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Weak Measurement and Bohmian Conditional Wave Functions

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It was recently pointed out (and demonstrated experimentally) by Lundeen *et al.* that the wave function of a particle (more precisely, the wave function possessed by each member of an ensemble of identically-prepared particles) can be "directly measured" using weak measurement. Here it is shown that if this same technique is applied, with appropriate post-selection, to one particle from a (perhaps entangled) multi-particle system, the result is precisely the so-called "conditional wave function" of Bohmian mechanics. Thus, a plausibly operationalist method for defining the wave function of a quantum mechanical sub-system corresponds to the natural definition of a sub-system wave function which Bohmian mechanics (uniquely) makes possible. Similarly, a weak-measurement-based procedure for directly measuring a sub-system's density matrix should yield, under appropriate circumstances, the Bohmian "conditional density matrix" as opposed to the standard reduced density matrix. Experimental arrangements to demonstrate this behavior – and also thereby reveal the non-local dependence of sub-system state functions on distant interventions – are suggested and discussed.

I. INTRODUCTION

The notion of "weak measurement", first introduced in [1] and recently reviewed in [2], has become an important tool for exploring foundational questions in quantum mechanics. For example, the recent theorem of Pusey, Barrett, and Rudolph [3] – according to which the quantum state or wave function must be understood as having an ontological (as opposed to merely epistemic) character – is nicely supported and supplemented by the rather different recent work of Lundeen *et al.* [4] showing that the quantum wave function can be "directly measured" using weak measurement techniques. (This is "direct" in contrast to the indirect or reconstructive approaches involved in quantum state tomography – but see also [5]).

The procedure goes as follows. In a weak measurement, one lets a system in state $|\psi\rangle$ couple weakly to a pointer whose position, if the coupling were stronger, would unambiguously register the value associated with observable $\hat{A}$. With the weak coupling, however, the pointer's registration remains quite ambiguous; but this can be made up for by repeating the process many times (on identically-prepared systems) and averaging. After the system couples weakly to the pointer, one may also make a (normal, strong) measurement of some other observable $\hat{B}$ and post-select on the outcome. In the weak-coupling limit, the average value of the pointer's reading (when the final measurement has outcome b) is the real part of the (here, complex) "weak value"

$$\langle \hat{A} \rangle_W^b = \frac{\langle b|\hat{A}|\psi\rangle}{\langle b|\psi\rangle} \tag{1}$$

whose imaginary part is also accessible via measurements of the pointer's conjugate momentum.

In the scheme introduced by Lundeen *et al.*, one lets $\hat{A} = \hat{\pi}_x = |x\rangle\langle x|$ and $\hat{B} = \hat{p}_x$. We then have that

$$\langle \hat{\pi}_x \rangle_W^{p_x} = \frac{\langle p_x|x\rangle\langle x|\psi\rangle}{\langle p_x|\psi\rangle} = \frac{e^{-ip_x x/\hbar}\psi(x)}{\tilde{\psi}(p_x)}. \tag{2}$$

For the particular case $p_x = 0$ we thus have that the weak value is proportional to the particle's wave function:

$$\langle \hat{\pi}_x \rangle_W^{p_x=0} \sim \psi(x). \tag{3}$$

Lundeen *et al.* [4] used this technique to directly measure the transverse wave function of a(n ensemble of identically prepared) photon(s).

Another recent example of the use of weak measurements to probe foundational questions involves Bohmian mechanics. Wiseman [6] pointed out that a certain naively plausible operational approach to experimentally determining the trajectory of a quantum particle – namely, defining the velocity of a particle at a certain position in terms of the difference between the weak value of its position at time t and the strong value at $t + dt$ – yields precisely the Bohmian expression for the particle's velocity (see also [7, 8]). Steinberg *et al.* [9] implemented this scheme to reconstruct the average trajectories for photons in the 2-slit experiment. The beautiful experimentally-reconstructed trajectories are indeed congruent with the iconic images of 2-slit Bohmian trajectories [10]. And it was recently pointed out by Braverman

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and Simon [11] that such measurements, if performed on one particle from an entangled pair, should allow an empirical demonstration of the non-local character of the Bohmian trajectories.

Following Braverman and Simon, the goal of the present work is to address the following seemingly natural question: what happens if the Lundeen *et al.* technique, for "directly measuring" the wave function of a particle, is applied to a particle which does not, according to ordinary quantum theory, *have* a wave function of its own, because it is entangled with some other particle(s)? The answer turns out to be that, under suitable conditions, the "directly measured" one-particle wave function corresponds exactly to the so-called "conditional wave function" of Bohmian mechanics [12] . Since this is undoubtedly an unfamiliar concept to most physicists, we review it in Section II before explaining, in Section III, this central claim and suggesting an experimental setup in which it should be demonstrable. Section IV then outlines a parallel result, regarding density matrices, especially appropriate for the (spin) states of particles with discrete degrees of freedom. Section V offers conclusions, focusing especially on questions surrounding the claimed observability of non-locality.

II. BOHMIAN CONDITIONAL WAVE FUNCTIONS

Consider for simplicity a system of two spin-0 particles (masses m_1 and m_2, coordinates x and y) each moving in one spatial dimension. According to ordinary quantum mechanics (OQM) the wave function $\Psi(x, y, t)$, obeying an appropriate two-particle Schrödinger equation, provides a complete description of the state of the system. According to Bohmian mechanics (BM), however, the description provided by the wave function alone is decidedly incomplete; a complete description requires specifying in addition the actual particle positions $X(t)$ and $Y(t)$. For BM the wave function $\Psi(x, y, t)$ obeys the usual Schrödinger equation, while $X(t)$ evolves according to

$$\frac{dX(t)}{dt} = \frac{\hbar}{2m_1 i} \frac{\Psi^* \frac{\partial}{\partial x}\Psi - \Psi \frac{\partial}{\partial x}\Psi^*}{\Psi^*\Psi}\Bigg|_{x=X(t),\, y=Y(t)} \tag{4}$$

and similarly for $Y(t)$. It is a joint property of the time-evolution laws for the wave and particles that, if the particle positions X and Y are random and $|\psi|^2$-distributed at some initial time (this is the so-called quantum equilibrium hypothesis, QEH), they will remain $|\psi|^2$ distributed for all times. This so-called "equivariance" property is crucial for understanding how BM reproduces the statistical predictions of OQM [12].

