{l_1}$. Thus,

$$S = \{s, 2s, \ldots, l_0 s\} \cup \{a_1, a_1 + s, \ldots, a_1 + (l_1 - 1)s\}.$$

Consider the difference set of s,

$$D(s) = \{a_1 - s, a_1, a_1 + s, \ldots, a_1 + (l_1 - 2)s\} \cup \{s, 2s, \ldots, (l_0 - 1)s\}.$$

Since $[a_1]_s^{l_1}$ is a maximal arithmetic progression, $a_1 - s$ cannot be in the signature. Note that all terms in $D(s)$ besides $a_1 - s$ are in the signature, and $\deg(s) = n-2$. This contradicts every vertex of G_n has degree $n - 3$.

Case (b) Three arithmetic progressions. Let the three progressions be $[1]_s^{l_0}$, $[a_1]_s^{l_1}$, and $[a_2]_s^{l_2}$, where we assume without loss of generality that $s < a_1 < a_2$. Note that s is not adjacent to either a_1 or a_2 since $[a_1]_s^{l_1}$ and $[a_2]_s^{l_2}$ are maximal progressions. By Observation 11, at most two members are not present in the set $\{s, 2s, \ldots, (l-1)s\}$.

We will first prove that $a_1 > (l_0 + 1)s$. Since a_1 is the first term of an arithmetic progression, $a_1 \neq (l_0 + 1)s$. Assume by contradiction that $a_1 < (l_0 + 1)s$. Then, $ps < a_1 < (p+1)s$, where $1 \leq p \leq l_0 s$. Note that the differences of a_1 with all elements in $[1]_s^{l_0}$ are less than $l_0 s$. Since a_1 is not a multiple of s, the differences of a_1 with all elements in $[1]_s^{l_0}$ are not multiples of s and not in S. That a_1 is not adjacent to any vertex in $[1]_s^{l_0}$ contradicts that $\deg(a_1) = n - 3$ unless $l_0 = 2$. However, if $l_0 = 2$ and $a_1 < (l_0 + 1)s$, then a_1 is not adjacent to s and $2s$. Since a_1 is adjacent to every element of $[a_1]_s^{l_1}$, $l_1 = 2$. Then $l_2 \geq 4$. Since a_1 is adjacent to every element of $[a_2]_s^{l_2}$, S contains the arithmetic progression of length $l_2 \geq 4$: $a_2 - a_1, a_2 + s - a_1, \ldots, a_2 + (l_2 - 1)s - a_1$, contradicting that $l_0 = l_1 = 2$. Therefore, $a_1 > (l_0 + 1)s$. By similar arguments, $a_2 > a_1 + l_1 s$.

Consider the ith term of the arithmetic progression $[1]_s^{l_0}$ *is* for $1 \leq i \leq \min\{l_0, l_1, l_2\}$. Since $a_1 - s$ and $a_2 - s$ are not in the signature and $\deg(is) = n-3$, is is adjacent to every vertex other than $a_1 + (i-1)s$ and $a_2 + (i-1)s$. In particular, if $2s \in S$, then $2s$ is adjacent to a_1 and a_2, and thus $|a_1 - 2s|$ and $|a_2 - 2s|$ are in S.

Case (i) $2s \in S$. Since $a_1 > (l_0 + 1)s$, $|a_1 - 2s| = a_1 - 2s$. Since $a_1 - 2s < a_1$ and $|a_1 - 2s| \in S$, then $a_1 - 2s \in [1]_s^{l_0}$. Let $a_1 - 2s = ks$, where $1 \le k \le l_0$. Since a_1 is the first term of an arithmetic progression, $a_1 = (k + 2)s > (l_0 + 1)s$, and hence $k > l_0 - 1$. Thus, $k = l_0$ and $a_1 = (l_0 + 2)s$. The arithmetic progression $[a_1]_s^{l_1}$ can be written as $\{(l_0 + 2)s, (l_0 + 3)s, \dots, (l_0 + l_1 + 1)s\}$.

By a similar reasoning, $|a_2 - 2s| = a_2 - 2s < a_2$ is in S and thus, is a multiple of s. Let $a_2 - 2s = js$, where $1 \le j \le l_0$ or $l_0 + 2 \le j \le l_0 + l_1 + 1$. Note that $a_2 = (j + 2)s > (l_1 + l_0 + 2)s$, hence $j > l_0 + l_1$. Thus, $j = l_1 + l_0 + 1$ and $a_2 = (l_0 + l_1 + 3)s$. Therefore,

$$S = \{s, 2s, \dots, l_0 s\} \cup \{(l_0 + 2)s, (l_0 + 3)s, \dots, (l_0 + l_1 + 1)s\}$$
$$\cup \{(l_0 + l_1 + 3)s, (l_0 + l_1 + 4)s, \dots, (l_0 + l_1 + l_2 + 2)s\}.$$

Note that S is an arithmetic progression with two deletions. Lemma 22 shows that this is not a signature for G_n.

