10-20-2010

Flat Zipper-Unfolding Pairs for Platonic Solids

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October 21, 2010

Abstract

We show that four of the five Platonic solids' surfaces may be cut open with a Hamiltonian path along edges and unfolded to a polygonal net each of which can "zipper-refold" to a flat doubly covered parallelogram, forming a rather compact representation of the surface. Thus these regular polyhedra have particular flat "zipper pairs." No such zipper pair exists for a dodecahedron, whose Hamiltonian unfoldings are "zip-rigid." This report is primarily an inventory of the possibilities, and raises more questions than it answers.

1 Introduction

It has been known since the time of Alexandrov—and it was certainly known to him—that the surface of a polyhedron could sometimes be cut open to a net and refolded to a doubly covered polygon, which we will henceforth call a *flat polyhedron*. Such flat polyhedra are explicitly countenanced in Alexandrov's 1941 gluing theorem.[1] Perhaps the first specific example of this possibility occurred in [LO96], which included the example illustrated in Figure 1: the familiar Latin-cross unfolding of the cube may be refolded to a flat convex quadrilateral polyhedron. This is one of the two flat convex polyhedron that may be folded from the Latin cross [DO07, Fig. 25.32].

Let us say that two polyhedra Q_1 and Q_2 form a *net pair* if they may be unfolded to a common polygonal net. In Figure 1, the cube is cut along edges to unfold to the Latin cross polygon, but the flat quadrilateral must have *face cuts* through the interior of its two faces to unfold to the same Latin cross.

In general there is little understanding of which polyhedra form net pairs. See, for example, Open Problem 25.6 in [DO07]. Here we explore a narrow question on net pairs, narrow enough to obtain a complete answer.

The cuts to unfold a convex polyhedron to a single polygon form a spanning tree of the polyhedron's vertices [DO07, Sec. 22.1.3]. Shephard explored the special case where the spanning tree is a Hamiltonian path of the 1-skeleton

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[1] See [DO07, Sec. 23.3] and [Pak10, Sec. 37] for descriptions of this theorem.

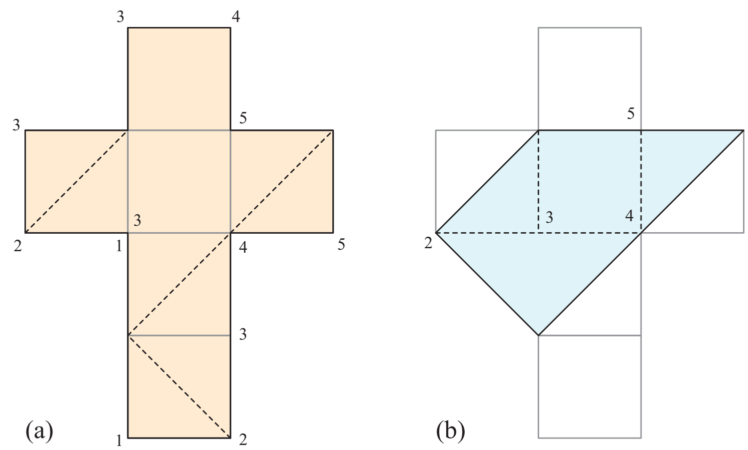

Figure 1: Folding the Latin cross cube net to a flat quadrilateral polyhedron. Points with the same label in (a) are identified in the refolding (b).

of the polyhedron, i.e., all cuts are along polyhedron edges [She75]. The result is a *Hamiltonian unfolding* of the polyhedron. (Note the cube unfolding that produces Figure 1(a) is not a Hamiltonian unfolding: the cut tree has four leaves.) Some combinatorial questions on Hamiltonian unfoldings were explored in [DDLO02]; see [DO07, Fig. 25.59]. In particular, there are polyhedra that have an exponential number of combinatorially distinct Hamiltonian unfoldings: $2^{\Omega(n)}$ for a polyhedron with n vertices.

Another variant is provided by the class of *perimeter-halving* foldings [DO07, Sec. 25.1.2], which correspond to spanning cut paths that may employ face cuts rather than solely following polyhedron edges. In [LDD$^+$10] these paths were memorably rechristened as *zipper paths*, producing *zipper unfoldings*. We will adopt that nomenclature, including the verbs *zip* and *unzip* to mean folding and unfolding (respectively) along zipper paths. We reserve *Hamiltonian path* to be a zipper path along polyhedron edges. Finally, if two polyhedra each unzip to a common polygonal net, we say they form a *zipper pair*.

The narrow question we explore is this:

> **Question:** Does each of the Platonic solids form a zipper pair with a flat convex polyhedron, with the zipper path on the regular polyhedron forming a Hamiltonian path of its edges?