The Bohmian "conditional wave function" (CWF) – for, say, the first particle – is simply the ("universal") wave function $\Psi(x, y, t)$ evaluated at $y = Y(t)$:

$$\chi_1(x, t) = \left.\Psi(x, y, t)\right|_{y=Y(t)} . \tag{5}$$

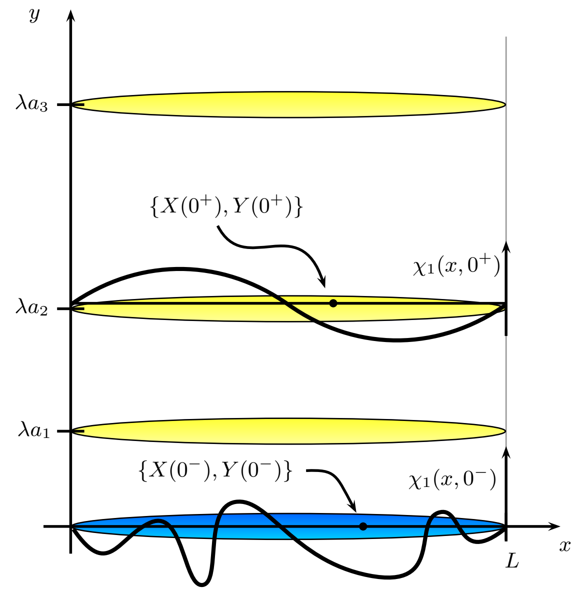

FIG. 1: Illustration of the collapse of the Bohmian CWF during an energy measurement ($\hat{A} = \hat{H}$) on a particle in a box. The initial two-particle wave function $\Psi(x, y, 0^-) = \psi(x)\phi_0(y)$ has support in the blueish region of the configuration space. Since this initial state factorizes, the CWF at $t = 0^-$ is (up to a multiplicative constant) just $\chi_1(x) = \sum_n c_n \psi_n(x)$, indicated with the bolded lower curve. The Schrödinger evolution from 0^- to 0^+ produces a two-particle wave function with localized islands of support in the configuration space, indicated by the yellowish regions. Each of the yellowish blobs is a Gaussian in y (centered at one of the possible post-interaction pointer positions λa_n) multiplied by one of the energy eigenfunctions (here $\psi_n(x) \sim \sin(n\pi x/L)$). And so if (for example, as shown) the actual configuration point $\{X(0^+), Y(0^+)\}$ ends up in the support of the yellowish blob at $y = \lambda a_2$ – something which will occur with probability $|c_2|^2$ with random initial configuration $\{X(0^-), Y(0^-)\}$ in accord with the QEH – then the post-interaction CWF of particle 1 will be $\chi_1(x, 0^+) \sim \psi_2(x)$ (as shown).

This is the obvious and natural way to construct a "single particle wave function" given the resources that BM provides. (OQM, with fewer resources at hand, provides no such natural – or even an unnatural – construction.) What makes this definition natural is that the evolution law for the position $X(t)$ of particle 1, Equation (4), can be re-written in terms of particle 1's CWF as follows:

$$\frac{dX(t)}{dt} = \frac{\hbar}{2m_1 i} \frac{\chi_1^* \frac{\partial}{\partial x}\chi_1 - \chi_1 \frac{\partial}{\partial x}\chi_1^*}{\chi_1^*\chi_1}\Bigg|_{x=X(t)} . \tag{6}$$

It is thus appropriate to think of $\chi_1(x, t)$ as the guiding- or pilot-wave that directly influences the motion of particle 1.

It is important to appreciate that $\chi_1(x, t)$ depends on time in two different ways – through the t-dependence of Ψ and also through the t-dependence of Y. Thus, in general, $\chi_1(x, t)$ does not obey a simple one-particle

Schrödinger equation, but obeys instead a more complicated pseudo-Schrödinger equation [12, 13]. In particular, it is easy to see that, under the appropriate measurement-like circumstances, $\chi_1(x,t)$ will *collapse*. Suppose for example that particle 1 has initial wave function

$$\psi(x) = \sum_n c_n \psi_n(x) \tag{7}$$

where the $\psi_n(x)$ are eigenstates of some observable $\hat{A}$ with eigenvalues a_n. And suppose that particle 2 is the pointer on an $\hat{A}$-measuring device, initially in the state

$$\phi_0(y) \sim e^{-y^2/2w^2}. \tag{8}$$

Now suppose the particles experience a (for simplicity, impulsive) interaction

$$\hat{H}_{int} = \lambda\, \delta(t)\, \hat{A}\, \hat{p}_y. \tag{9}$$

The usual unitary Schrödinger evolution of the initial wave function $\Psi(x,y,0^-) = \psi(x)\phi_0(y)$ then takes it into

$$\Psi(x,y,0^+) = \sum_n c_n \psi_n(x)\phi_0(y - \lambda a_n). \tag{10}$$

That is, the two-particle wave function after the interaction can be understood as an entangled superposition of terms, each of which has particle 1 in an eigenstate of $\hat{A}$ and particle 2 in a new position that registers the corresponding value a_n. (Note that we assume here that λ is sufficiently large that the separation between adjacent values of λa_n is large compared to the width w of the pointer packet. This is thus a "strong" measurement.) From the point of view of OQM, Equation (10) exhibits the standard problem of Schrödinger's cat: instead of resolving the superposition of distinct a-values, the measuring device itself gets infected with the superposition. In OQM (where there is nothing but the wave function at hand) one thus needs to introduce additional dynamical ("collapse") postulates to account for the observed (apparently non-superposed) behavior of real laboratory equipment.

