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# $\pi/2$ -Angle Yao Graphs are Spanners

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# $\pi/2$ -Angle Yao Graphs are Spanners

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## Abstract

We show that the Yao graph  $Y_4$  in the  $L_2$  metric is a spanner with stretch factor  $8(29+23\sqrt{2})$ . Enroute to this, we also show that the Yao graph  $Y_4^\infty$  in the  $L_\infty$  metric is a planar spanner with stretch factor 8.

## 1 Introduction

Let  $V$  be a finite set of points in the plane and let  $G = (V, E)$  be the complete Euclidean graph on  $V$ . We will refer to the points in  $V$  as *nodes*, to distinguish them from other points in the plane. The *Yao graph* [6] with an integer parameter  $k > 0$ , denoted  $Y_k$ , is defined as follows. At each node  $u \in V$ , any  $k$  equally-separated rays originating at  $u$  define  $k$  cones. In each cone, pick a shortest edge  $uv$ , if there is one, and add to  $Y_k$  the directed edge  $\vec{uv}$ . Ties are broken arbitrarily. Most of the time we ignore the direction of an edge  $uv$ ; we refer to the directed version  $\vec{uv}$  of  $uv$  only when its origin ( $u$ ) is important and unclear from the context. We will distinguish between  $Y_k$ , the Yao graph in the Euclidean  $L_2$  metric, and  $Y_k^\infty$ , the Yao graph in the  $L_\infty$  metric. Unlike  $Y_k$  however, in constructing  $Y_k^\infty$  ties are broken by always selecting the most counterclockwise edge; the reason for this choice will become clear in Section 2.

For a given subgraph  $H \subseteq G$  and a fixed  $t \geq 1$ ,  $H$  is called a *t-spanner* for  $G$  if, for any two nodes  $u, v \in V$ , the shortest path in  $H$  from  $u$  to  $v$  is no longer than  $t$  times the length of  $uv$ . The value  $t$  is called the *dilation* or the *stretch factor* of  $H$ . If  $t$  is constant, then  $H$  is called a *length spanner*, or simply a *spanner*.

The class of graphs  $Y_k$  has been much studied. Bose et al. [1] showed that, for  $k \geq 9$ ,  $Y_k$  is a spanner with stretch factor  $\frac{1}{\cos \frac{2\pi}{k} - \sin \frac{2\pi}{k}}$ . In the appendix, we improve the stretch factor and show that, in fact,  $Y_k$  is a spanner for any  $k \geq 7$ . Recently, Molla [4] showed that  $Y_2$  and  $Y_3$  are not

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spanners, and that  $Y_4$  is a spanner with stretch factor  $4(2 + \sqrt{2})$ , for the special case when the nodes in  $V$  are in convex position (see also [2]). The authors conjectured that  $Y_4$  is a spanner for arbitrary point sets. In this paper, we settle their conjecture and prove that  $Y_4$  is a spanner with stretch factor  $8(29 + 23\sqrt{2})$ .

The paper is organized as follows. In Section 2, we prove that the graph  $Y_4^\infty$  is a spanner with stretch factor 8. In Section 3, we prove, in a sequence of Lemmas, several properties for the graph  $Y_4$ . Finally, in Section 4, we use the properties of Section 3 to prove that for every edge  $ab$  in  $Y_4^\infty$ , there exists a path between  $a$  and  $b$  in  $Y_4$ , whose length is not much more than the Euclidean distance between  $a$  and  $b$ . By combining this with the result of Section 2, it follows that  $Y_4$  is a spanner.

## 2 $Y_4^\infty$ : in the $L_\infty$ Metric

In this section we focus on  $Y_4^\infty$ , which has a nicer structure compared to  $Y_4$ . First we prove that  $Y_4^\infty$  is planar. Then we use this property to show that  $Y_4^\infty$  is an 8-spanner. To be more precise, we prove that for any two nodes  $a$  and  $b$ , the graph  $Y_4^\infty$  contains a path between  $a$  and  $b$  whose length (in the  $L_\infty$ -metric) is at most  $8|ab|_\infty$ .