We show that the tetrahedron,[2] the cube, the octahedron, and the icosahedron all form such zipper pairs with flat parallelogram polyhedra. The dodecahedron has no such zipper mate. Note that it would be too restrictive to insist

[2] We drop the modifier "regular" to shorten the names of the five regular polyhedra.

that both zippers are Hamiltonian paths of the 1-skeletons, because for a flat polyhedron, the 1-skeleton is the single cycle bounding the polygon, and so a Hamiltonian unfolding is just two copies of the convex polygon joined along one edge.

2 Flat Zipper Pairs

2.1 Tetrahedron

The regular tetrahedron has only one Hamiltonian path (up to symmetries), which unfolds to the 2×1 parallelogram shown in Figure 2(b) (in Fig. 2 in [LDD+10]). Because this net is a convex polygon, Thm. 25.1.4 in [DO07] establishes that it has an infinite number of zippings to various convex polyhedra. The zipping shown in Figure 2(c) folds it to a doubly covered rhombus.

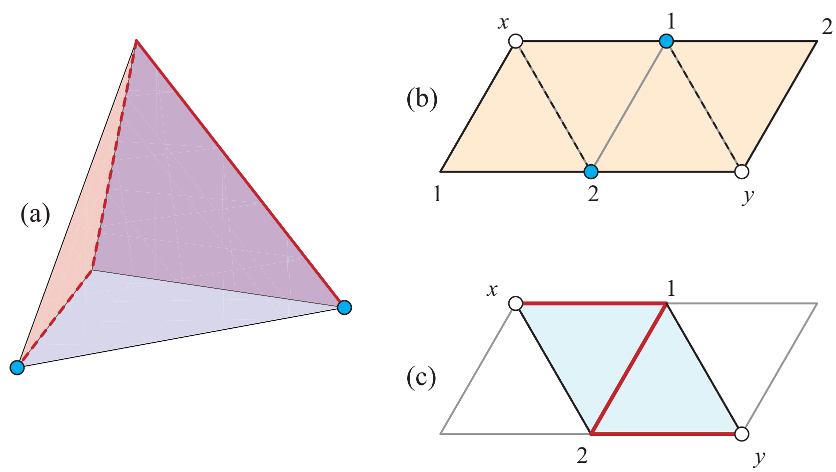

Figure 2: (a) The Hamiltonian cut path on a tetrahedron leads to the Hamiltonian unfolding (b), which zips from x to y (identifying the labeled points) to a flat rhombus polyhedron of side length 1 (c).

2.2 Cube

The cube has three distinct Hamiltonian unfoldings (Fig. 1 in [LDD+10]): one with the path endpoints at opposite cube corners, and two with the path endpoints at either end of a cube edge. One of the latter (shown in Figure 3) produces a 'T'-shape that has no zippings except back to the cube. We call such a zipper unfolding *zip-rigid*. We defer an explanation of how it is known that this unfolding is zip-rigid to Section 2.3 below.

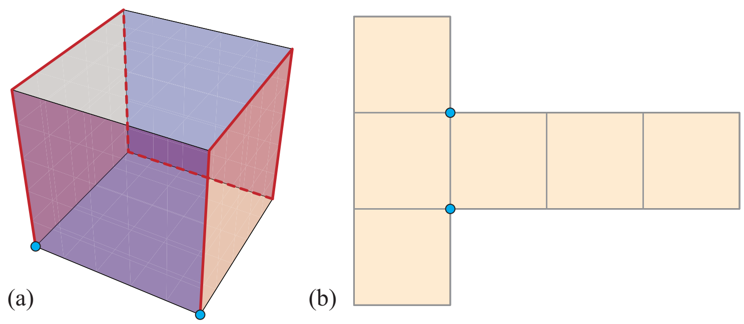

Figure 3: (a) The first Hamiltonian cut path leads to (b) a zip-rigid Hamiltonian 'T'-unfolding of the cube.

The other two Hamiltonian unfoldings of the cube, which we call the 'S'- and the 'Z'-unfoldings, both zip to the same doubly covered parallelogram, as shown in Figures 4 and 6. An animation of the 'S'-folding is shown in Figure 5.

2.3 A Zipping Algorithm

Let P be a polygon, the polygonal net corresponding to a zip-pair of polyhedra Q_1 and Q_2. Each of the two zippings of P are perimeter-halving foldings, with the endpoints of the zip path bisecting the perimeter. If we normalize the perimeter of P to 1 and parametrize it from 0 to 1, we can view the two zippings abstractly as in Figure 7. One of the zip-path endpoints are at 0 and $\frac{1}{2}$, and the other zip-path endpoints are at x and $y = x + \frac{1}{2}$. We seek to find all the locations x that determine a zipping to some convex polyhedron.