In BM, however, there is no such problem. The observable outcome of the measurement is not to be found in the wave function, but instead in the actual position $Y(0^+)$ of the pointer after the interaction. It is easy to see that (with appropriate random initial conditions) this will, with probability $|c_n|^2$, lie near the value λa_n which indicates that the result of the measurement was a_n. Furthermore, it is easy to see that if $Y(0^+)$ is near the value λa_n, then the CWF of particle 1 will be (up to a multiplicative constant) the appropriate eigenfunction:

$$\chi_1(x,0^+) \sim \psi_n(x). \tag{11}$$

That is, the CWF of particle 1 *collapses* (from a superposition of several ψ_ns to the particular ψ_n which corresponds to the actually-realized outcome of the measurement) as a result of the interaction with the measuring device, even though the dynamics for the "universal" wave function Ψ is completely unitary. See Figure 1 and its caption for an illustration.

We have here explained the idea of (and one important and perhaps surprising property of the dynamical evolution of) Bohmian CWFs as if the particle of interest were interacting with the particle or particles constituting a *measuring device*. That is of course the crucial kind of situation if one is worrying about the so-called quantum measurement problem. But more important for our purposes here is the fact that the Bohmian CWF (for a single particle) is perfectly well-defined at all times for any particle that is part of a larger (multi-particle) quantum system. Indeed, BM only really provides a solution of the measurement problem because it treats "measurements" as just ordinary physical interactions, obeying the same universal dynamical laws as all interactions. It should thus be clear that, according to BM, collapses (like the one we just described happening as a result of an interaction with a measuring device) will actually be happening all the time, as particles interact with each other. It is the goal of the following analysis to show how this feature of the Bohmian theory can be experimentally manifested using weak measurement.

III. DIRECT MEASUREMENT OF SINGLE PARTICLE WAVE FUNCTIONS

Let us then turn to the main result of the present paper. Suppose we carry out the Lundeen-type "direct measurement of the wave function" procedure on one particle of a two-particle system. As a reality check, suppose to begin with that the two-particle system has a factorizable quantum state

$$|\Psi\rangle = |\psi\rangle|\phi\rangle \tag{12}$$

where the first and second factors on the right refer to particles 1 and 2 respectively. The Lundeen-type procedure involves post-selecting on the final momentum p_x of the particle whose wave function we are trying to measure (here, particle 1). Let us also post-select on the final position Y of particle 2 [14]. It is then straightforward to calculate that

$$\langle\hat{\pi}_x\rangle_W^{p_x,y=Y} = \frac{\langle Y|\phi\rangle\langle p_x|x\rangle\langle x|\psi\rangle}{\langle Y|\phi\rangle\langle p_x|\psi\rangle} = \frac{e^{-ip_x x/\hbar}\,\psi(x)}{\tilde{\psi}(p_x)} \tag{13}$$

which is, as expected, identical to Equation (2).

If, however, particle 1 is in a general, entangled state with particle 2, as in

$$|\Psi\rangle = \int dx'dy'\,\Psi(x',y')|x'\rangle|y'\rangle \tag{14}$$

then the operational determination of particle 1's wave function (post-selected on the final strongly-measured position Y of particle 2) yields

$$\langle\hat{\pi}_x\rangle_W^{p_x,y=Y} = \frac{\langle p_x|x\rangle\langle x,Y|\Psi\rangle}{\langle p_x,Y|\Psi\rangle} = \frac{e^{-ip_x x/\hbar}\,\Psi(x,Y)}{\int dx'\Psi(x',Y)\,e^{-ip_x x'/\hbar}} \quad (15)$$

Restricting, as before, our attention to the cases in which the final measured momentum p_x is zero, we have that

$$\langle\hat{\pi}_x\rangle_W^{p_x=0,y=Y} \sim \Psi(x,Y) = \chi_1(x) \quad (16)$$

where the right hand side is precisely the Bohmian CWF for particle 1. Note that we have tacitly relied on the fact that, for Bohmian mechanics, position is a *non-contextual* ("hidden") variable. Thus the final position measurement on particle 2 simply reveals, for Bohmian mechanics, the actual pre-existing location Y of that particle. In short, the two Ys in the analysis – the one representing the outcome of the final position measurement of particle 2, and the one, used in the definition of the Bohmian CWF, representing the actual position of particle 2 – are, for Bohmian mechanics, the same.

So far we have basically ignored the issue of the exact timing of the various measurements on the two-particle system. Let us then examine this in the context of a somewhat concrete example. Consider two photons prepared in some kind of entangled state (to be specified shortly) and propagating in roughly opposite directions. Let the variable x refer to the transverse spatial degree of freedom of photon 1 (propagating, say, to the right) and the variable y refer to the transverse spatial degree of freedom of photon 2 (propagating to the left). We imagine a setup like that reported in [4] in which the weak measurement is effected using an extremely narrow half-wave plate ("$\lambda/2$ sliver") and the $p_x = 0$ post-selection is effected by accepting only those photons which pass a narrow slit downstream from and along the axis of a Fourier Transform lens. The remaining photons are then passed through an appropriate quarter (or half) wave plate; the imbalance between the two polarization states then yields the real (or, respectively, imaginary) part of the transverse wave function at the location of the $\lambda/2$ sliver. See [4] for details.