Case (ii) $2s \notin S$. The three arithmetic progressions that partition S are:

$$S = \{s\} \cup [a_1]_s^{l_1} \cup [a_2]_s^{l_2}.$$

Without loss of generality, we assume that $l_1 > l_2$, and so $l_1 \ge 4$ and $a_1 + 2s, a_1 + 3s \in S$. Since $2s, a_1 - s \notin S$, a_1 is not adjacent to s and $a_1 + 2s$. Then, a_1 must be adjacent to $a_1 + 3s$, and so $3s \in S$. Thus,

$$S = \{s\} \cup \{3s, 4s, \dots, (l_1 + 2)s\} \cup [a_2]_s^{l_2}.$$

We know $3s$ is not adjacent to s and $5s$, and so $3s$ must be adjacent to a_2. By assumption, $a_2 > 3s$. We suppose $a_2 - 3s = qs$, where $3 \le q \le l_1 + 2$ or $q = 1$. Since a_2 is the first term of an arithmetic progression, $a_2 = (q + 3)s > (l_1 + 3)s$, thus $q > l_1$. This implies that q can only be $l_1 + 1, l_1 + 2$. If $q = l_1 + 1$, then S is an arithmetic progression with two deletions of $2s$ and $(l_1 + 3)s$. By Lemma 22, this is not a signature for G_n. Thus, $q = l_1 + 2$, and hence $a_2 = (l_1 + 5)s$. Therefore,

$$S = \{s\} \cup \{3s, 4s, \dots, (l_1 + 2)s\} \cup \{(l_1 + 5)s, (l_1 + 6)s, \dots, (n + 3)s\}$$
$$= [1]_s^{n+3} \setminus \{2s, (l_1 + 3)s, (l_1 + 4)s\}.$$

But this can never be a signature for G_n. If $1 \le l_1 \le n - 4$, then $l_1 + 7 \le n + 3$, and thus $(l_1 + 7)s, (l_1 + 6)s$ are both in S. Then, the vertex labeled $3s$ will not be adjacent to $s, (l_1 + 6)s$, and $(l_1 + 7)s$, contradicting that $\deg(3s) = n - 3$. On the other hand, if $n - 3 \le l_1 \le n - 2$, then $l_1 + 7 > n + 3$ and the vertex labeled $(l_1 + 5)s$ is only not adjacent to s, contradicting that $\deg((l_1 + 5)s) = n - 3$.

We have exhausted all possible signatures for G_n, without finding one. This completes the proof.   ∎

5.   CONCLUSION

The study of autographs offers some insight into what makes certain graphs difficult or impossible to express in this particular abbreviated form. The demonstration of these specific non-autographs confirms the intuition of [1] that graphs whose complements have low degree are not likely to be autographs.

5.1.   Discussion

At first glance, one might be tempted to hope that autographs might provide a mechanism for determining if two graphs are isomorphic, since it is easy to verify that two signatures are identical. However, such a mechanism could provide a solution to the Graph Isomorphism Problem, which, while not known to be NP-complete, has no known polynomial time solution [4]. Thus, it is unlikely that a polynomial time algorithm for the SIGNATURE problem can be found. Indeed, the question of whether possible signature values for a graph could be bounded was raised by [3] and addressed for certain cases by [1]. Thus, algorithmic searches for signatures for graphs are compromised by the facts that: (i) signatures are not unique; (ii) bounds for possible signature values are not known; and (iii) there is no polynomial time solution for determining whether a candidate signature realizes a target graph. Nevertheless, the Graph Isomorphism Problem can be solved in polynomial time for graphs of bounded degree [9], so there is some hope that algorithmic approaches could be viable in finding signatures for some autographs.

5.2.   Open Problems

Many interesting problems concerning autographs remain. Namely:

1. Is the signature problem GI-complete? Such a result would clarify the computational complexity of this line of inquiry.
2. Is it possible to derive bounds for signature values in autographs that would enable polynomial time algorithmic searches for signatures?
3. Are there other interesting families of non-autographs?
4. Is there a property that determines whether a graph is an autograph?

Acknowledgments

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