As previously mentioned, if P is convex, then every x determines a convex polyhedron (Thm. 25.1.4 in [DO07]), so we henceforth exclude that case. If P is not convex, it has at least one reflex vertex v with internal angle $\beta > \pi$. Now there are only two options at v: (1) v can serve as x, so the zipping starts at $v = x$; or (2) Some strictly convex vertex u_i whose internal angle α_i satisfies $\alpha_i + \beta \leq 2\pi$ is glued to v. If more than one vertex is glued to v, then the folding would not be a zipping, as v would then constitute a junction of degree > 2 in the gluing tree ([DO07, Sec. 25.3]). Note that if u_i glues to v, then x is determined: halfway between u_i and v along the perimeter of P. Thus we only need try each u_i in turn, and check that Alexandrov's conditions hold for the uniquely determined zipping [DO07, Thm. 23.3.1]). This incidentally shows that any P with a reflex vertex admits only $O(n)$ zippings.

For example, applying this algorithm to the cube 'Z'-unfolding in Figure 6 results in six zippings: two copies of the one shown in that figure, two copies of a tetrahedron, one 5-vertex and one 6-vertex polyhedron.

Although this provides a linear-time algorithm for determining all zippings of P, it does not tell us which of these zippings lead to flat polyhedra. Although

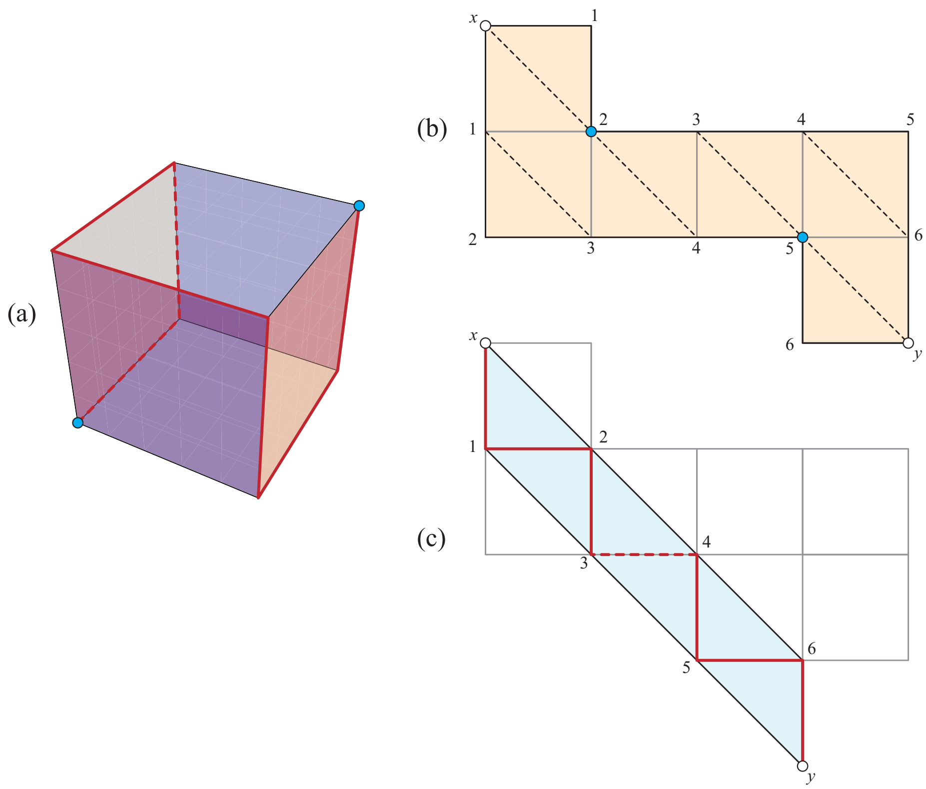

Figure 4: (a) The second Hamiltonian cut path on a cube. (b) The resulting Hamiltonian 'S' unfolding. (c) Zipped according to the indicated point identifications to a parallelogram polyhedron of side lengths 1 and $3\sqrt{2}$.