We now consider the possibility that each photon that enters the device is entangled with a second photon:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\psi_1\rangle|\phi_1\rangle + |\psi_2\rangle|\phi_2\rangle\right). \quad (17)$$

It is important (for the proper functioning of the wave function measurement) that $|\psi_1\rangle$ and $|\psi_2\rangle$ have the same (say, linear) polarization. But let them have distinct (transverse) spatial profiles – for example, and most simply, suppose that $|\psi_1\rangle$ has transverse spatial support just *above* the z-axis (i.e., for $x > 0$) while $|\psi_2\rangle$ has transverse spatial support just *below* the z-axis ($x < 0$). See Figure 2(a).

As to photon 2, suppose that (at least initially) $|\phi_1\rangle$ and $|\phi_2\rangle$ have identical transverse profiles and completely

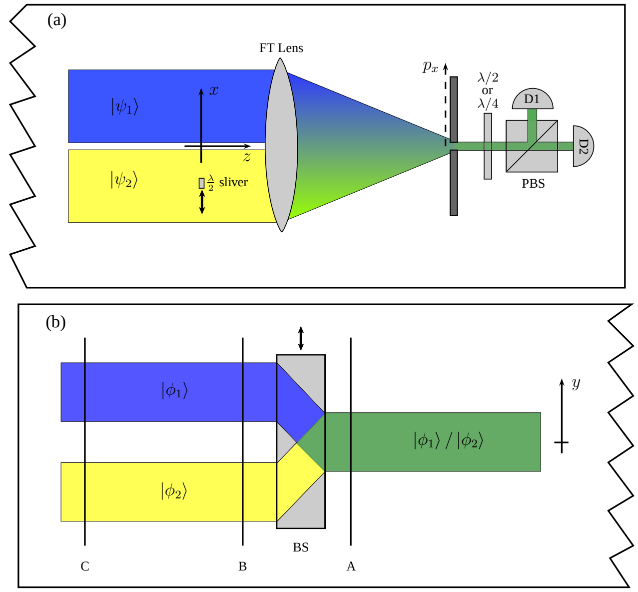

FIG. 2: Schematic diagram of the proposed experiment. Frame (a), following [4], shows photon 1 propagating to the right and undergoing first a weak measurement (effected by the $\lambda/2$ sliver), then a $p_x = 0$ measurement/post-selection, and finally the readout of the weak measurement. Frame (b) shows photon 2 propagating to the left. The two-photon state is $|\Psi\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle|\phi_1\rangle + |\psi_2\rangle|\phi_2\rangle)$ where $|\psi_1\rangle$ and $|\psi_2\rangle$ have distinct (transverse, i.e., x) spatial profiles – for example, as shown here, perhaps $|\psi_1\rangle$ has support only for $x > 0$ while $|\psi_2\rangle$ has support only for $x < 0$. For photon 2, $|\phi_1\rangle$ and $|\phi_2\rangle$ are initially overlapping spatially but are distinct in some property (such as energy or polarization) which allows the beams to be separated, by some kind of (removable) beam-splitter (BS), as shown. Three different possible detection planes – A, B, and C – for the final measurement/post-selection of the transverse position Y of photon 2 are shown and discussed in the main text.

overlap spatially, but are distinct in some way (for example, they could be orthogonally polarized, or could have different energies) that allows the two parts of the beam to be separated by some type of beam splitter (BS). See Figure 2(b).

Let us now consider several different spatial locations for, and time-orderings involving, the final measurement of the position Y of photon 2.

To begin with, let us first imagine that the measurement/post-selection on photon 2's transverse coordinate y occurs (in, say, the lab frame) *before* the weak measurement on photon 1 and at a plane like A in Figure 2 (i.e., before photon 2 has passed any beam splitter). According to OQM, this measurement of Y will collapse the 2-particle wave function and leave photon 1 with a definite (non-entangled) wave function of its own. Because $|\phi_1\rangle$ and $|\phi_2\rangle$ overlap at A, however, the position measurement gives no information about $|\phi_1\rangle$ vs. $|\phi_2\rangle$ and so leaves photon 1 in the state $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$. And this of course coincides with the predicted result of

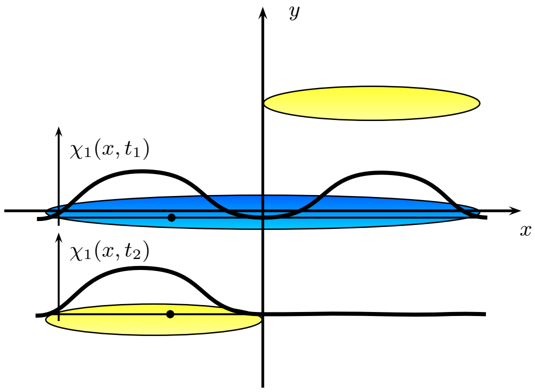

FIG. 3: In the (x, y) configuration space, the entangled two-photon wave function initially has support in the blueish region and the CWF for photon 1 – $\chi_1(x, t_1)$ – looks something like the two-hump curve shown. The passage of photon 2 through the beam-splitter (BS) separates the two-photon wave function into the two yellowish islands. The actual configuration point (shown here as a black dot) ends up (depending on its random initial position for each photon pair) in one of the two yellowish islands, and the CWF at t_2 will thus have collapsed to either $\langle x|\psi_1\rangle$ or $\langle x|\psi_2\rangle$. (The latter case is shown here.)

the direct measurement protocol: for measurement/post-selection at A

$$\langle\hat{\pi}_x\rangle_W^{p_x=0,y=Y} = \frac{\psi_1(x)\langle Y|\phi_1\rangle + \psi_2(x)\langle Y|\phi_2\rangle}{\tilde{\psi}_1(0)\langle Y|\phi_1\rangle + \tilde{\psi}_2(0)\langle Y|\phi_2\rangle} \sim \frac{1}{\sqrt{2}}\left(\psi_1(x) + \psi_2(x)\right) \tag{18}$$

where we have used the fact that, for detection at the A plane, $\langle Y|\phi_1\rangle = \langle Y|\phi_2\rangle$.