We need a few definitions. We say that two edges  $ab$  and  $cd$  *properly cross* (or *cross*, for short) if they share a point other than an endpoint ( $a, b, c$  or  $d$ ); we say that  $ab$  and  $cd$  *intersect* if they share a point (either an interior point or an endpoint). Let  $Q_1(a), Q_2(a), Q_3(a)$  and  $Q_4(a)$  be the

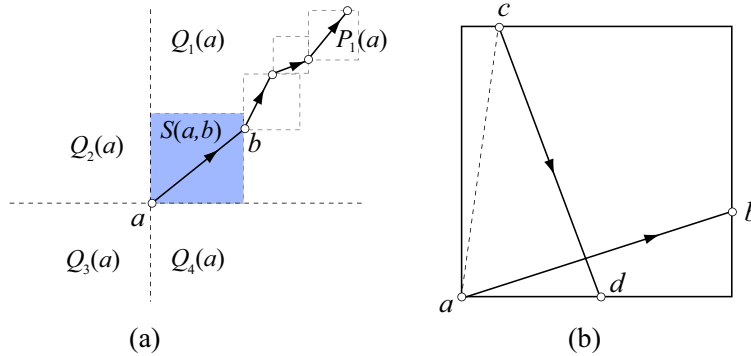


Figure 1: (a) Definitions:  $Q_i(a), P_i(a)$  and  $S(a, b)$ . (b) Lemma 1:  $ab$  and  $cd$  cannot cross.

four quadrants at  $a$ , as in Figure 1a. Let  $P_i(a)$  be the path that starts at point  $a$  and follows the directed Yao edges in quadrant  $Q_i$ . Let  $P_i(a, b)$  be the subpath of  $P_i(a)$  that starts at  $a$  and ends at  $b$ . Let  $|ab|_\infty$  be the  $L_\infty$  distance between  $a$  and  $b$ . Let  $sp(a, b)$  denote a shortest path in  $Y_4^\infty$  between  $a$  and  $b$ . Let  $S(a, b)$  denote the open square with corner  $a$  whose boundary contains  $b$ , and let  $\partial S(a, b)$  denote the boundary of  $S(a, b)$ . These definitions are illustrated in Figure 1a. For a node  $a \in V$ , let  $x(a)$  denote the  $x$ -coordinate of  $a$  and  $y(a)$  denote the  $y$ -coordinate of  $a$ .

**Lemma 1**  $Y_4^\infty$  is planar.

**Proof.** The proof is by contradiction. Assume the opposite. Then there are two edges  $\vec{ab}, \vec{cd} \in Y_4^\infty$  that cross each other. Since  $\vec{ab} \in Y_4^\infty$ ,  $S(a, b)$  must be empty of nodes in  $V$ , and similarly for  $S(c, d)$ . Let  $j$  be the intersection point between  $ab$  and  $cd$ . Then  $j \in S(a, b) \cap S(c, d)$ , meaning that  $S(a, b)$

and  $S(c, d)$  must overlap. However, neither square may contain  $a, b, c$  or  $d$ . It follows that  $S(a, b)$  and  $S(c, d)$  coincide, meaning that  $c$  and  $d$  lie on  $\partial S(a, b)$  (see Figure 1b). Since  $cd$  intersects  $ab$ ,  $c$  and  $d$  must lie on opposite sides of  $ab$ . Thus either  $ac$  or  $ad$  lies counterclockwise from  $ab$ . Assume without loss of generality that  $ac$  lies counterclockwise from  $ab$ ; the other case is identical. Because  $S(a, c)$  coincides with  $S(a, b)$ , we have that  $|ac|_\infty = |ab|_\infty$ . In this case however,  $Y_4^\infty$  would break the tie between  $ac$  and  $ab$  by selecting the most counterclockwise edge, which is  $\vec{ac}$ . This contradicts the fact that  $\vec{ab} \in Y_4^\infty$ . ■

It can be easily shown that each face of  $Y_4^\infty$  is either a triangle or a quadrilateral (except for the outer face). We skip this proof however, since we do not make use of this property in this paper.