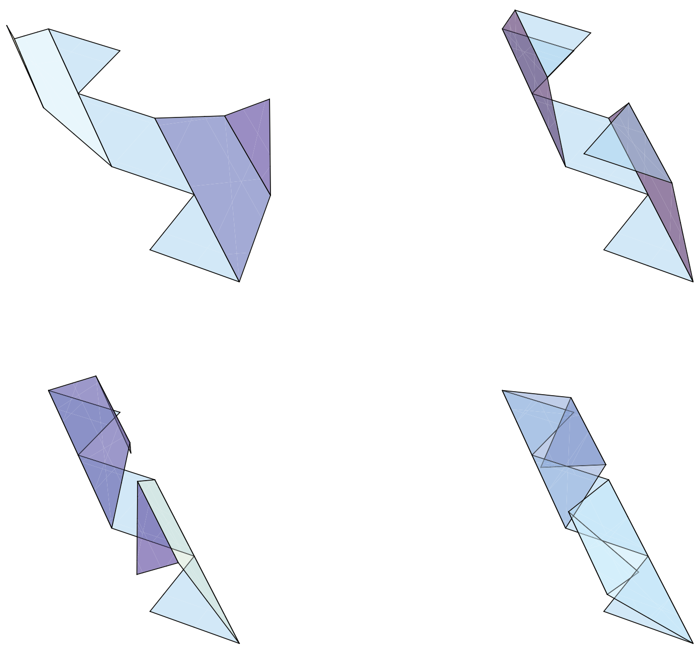

Figure 5: Snapshots from an animation folding the parallelogram in Figure 4(b,c).

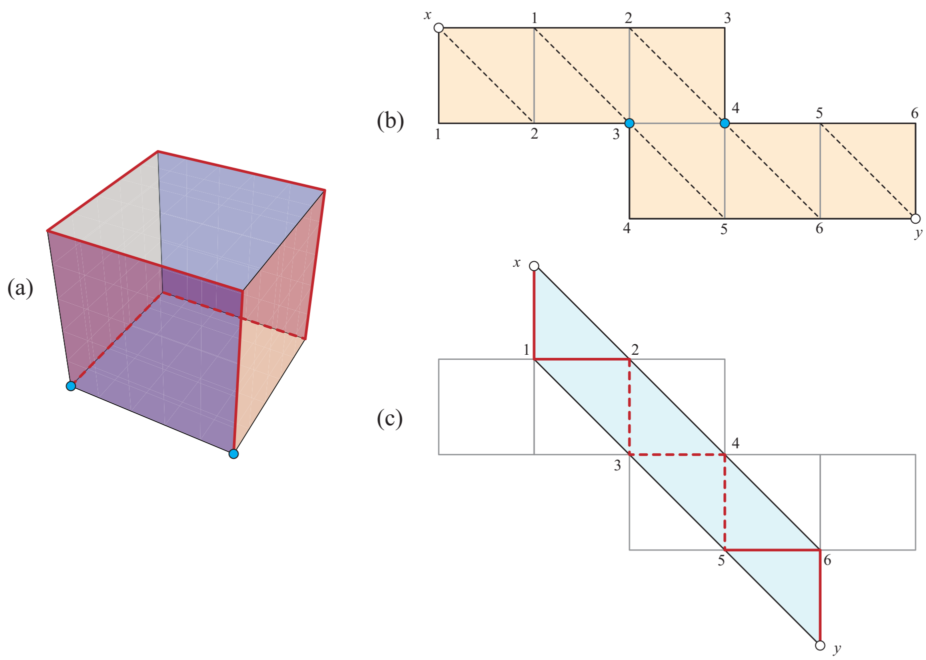

Figure 6: (a) The third Hamiltonian cut path on a cube. (b) The resulting Hamiltonian 'Z' unfolding. (c) Zipped according to the indicated point identifications to a parallelogram polyhedron of side lengths 1 and $3\sqrt{2}$.

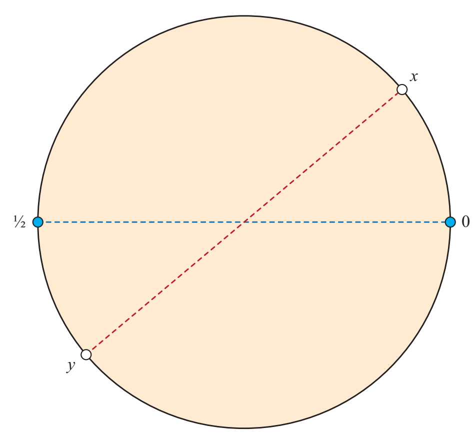

Figure 7: A zip-pair, abstractly. The perimeter has been normalized to 1.

there is an O(n^3) algorithm for deciding if an Alexandrov gluing is flat [O'R10], this remains unimplemented. We resorted to manual folding of the zippings.

2.4 Octahedron

The octahedron also has three distinct Hamiltonian paths,[3] one between the top and bottom vertices (separated by distance 2 in the 1-skeleton), and two paths between adjacent (distance-1) vertices. The first Hamiltonian unfolding both zips to a rectangle as shown in Figure 8, and zips to a parallelogram, Figure 9. I find the rectangle zipping especially surprising, as it derives from a shape all of whose angles are multiples of $\pi/3 = 60°$.