On the other hand, if the measurement/post-selection on photon 2 is performed at plane B (after photon 2 has passed the beam splitter, but still *before* the measurement protocol has been carried out on photon 1) then according to OQM the wave function of photon 1 will collapse to *either* $|\psi_1\rangle$ (if Y is found in the support of $\langle y|\phi_1\rangle$) *or* $|\psi_2\rangle$ (if Y is found in the support of $\langle y|\phi_2\rangle$). This again coincides with the expected results of the direct measurement: for detection at the B plane, one or the other of $\langle Y|\phi_1\rangle$ and $\langle Y|\phi_2\rangle$ will be zero. We will thus find that

$$\langle\hat{\pi}_x\rangle_W^{p_x=0,y=Y} \sim \psi_m(x) \tag{19}$$

for $Y \in \text{supp}(\langle y|\phi_m\rangle)$.

For both of those two scenarios, ordinary QM attributes a one-particle wave function to particle 1 at the time in question; this wave function coincides with the Bohmian CWF and the expected results of the weak measurement technique. In short, there is nothing surprising or interesting here from the point of view of OQM.

Consider, however, a scenario involving measurement/post-selection of photon 2's position at plane C (such that this measurement takes place *well after* the weak measurement on photon 1 has already gone fully to completion). It is trivial to see that the results of the weak measurement will again be given by Equation (19) – that is, dramatically different ("collapsed") wave functions for photon 1 will be found depending on whether particle 2 is (later!) found in the support of $|\phi_1\rangle$ or $|\phi_2\rangle$. From the point of view of OQM, however, it is rather difficult to understand why, prior to any actual measurement (meaning here an interaction involving macroscopic amplification) that would trigger a collapse, one should find collapsed one-particle wave functions. On the other hand, this is perfectly natural from a Bohmian point of view: as sketched in Figure 3 the conditional wave function (CWF) for particle 1 collapses as soon as the packets separate in the two-dimensional configuration space. This happens when the components $|\phi_1\rangle$ and $|\phi_2\rangle$ are split at the BS; no actual position *measurement* is required. (Note that the claim here is *not* that OQM makes the wrong predictions. Undoubtedly it makes the right predictions – indeed we have used nothing but orthodox quantum ideas to calculate what should be observed in the experiment. The point is rather only that, from the OQM perspective, it is at best obscure what one ought to expect the operationalist determination of photon 1's "one-particle wave function" to yield at times when photon 1 remains entangled with photon 2. Whereas, from the Bohmian point of view, the results are the obvious, natural thing.)

The proposed experiment, then, should involve a fixed detection plane, like plane C in the Figure, at a greater optical distance from the two-particle source than that of the measuring apparatus for particle 1. The removable beam splitter BS should, on the other hand, ideally be slightly closer to the two-particle source than the particle 1 apparatus. With the BS in place, the arrangement would be like that shown in the Figure and the "direct measurement" on photon 1 would yield (after appropriate post-selection on Y) collapsed photon 1 wave functions (even though the actual position measurement on photon 2 would occur well after photon 1 was already measured). On the other hand, with the BS removed, the situation would be equivalent to detecting photon 2 at plane A (except that this too would only occur later, after the measurements on photon 1 had occured) and the measurement of photon 1 would reveal the uncollapsed wave function. One could thus in some sense observe the collapse of the Bohmian CWF of photon 1 as a direct result of the insertion (perhaps at space-like separation) of the BS into the path of photon 2.

This proposed setup should be realizable in practice along the following lines. Type II spontaneous parametric down-conversion yields a pair of photons in an entangled polarization state $\left(|H\rangle|H\rangle + |V\rangle|V\rangle\right)/\sqrt{2}$ with the individual photons being coupled into single-mode optical fibres. The first photon should then be split by a polarizing beam splitter (PBS), with, say, the $|H\rangle$ component being shunted into the $+z$ direction with $x > 0$ and the $|V\rangle$ component passed through a $\lambda/2$ plate (rotating its polarization to $|H\rangle$) before being shunted into the $+z$ direction with $x < 0$. This prepares photon 1 as suggested in Figure 2(a) and the subsequent measurements may then be carried out exactly as in [4]. The second photon can be directly shunted into the $-z$ direction so that the two orthogonal polarization components have identical and perfectly overlapping transverse spatial profiles, as in Figure 2(b). The overall (transverse spatial and polarization) two-photon state after such preparation can thus be written

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|+H\rangle|\emptyset H\rangle + |-H\rangle|\emptyset V\rangle\right) \tag{20}$$

where $+$ indicates that the transverse state has support for $x > 0$, $-$ indicates that the support lies in $x < 0$, and $\emptyset$ indicates that the support is centered at $x = 0$. (It is also of course understood that photon 1 is moving in the $+z$ direction and photon 2 in the $-z$ direction.) The perfect correspondence between Equations (20) and (17) should be clear. Note that for this type of implementation, the (generic) "BS" in Figure 2(b) can be a standard PBS.

