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On Folding a Polygon to a Polyhedron

Joseph O'Rourke[*]

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Abstract

We show that the open problem presented in *Geometric Folding Algorithms: Linkages, Origami, Polyhedra* [DO07] is solved by a theorem of Burago and Zalgaller [BZ96] from more than a decade earlier.

1 Introduction

In [DO07, p. 384] we formulated this open problem:

> OPEN PROBLEM 25.1: Folding Polygons to (Nonconvex) Polyhedra
>
> Does every simple polygon fold (by perimeter gluing) to some simple polyhedron? In particular, does some perimeter halving always lead to a polyhedron?

This note explains how this problem is solved (positively) by a theorem in [BZ96]. There is nothing original in this note. It is merely a description of earlier work, interpreting it in the context of the folding problem.

The theorem of Burago and Zalgaller [BZ96, p. 370] that solves the problem is this:

Theorem 1 (Burago-Zalgaller 1.7) *"Every polyhedron M admits an isometric piecewise-linear $\mathcal{C}^0$ immersion into $\mathbb{R}^3$. If M is orientable or has a nonempty boundary, then M admits an isometric piecewise-linear $\mathcal{C}^0$ embedding into $\mathbb{R}^3$."*

Interpreting this theorem (henceforth, the BZ theorem) and connecting it to the folding problem requires an analysis of its technical terms.

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2 Definition of Terms

First we explain the folding problem from [DO07]. Folding a polygon by perimeter gluing means identifying points of the boundary of the polygon with other points so that the resulting manifold is topologically a sphere, that is, closed, orientable, and with genus zero. Isolated points might have no match, which occurs where the boundary is "zipped" in the neighborhood of such a point. This is perhaps clearest for a perimeter halving. Let x and y be points on the boundary of the polygon that partition the perimeter into two equal-length halves. Then *perimeter halving* identifies the two halves of the boundary, zipping in the neighborhoods of x and y. (See ahead to Figure 1.) Our question was whether every perimeter gluing, and more specifically, every perimeter halving, produces (is realized by) a simple (non-self-intersecting) polyhedron.

This question is an analog of Alexandrov's 1941 theorem, described in [DO07, Sec. 23.3], [Pak10, Sec. 37], and Alexandrov's book [Ale05]. Alexandrov's theorem adds one more condition—that the gluing creates no more than 2π surface angle surrounding any point of the resulting manifold—and concludes that the result is a unique *convex* polyhedron, where "polyhedron" is interpreted to include a flat doubly covered convex polygon. The focus of our problem was to ask for an extension without the 2π convexity condition. The restriction to folding a single polygon is not essential to the problem. Alexandrov considered a more general gluing of a collection of polygons, which has been translated as a gluing of a "development" [Ale05, p. 50ff], terminology not always followed by later authors.

We will draw several definitions from [Pak06] (many also incorporated into [Pak10]), for they seem the clearest on this topic. In particular, the notion of a "gluing" is captured in this definition [Pak06, Sec. 1.3]:

> "Let S be an *abstract 2-dimensional polyhedral surface* defined as a collection of triangles $T_1, \ldots, T_m$ with combinatorial gluing rules. Here each triangle T_i is given by its edge lengths, and whenever two edges are glued, they have equal length. We always assume that S is a connected simplicial complex, and that it is closed (has no boundary) and orientable, i.e. homeomorphic to a sphere with $g \geq 0$ handles."

Restricting to $g = 0$ yields exactly the gluings that are our main focus. Our polygon can be partitioned into triangles so that the perimeter gluing is captured by edge-to-edge gluings of the triangles. To simplify the language, we will call Pak's "abstract 2-dimensional polyhedral surface" a *gluing of polygons*. Next we concentrate on the phrase "every polyhedron M" in the BZ theorem, and show that it refers to gluings of polygons. From [BZ96, p. 369]:

> "By a two-dimensional manifold with polyhedral metric (in brief, a *polyhedron*), we mean a metric space endowed with the structure of a connected compact two-dimensional manifold (possibly with boundary) every point x of which has a neighborhood isometric to

the vertex of a cone. ... The metric is locally flat everywhere except
for a finite collection of points; these points are the 'true' vertices."

Although their naming this concept a "polyhedron" is nonstandard, it is clear
this coincides with the notion of a gluing of polygons. The "true vertices" V
of such a gluing are the points whose total angle differs from 2π, at which the
discrete curvature is concentrated.

For definitions of "immersion" and "embedding," we again turn to
Pak [Pak06]:

> "A (3-dimensional) *realization* of S is defined as a map $f : V \mapsto \mathbb{R}^3$
> such that the Euclidean distance $||v_1, v_2||$ between vertices is equal
> to the edge length $|v_1, v_2|$ of any triangle T_i which contains v_1 and
> v_2.
>
> An *immersion* is a realization where no two triangles have a 2-
> dimensional intersection. For example, a doubly covered triangle is
> a realization in $\mathbb{R}^3$ of a surface homeomorphic to a sphere, but not
> an immersion. An *embedding* is a realization where two triangles
> intersect only by an edge or by a vertex they share. We always con-
> sider surfaces S up to isometry, so we speak of isometric immersions
> and isometric embeddings."