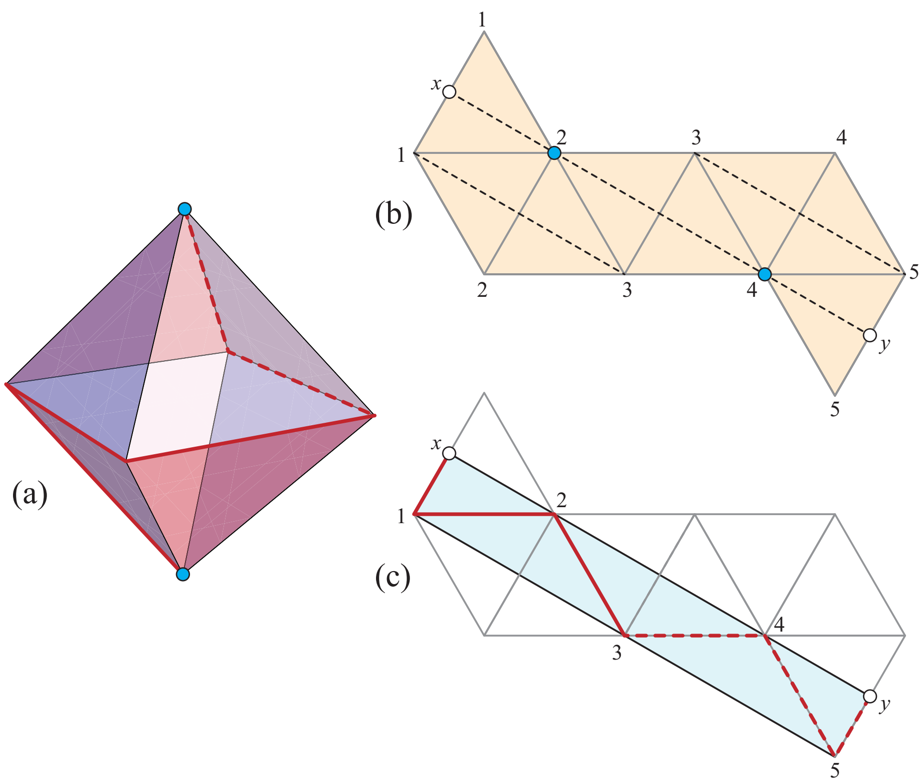

Figure 8: (a) Hamiltonian cut path on an octahedron. (b) Its corresponding Hamiltonian unfolding. (c) Zipping folds it to a flat doubly covered rectangle of dimensions $\frac{1}{2} \times 2\sqrt{3}$.

One of the other Hamiltonian unfoldings of the octahedron, shown in Figure 10, zips to a parallelogram. The other Hamiltonian unfolding does not have

[3] This is natural because the cube and octahedron are duals. However, it is shown in [LDD+10, Fig. 4] that the dual of a Hamiltonian unfolding is not necessarily a Hamiltonian path through the faces of that unfolding.

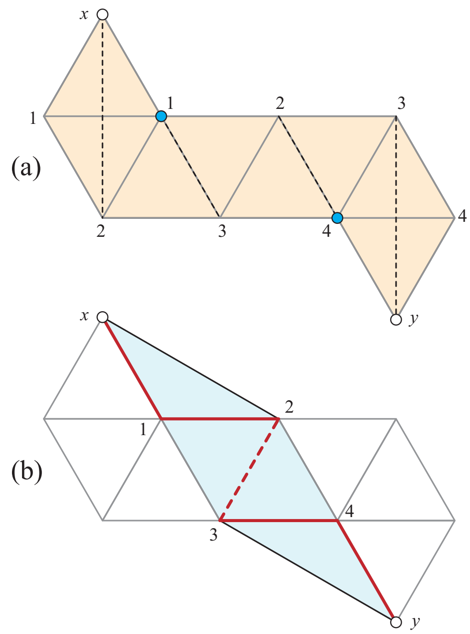

Figure 9: (a) Another zipping of the same unfolding from Figure 8 leads to (b) a $1 \times \sqrt{3}$ parallelogram polyhedron.

a flat zipping, although it does have zippings, e.g., to a tetrahedron all four of whose vertices have curvature π.

I cannot resist mentioning that this last net folds to a flat rectangular polyhedron, whose cut tree, however, is not a zipping: Figure 11.

2.5 Dodecahedron

Every Hamiltonian unfolding of the dodecahedron is zip-rigid, and therefore it has no flat zip pair in the sense posed in our Question above. The reason is as follows. Let x and y be the endpoints of the Hamiltonian path that unfolds the dodecahedron. Then the reflex angle of the net at x and y is $3 \cdot \frac{3}{5}\pi = 324°$, leaving an external angle of $36°$ there. The smallest convex angle in any edge unfolding of the dodecahedron is $\frac{3}{5}\pi = 108°$, so no vertex can glue into x or y. Therefore, a zipping must zip at x and y, leading directly back to the dodecahedron.