IV. DENSITY MATRICES

In the Lundeen *et al.* procedure for measuring the wave function, it is the particles' spatial degree of freedom that is probed, with the polarization serving as the pointer. But the polarization state of a particle can also be measured using weak measurement techniques, with the position degree of freedom (or in principle some other extrinsic degree of freedom) playing the role of the pointer. For example, in a scheme recently proposed and demonstrated by Lundeen and Bamber [15, 16] (see also [17]) the polarization density matrix of a particle can be measured as follows. For a photon in a mixed state described by density operator $\hat{\rho}$, the weak value of an observable $\hat{A}$ is given by

$$\langle \hat{A} \rangle_W^b = \frac{\langle b|\hat{A}\hat{\rho}|b\rangle}{\langle b|\hat{\rho}|b\rangle} \tag{21}$$

where $|b\rangle$ is the final post-selected state, as in Equation (1). In the case of a pure state, $\hat{\rho} = |\psi\rangle\langle\psi|$, Equation (21) reduces to (1), whereas for a genuinely mixed state, (21) is the appropriate weighted average. In the event of no post-selection, a further averaging gives

$$\langle \hat{A} \rangle_W = \mathrm{Tr}\left[\hat{A}\hat{\rho}\right]. \tag{22}$$

The Lundeen/Bamber procedure can then be most simply understood as follows. Defining operators

$$\hat{\pi}_{ij} = |i\rangle\langle j| \tag{23}$$

(with $i, j \in \{H, V\}$) on the two-dimensional polarization Hilbert space for a photon, one sees that their weak values correspond to the entries in the polarization density matrix:

$$\langle \hat{\pi}_{ij} \rangle_W = \mathrm{Tr}\left[\hat{\pi}_{ij}\hat{\rho}\right] = \langle j|\hat{\rho}|i\rangle. \tag{24}$$

Of course, for $i \neq j$, $\hat{\pi}_{ij}$ is not a Hermitian operator, so measuring it – even weakly – raises some questions. But the worrisome matrix elements can be re-expressed in terms of weak values of perfectly reputable operators by introducing post-selection, e.g.,

$$\begin{aligned} \rho_{VH} &= \langle H|\hat{\rho}|V\rangle \\ &= P(D)\,\langle \hat{\pi}_{HH} \rangle_W^D - P(A)\,\langle \hat{\pi}_{HH} \rangle_W^A \end{aligned} \tag{25}$$

where $|D\rangle = (|H\rangle + |V\rangle)/\sqrt{2}$ and $|A\rangle = (|H\rangle - |V\rangle)/\sqrt{2}$, and $P(D) = \langle D|\hat{\rho}|D\rangle$ and $P(A) = \langle A|\hat{\rho}|A\rangle$ are respectively the rates of successful post-selection on the $|D\rangle$ and $|A\rangle$ states. See [15] for further details.

Consider now a two-particle system in state

$$|\psi_o\rangle = \int dy \sum_{i,j} \psi_{i,j}(y)|i\rangle_1|j\rangle_2|y\rangle_2 \tag{26}$$

where, as before, $i, j \in \{H, V\}$ are one-particle polarization eigenstates, and $|y\rangle_2$ is a position eigenstate of particle 2. (We suppress, for simplicity, the position degree of freedom of particle 1; recall that it may be used as the pointer variable to weakly measure the polarization density matrix.) The two-particle state thus has density operator

$$\hat{\rho} = |\psi_0\rangle\langle\psi_0| \tag{27}$$

in terms of which one can define the reduced density matrix (RDM) of particle 1 by tracing over the degrees of freedom associated with particle 2:

$$\hat{\rho}_1^{\mathrm{red}} = \int dy\ \mathrm{Tr}_2\left[\langle y|\hat{\rho}|y\rangle\right] = \int dy \sum_j \langle y, j|\hat{\rho}|y, j\rangle. \tag{28}$$

The RDM is of course the standard way of defining the "state" of a (perhaps-entangled) subsystem.

As discussed above, for systems with complex-valued wave functions, Bohmian mechanics allows one to define the conditional wave function of a sub-system, in terms of which the guidance law for the particles comprising the sub-system can be re-expressed. For systems with discrete (spin, polarization) degrees of freedom, however, the Bohmian CWF for each one-particle sub-system would carry the discrete indices for *all* particles in the system. It thus cannot really be regarded as a "wave function for a single particle". In such situations, Bohmian mechanics thus follows ordinary quantum mechanics in defining the state of the sub-system in terms of an appropriate density matrix. But as was first pointed out by Bell [18], the correct Bohmian particle trajectories (needed to reproduce the statistical predictions of ordinary QM) cannot be expressed in terms of the usual reduced density matrix. Instead, one needs to introduce the "conditional density matrix" (CDM), which involves tracing over the discrete indices associated with particles outside the sub-system in question, but then evaluating the spatial variables (again, associated with particles outside the sub-system in question) at the actual locations of the Bohmian particles. (See [19] for a detailed discussion.) Thus, for the system introduced just above, we would have

$$\hat{\rho}_1^{\text{cond}} = \text{Tr}_2 \left[\langle Y | \hat{\rho} | Y \rangle \right]. \tag{29}$$

Note that a sub-system will in general possess both a RDM and a CDM, but that these will not in general be equal: the RDM can be understood as the *average* of all possible CDMs. (If we had normalized the CDM, then the RDM would be given by the average of all possible CDMs, weighted by the usual quantum probability for $y = Y$.)