Our interest in the folding problem is focused on embeddings (with the ex-
ception of doubly covered polygons): because we imagine performing the folding
with paper, we do not want the paper to penetrate itself. So we henceforth con-
centrate on the second half of the BZ theorem, which describes embeddings.

In our context, that the embedding is $\mathcal{C}^0$ simply means that it is continu-
ous, that is, without tearing. I believe they include this qualification primarily
to distinguish their result from the famous $\mathcal{C}^1$-smooth embedding theorem of
Nash [Nas54] (improved by Kuiper [Kui55]). This Nash-Kuiper result cannot
be improved to $\mathcal{C}^2$. Histories of immersion and embedding theorems can be
found in [Spr05] and [Sau10].

The key phrase in the BZ theorem is that the embedding is "isometric
piecewise-linear." Isometric simply means distances within the surface are main-
tained. Piecewise-linear means that each triangle T_i from the gluing of polygons
is mapped to a finite collection of triangles in $\mathbb{R}^3$. Continuous isometric defor-
mations are often called *bendings* in the literature (e.g., [Pak10, Sec. 38.1]). The
addition of "piecewise-linear" means that the bending is accomplished with flat
pieces. A nice phrase used in [BZ96, Lem. 2.2] to describe the mapping of one
triangle is that it becomes a "'pleated' surface" in $\mathbb{R}^3$. The triangle gets creased
along a network of straight segments. From the construction in that paper, it
is clear that only a finite number of pieces are used to embed each triangle T_i.

So we may rephrase the embedding part of the BZ theorem that is our focus
as follows:

Theorem 2 *Every gluing of polygons to form an oriented manifold has an iso-
metric embedding as a polyhedral surface (composed of a finite number of flat
triangles) in $\mathbb{R}^3$.*

Because perimeter gluing, and perimeter halving in particular, constitutes a gluing of polygons, this theorem provides a positive answer to Open Problem 25.1.

I presented Figure 1 at the 20th *Symposium on Computational Geometry* (the proceedings cover image) as a challenge to fold to a polyhedron. The figure marks out a particular perimeter-halving folding, which, by BZ's theorem, must fold to a simple polyhedron.
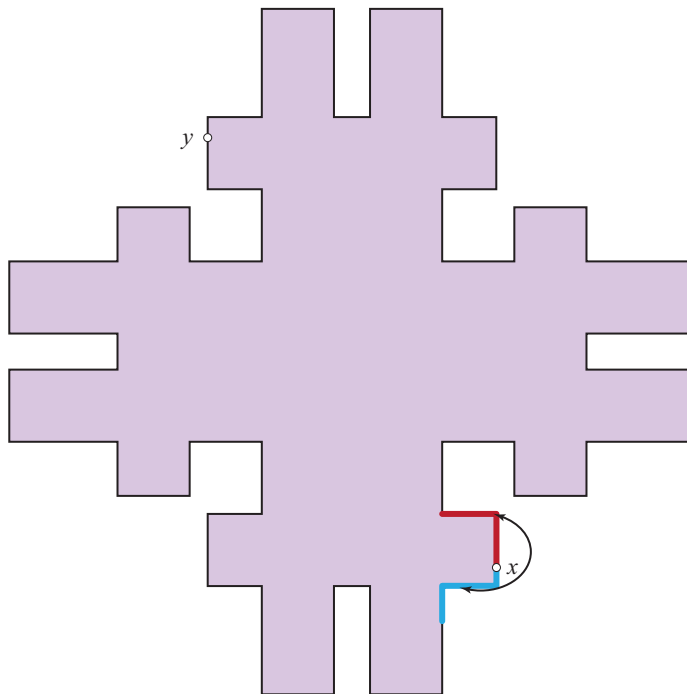

Figure 1: Points x and y are perimeter-halving points. The folding "zips" the perimeter closed from x to y. A short initial segment of the gluing in the neighborhood of x is indicated: the red portion of the boundary glues to the symmetric blue portion. This perimeter gluing continues from x to y.

3 Discussion

One of the motivations of the work in [BZ96] was to show that one can increase the volume of a convex polyhedron by making it nonconvex, but otherwise remaining isometric to the original. This question is explored in great depth in [Pak06] and [Pak08]. The constructions are piecewise-linear, that is, polyhedral.

Although I have been citing [BZ96] as resolving the folding problem, in fact earlier (as yet untranslated) work of Burago and Zalgaller [BZ60] apparently

already sufficed for the special case when the manifold is homeomorphic to a sphere. This is cited in support of Exercise 39.13(a) in [Pak10].

Although the problem is solved by the BZ theorem, their proof is sufficiently complex that it is difficult to see what their construction will produce for a specific example, such as the folding in Figure 1. Saucan has a useful summary of the Burago-Zalgaller construction in [Sau10]. The construction incorporates ideas from Nash's $\mathcal{C}^1$ embedding construction. In particular, Nash's "'spiralling' perturbations" [Nas54, p. 383] of the surface results in a polyhedral surface that is "strongly 'corrugated'," to quote Saucan's apt description. Although the number of triangular facets of the final embedded polyhedron is finite, it does not seem straightforward to provide an explicit upper bound on the number of facets. The work of Bern and Hayes in [BH09] achieves a flat embedding in $\mathbb{R}^2$ with just $O(n)$ facets, where n is the number of triangles of the polygon gluing. Perhaps their work will lead to a more efficient construction in $\mathbb{R}^3$.

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