We should mention that loosening the criteria posed in our Question leads to a flat refolding of a Hamiltonian net for the dodecahedron. Figure 12 illustrates one such, using the unfolding in Fig. 2 in [LDD+10]. Here the refolding in Figure 12(c) is neither convex nor a zipper folding.

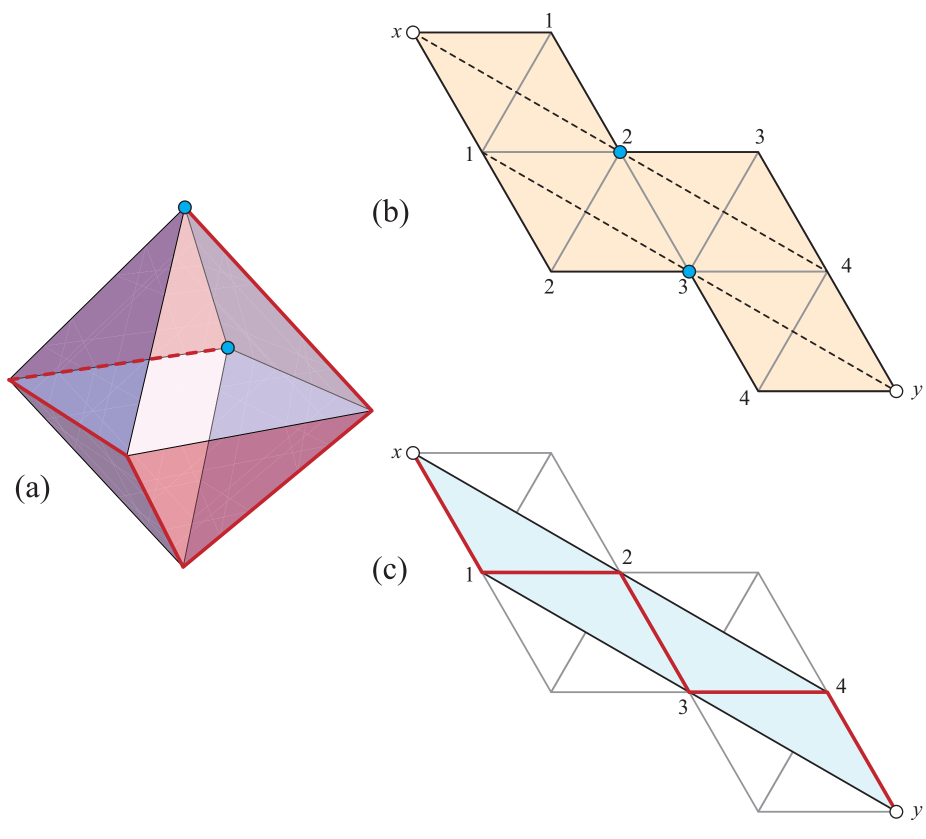

Figure 10: (a) Hamiltonian cut path on an octahedron. (b) Its corresponding Hamiltonian unfolding. (c) Zipping folds it to a flat doubly covered parallelogram of dimensions $1 \times 2\sqrt{3}$.

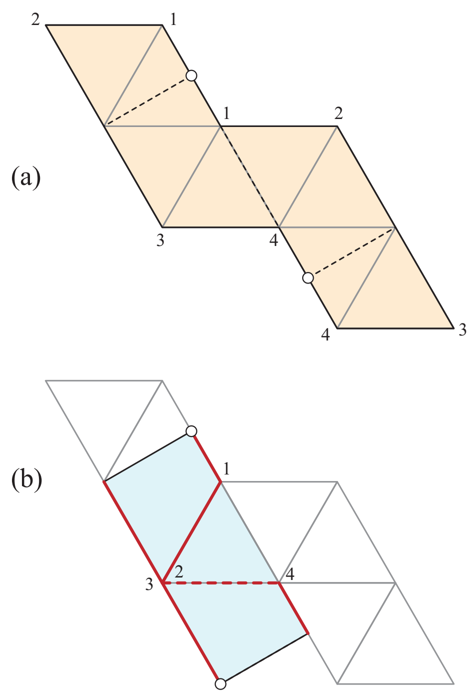

Figure 11: The same Hamiltonian unfolding from Figure 10 folds to a $\frac{\sqrt{3}}{2} \times 2$ doubly covered rectangle, but this folding is not a zipping.