Now, what should happen if one performs the Lundeen/Bamber procedure for directly measuring the polarization density matrix of one photon from an entangled pair, also – as in the previous section – post-selecting on the final position Y of the second particle? It is easy to see that under such conditions the weak value of an operator $\hat{A}$ (acting just on the particle 1 Hilbert space) is

$$\begin{aligned} \langle \hat{A} \rangle_W^Y &= \text{Tr}_{1,2} \Big[\langle Y | \, \hat{A} \, \hat{\rho} \, | Y \rangle \Big] \\ &= \text{Tr}_1 \Big[\hat{A} \; \text{Tr}_2 [\langle Y | \, \hat{\rho} \, | Y \rangle] \Big] \\ &= \text{Tr}_1 \Big[\hat{A} \, \hat{\rho}_1^{\text{cond}} \Big]. \end{aligned} \tag{30}$$

The last line is identical to Equation (22) except that the (one-particle) density matrix $\hat{\rho}$ is replaced by the Bohmian CDM from Equation (29). One thus expects that the operational procedure sketched above – in which $\hat{A} = \hat{\pi}_{ij}$ from Equation (23) – should yield the Bohmian conditional density matrix (and not the usual reduced density matrix) as the directly-measured one-particle density matrix.

Of course, just as with the setup discussed in the previous section, this result is not very interesting or surprising if the post-selection-basing measurement of particle 2's position Y occurs prior to the weak measurement procedure on particle 1. For then, the measurement of Y will have collapsed the two-particle state such that the ordinary RDM and the Bohmian CDM coincide.

On the other hand, if one arranges for the measurement of Y to occur only *after* the procedure on particle 1 has gone to completion, it is quite interesting indeed that the procedure should yield the Bohmian CDM as opposed to the ordinary RDM. Consider for example the following setup, very much in the spirit of the one proposed in the previous section. A two-photon system is prepared in the state

$$|\psi_1\rangle = |\phi^0\rangle \left(|H\rangle_1 |H\rangle_2 + |V\rangle_1 |V\rangle_2 \right) / \sqrt{2} \tag{31}$$

where $|\phi^0\rangle$, referring to the transverse spatial degree of freedom of particle 2, is (say) a Gaussian centered at $y = 0$. (The spatial degrees of freedom of particle 1, and the non-transverse spatial degrees of freedom of particle 2, are suppressed for simplicity.) By means of a polarizing beam splitter that can be inserted (or not) in the path of particle 2, the two-particle state may (or may not) be transformed into

$$|\psi_2\rangle = \big(|\phi^+\rangle |H\rangle_1 |H\rangle_2 + |\phi^-\rangle |V\rangle_1 |V\rangle_2 \big) / \sqrt{2} \tag{32}$$

where $|\phi^\pm\rangle$ is (say) a Gaussian displaced in the $\pm$ y-direction by an amount that is larger than its width.

The crucial point is then that, for both $|\psi_1\rangle$ and $|\psi_2\rangle$, the RDM for particle 1 is

$$\hat{\rho}_1^{\text{red}} \xrightarrow[|H\rangle, |V\rangle \text{ basis}]{} \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}. \tag{33}$$

The Bohmian CDM for particle 1 will be (proportional to) this same matrix if the state is $|\psi_1\rangle$. But if the beam splitter is inserted such that the state is $|\psi_2\rangle$, the CDM will have "collapsed", being now proportional to either

$$\hat{\rho}_1^{\text{cond}} \xrightarrow[|H\rangle, |V\rangle \text{ basis}]{} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ OR } \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{34}$$

depending on whether $Y \in \text{supp}(\langle y | \phi^+ \rangle)$ or $Y \in \text{supp}(\langle y | \phi^- \rangle)$.

According to Bohmian mechanics, one of these two possibilities is realized – and the particle 1 CDM collapses accordingly – as soon as particle 2 traverses the polarizing beam splitter (should it be inserted). Furthermore, once the $|H\rangle$ and $|V\rangle$ components of the particle 2 beam are split apart, the actual particle position Y will not, according to the theory, change. So the actual measurement of Y (for the purpose of post-selection) can wait as long as is desired – for example, until after the weak measurement procedure on particle 1 has been carried out. Still, when the dust settles and all the data is properly binned up, OQM predicts that it is a collapsed density matrix corresponding precisely to the Bohmian CDM, that should be revealed by the direct measurement of particle 1's state.

V. DISCUSSION

If the procedure of [4] for making a "direct measurement of the quantum wave function" is applied to one particle from an entangled pair (and regarded as a plausible operationalist definition of the "single particle wave function" for such a particle) the result, with suitable post-selection on the other particle, is precisely the "conditional wave function" (CWF) of Bohmian mechanics – that is, the natural theoretical concept of a "single particle wave function" that Bohmian mechanics (uniquely) makes possible. Similarly, the results of applying a related procedure – for directly measuring the density matrix associated with a single particle – to one particle from an entangled two-particle system, should yield the Bohmian "conditional density matrix" (CDM) as opposed to the more standard reduced density matrix (RDM). These results are particularly interesting when the weak measurement (that reveals the state of particle 1) is carried out prior to the strong position measurement on particle 2 on which post-selection will be based. Thus, in the same way that Braverman and Simon [11] have suggested that one can "observe the nonlocality of Bohmian trajectories with entangled photons" one should also be able to observe the nonlocal dependence of Bohmian single-particle states (wave functions and density matrices) on distant interventions such as the insertion (or not) of the BS in Figure 2.

Indeed, from the perspective suggested earlier, in which each particle's CWF (or CDM) is regarded as the object which directly guides or pilots the Bohmian particle, the present work can be seen as digging yet one level deeper beyond Braverman/Simon: instead of merely observing how the particle trajectories change as a result of some distant interventions, one may also observe how the "field" *responsible* for those changes itself changes. A successful experimental demonstration of this effect would thus in some sense reveal the non-local character of Bohmian mechanics in an unprecedentedly fundamental way.

But to formulate things in these ways is to invite several possible misunderstandings, so let us clarify a couple of points.