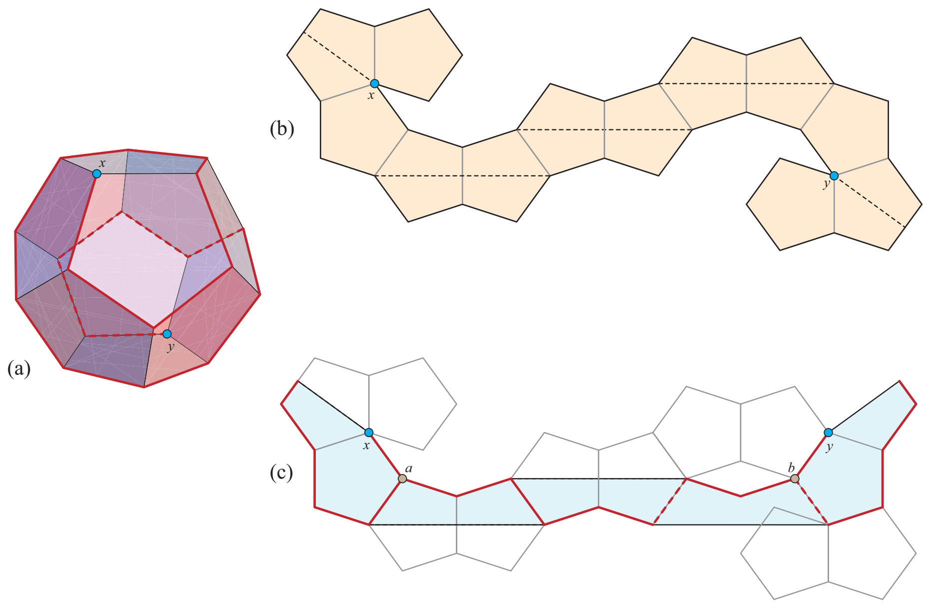

Figure 12: (a) Hamiltonian cut path on a dodecahedron. (b) Its corresponding Hamiltonian unfolding. (c) A non-zipper refolding to a doubly covered flat nonconvex polygon. The cut tree has degree 3 at vertices a and b.

2.6 Icosahedron

For the tetrahedron, cube, and octahedron, it was easy to explore all the Hamiltonian unfoldings, because there are so few (1, 3, and 3 respectively). The icosahedron, however, has hundreds of Hamiltonian unfoldings. At this writing, I do not know precisely how many geometrically distinct Hamiltonian unfoldings it possesses.

The diameter of the icosahedral graph is 3, so the end points of a Hamiltonian path are a distance 1, 2, or 3 apart. Fixing two vertices separated by a distance $d \in \{1, 2, 3\}$, I found that there are, respectively, 512, 608, and 720 *labeled* Hamiltonian paths between them.[4] Of course not all these labeled paths are distinct geometric paths because of symmetries. However, I have not carried out the more difficult enumeration of the number of geometrically distinct (incongruent as paths in $\mathbb{R}^3$) Hamiltonian paths on an icosahedron. But certainly this number is less than $512 + 608 + 720 = 1840$.

For each of these 1840 Hamiltonian unfoldings, I ran the zipping algorithm in Section 2.3, which determined that 82 of the unfoldings had at least one zipping, while all the others are zip-rigid ($82 = 12 + 20 + 50$ in the three classes, respectively). By visual inspection, 18 of these Hamiltonian unfoldings are

[4] It is a curious fact that the number of labeled Hamiltonian cycles through any fixed edge is $2^9 = 512$. The simplicity of this expression suggests there might be a combinatorial explanation, a question I asked on MathOverflow, `http://mathoverflow.net/questions/37788/`.

distinct; they are displayed in Figure 13.

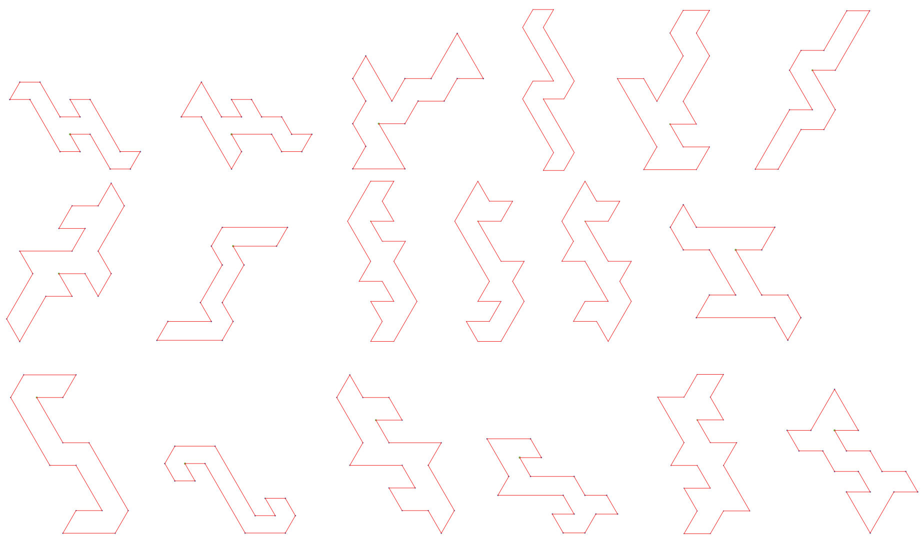

Figure 13: The 18 distinct Hamiltonian unfolding of the icosahedron that each have at least one zipping to another convex polyhedron (Not all are displayed to the same scale.)