First, as Einstein [20] famously remarked: "Whether you can observe a thing or not depends on the theory which you use. It is the theory which decides what can be observed." In particular, the claims above should only be understood to mean that *from the point of view of Bohmian mechanics*, the experimental procedures outlined here can be understood as *observations* of (for example) the interesting dynamical behavior of single particle CWFs. From the point of view of orthodox QM, on the other hand, the same procedures would evidently *not* be regarded as genuine observations of anything. The situation is thus completely parallel to that surrounding the reconstructed particle trajectories in [9]. The result of the empirical procedure coincides with certain things that are posited to actually exist by one candidate theory, Bohmian mechanics. But another candidate theory, which does not posit those things, also has no trouble accounting for the results. Indeed, insofar as the two theories are empirically equivalent, the results cannot really count as evidence for or against either theory. Nevertheless, the expected results remain somehow more natural from the Bohmian point of view! The situation can perhaps be summed up like this: *if* one accepts, from the outset, that empirically measured weak values correspond to some physically real features of the system in question, then the predicted results would be trivial to reconcile with Bohmian mechanics but difficult to reconcile with ordinary QM. But of course, whether one should accept such a premise is, to put it mildly, highly controversial.

And then, second: It would be easy to get the impression from the way things were put above that by inserting (or refraining from inserting) the BS in the path of photon 2, one can instantaneously affect the observable CWF/CDM of the distant particle 1. That is, it would be easy to get the impression that one could (in principle, if impractically) send a superluminal signal by running many copies of the experiment in parallel (so that the many trials required to build up sufficient data all occur simultaneously). But this, of course, should be impossible (whether one believes in BM or OQM or any other such empirically-equivalent theory).

There are two points to be understood here, one rather obvious and one more subtle. The obvious point is that Alice (on the right) must learn the outcome of Bob's position measurement (on the left) before she can know how to properly bin her data. And this information will have to be sent to her through a "classical" (i.e., here, subluminal) communication channel. So it is already clear that no actual superluminal signalling will be possible.

The more subtle, and more interesting, point is that the statistical relationship between Alice's and Bob's measurement results is actually independent of the exact temporal sequence of the measurements. This is certainly not surprising from the point of view of relativity, given that Alice's and Bob's measurements may well occur at spacelike separation. But it is somewhat surprising from the point of view of Bohmian mechanics, which involves a hidden (but dynamically relevant) privileged reference frame.

Essentially for reasons of drama, we have described the setup above, in the allegedly interesting cases, so that the temporal sequence is as follows: first Bob (or his assistant) decides whether to insert or not insert the BS into the path of particle 2; then Alice's measurement protocol on particle 1 occurs; and then finally Bob measures the final transverse position Y of particle 2. From the point of view of Bohmian mechanics, then, we may say the following. If this is the true temporal sequence – in the dynamically privileged reference frame posited by the theory – then things develop causally in the way we have suggested: the insertion of the BS (if and only if it is inserted) causes the two-particle wave function to divide in the configuration space as shown in Figure 3 and thus causes the CWF (or CDM) for photon 1 to collapse; the (here, subsequent) measurement protocol by Alice then simply reveals the true CWF/CDM of photon 1 at the time of that measurement; the final measurement/post-selection by Bob then plays the (dynamically) purely passive role of revealing, to Bob, what was already physically definite, in order that the already-acquired data can be properly binned.

But since the privileged frame is, for Bohmian mechanics, hidden, it is entirely possible that the "true" temporal sequence (i.e., the temporal sequence in the privileged frame) is instead as follows: Alice's measurement protocol on photon 1 occurs *first*; *then* comes Bob's (assistant's) decision to insert the BS or not, followed by measurement of the transverse position Y. If this is the "true" temporal sequence, the *statistics* will be unchanged, but the causal story will be somewhat different. To wit: instead of the passage (or not) of photon 2 through the BS affecting the CWF/CDM of photon 1, now it will be the measurement protocol on photon 1 which (at least sometimes) affects the CWF/CDM of photon 2 *and thus influences where, for a given Y, it will go, should it encounter the BS.* Concretely, there will exist possible initial conditions for the 2-particle system which have the following property: had the measurement protocol on particle 1 not been carried out, particle 2 would definitely have gone "up" at the BS (and would hence have been found with $Y > 0$), but given that the measurement protocol on particle 1 *was* carried out, particle 2 instead went "down" at the BS (and was hence found with $Y < 0$).

With this second possible "true" temporal sequence, it is no longer really the case that the weak measurement protocol on particle 1 is simply revealing the structure of particle 1's CWF at the time of the measurement. Instead, the measurement on particle 1 may actively affect the state (in particular, the CWF or CDM) of particle 2, making the subsequent post-selection on particle 2's position rather less benign, less passive. And this makes clear in principle why, despite the presence in the theory of a dynamically privileged reference frame in which instantaneous action-at-a-distance occurs, one is not only prevented from sending signals faster than light, but also prevented from putting any experimental limits on the speed of the laboratory with respect to the presumed underlying privileged reference frame. The statistical patterns in the data will – presumably – remain the same as the "true" temporal sequence is varied between the two possibilities discussed here, even as the Bohmian causal story changes rather dramatically.

It does, however, remain absolutely valid to say that – provided one adopts the Bohmian point of view – the experimental setup suggested here would allow for a direct empirical observation of the non-local dependence of a single particle's (Bohmian) CWF/CDM on distant interventions. It's just that – compared to the way we initially explained things – it is rather ambiguous whether one is observing the effect, on particle 1's CWF/CDM, of inserting or not inserting the BS in front of particle 2 ... or instead observing the effect, on particle 2's CWF/CDM, of carrying out the weak measurement protocol on particle 1. It thus remains appropriate to conclude that a realization of the proposed type of experiment would be quite interesting and would in particular help bring the Bohmian CWF/CDM into the light of experimental reality.

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