Exactly one of these unfoldings (the leftmost in the first row) zips to a parallelogram, as shown in Figure 14. None of the other zips to a parallelogram (but it remains possible that there is some zipping to flat polyhedron different from a doubly covered parallelogram).

3 Future Work

As is evident from the foregoing, there is little theory behind the unfoldings detailed here. The central open problem is to gain more insight into which polyhedra are net pairs, or more specifically, zipper pairs. Perhaps intuition can be strengthened by tackling specific subquestions that fall under this general umbrella. It is easy to list such questions, all of are open because of the lack of a general theory. For example, the Hamiltonian unfoldings of the Archimedean solids detailed in [LDD$^+$10] could be explored.

An interesting specific but tangential question raised by this work is to determine the exact number of geometrically distinct Hamiltonian paths on a regular icosahedron.

Acknowledgments. I thank Stephanie Annessi and Katherine Lipow for help in enumerating and folding the icosahedron Hamiltonian unfoldings.

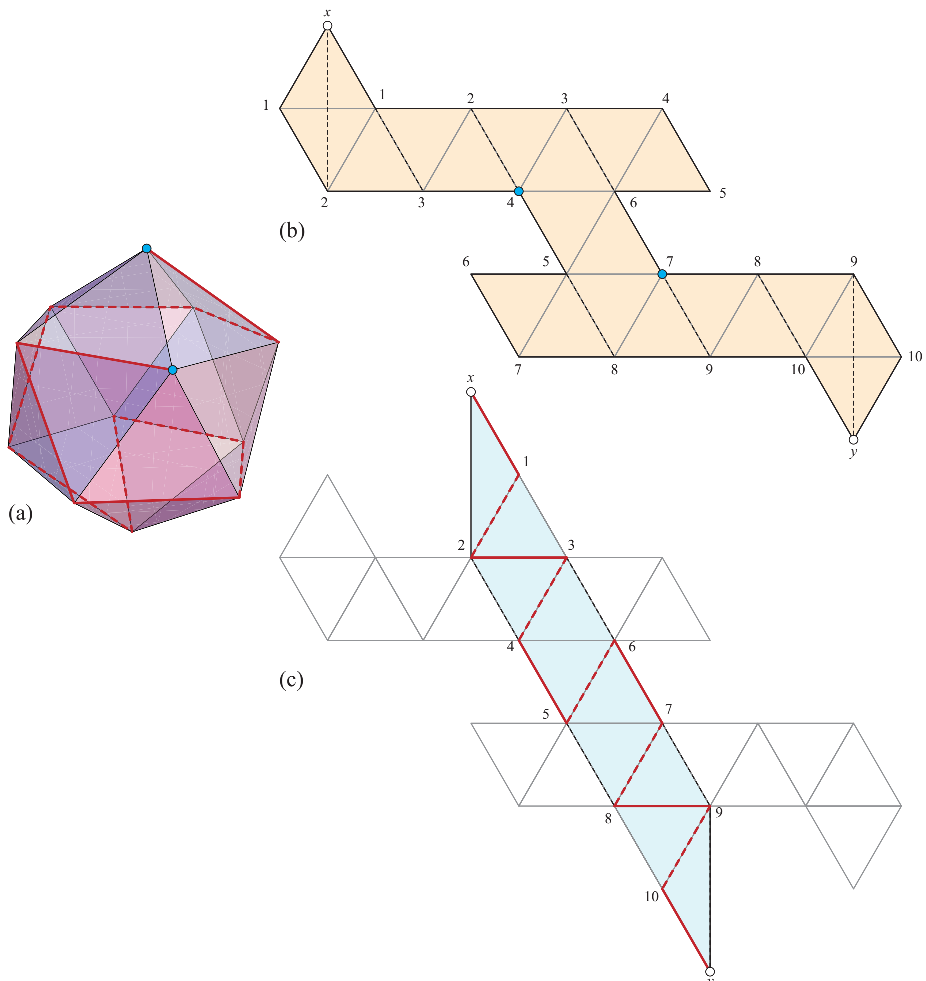

Figure 14: (a) Hamiltonian cut path on an icosahedron. (b) Its corresponding Hamiltonian unfolding. (c) Rezipping folds it to a flat doubly covered parallelogram of side lengths $\sqrt{3}$ and 5.

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