**Theorem 1**  $Y_4^\infty$  is an 8-spanner.

**Proof.** We show that, for any pair of points  $a, b \in V$ ,  $|sp(a, b)|_\infty < 8|ab|_\infty$ . The proof is by induction on the pairwise distance between the points in  $V$ . Assume without loss of generality that  $b \in Q_1(a)$ , and  $|ab|_\infty = |x(b) - x(a)|$ . Consider the case in which  $ab$  is a closest pair of points in  $V$  (the base case for our induction). If  $ab \in Y_4^\infty$ , then  $|sp(a, b)|_\infty = |ab|_\infty$ . Otherwise, there must be  $ac \in Y_4^\infty$ , with  $|ac|_\infty = |ab|_\infty$ . But then  $|bc|_\infty < |ab|_\infty$  (see Figure 2a), a contradiction.

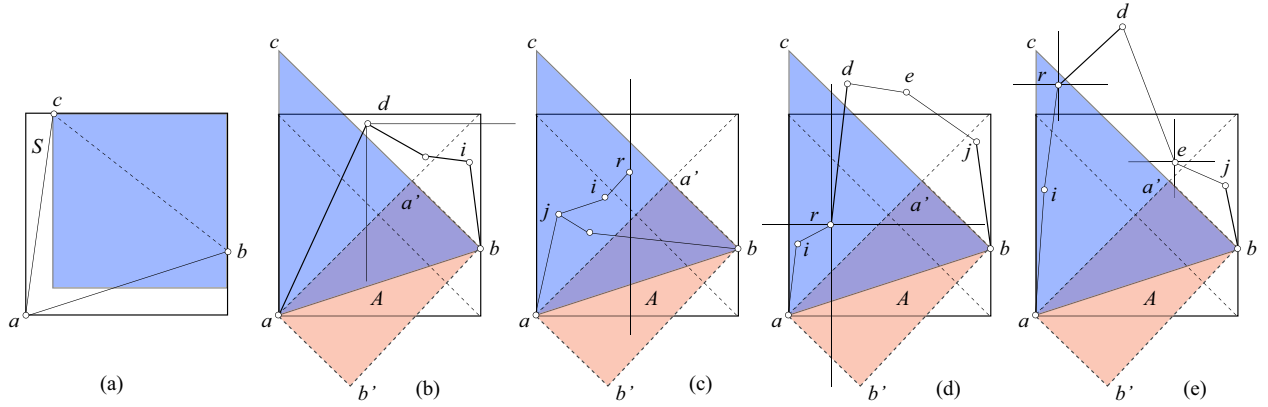


Figure 2: (a) Base case. (b)  $\triangle abc$  empty (c)  $\triangle abc$  non-empty,  $P_{ar} \cap P_2(b) = \{j\}$  (d)  $\triangle abc$  non-empty,  $P_{ar} \cap P_2(b) = \emptyset$ ,  $e$  above  $r$  (e)  $\triangle abc$  non-empty,  $P_{ar} \cap P_2(b) = \emptyset$ ,  $e$  below  $r$ .

Assume now that the inductive hypothesis holds for all pairs of points closer than  $|ab|_\infty$ . If  $ab \in Y_4^\infty$ , then  $|sp(a, b)|_\infty = |ab|_\infty$  and the proof is finished. If  $ab \notin Y_4^\infty$ , then the square  $S(a, b)$  must be nonempty.

Let  $A$  be the rectangle  $ab'ba'$  as in Figure 2b, where  $ba'$  and  $bb'$  are parallel to the diagonals of  $S$ . If  $A$  is nonempty, then we can use induction to prove that  $|sp(a, b)|_\infty \leq 8|ab|_\infty$  as follows. Pick  $c \in A$  arbitrary. Then  $|ac|_\infty + |cb|_\infty = |x(c) - x(a)| + |x(b) - x(c)| = |ab|_\infty$ , and by the inductive hypothesis  $sp(a, c) \oplus sp(c, b)$  is a path in  $Y_4^\infty$  no longer than  $8|ac|_\infty + 8|cb|_\infty = 8|ab|_\infty$ ; here  $\oplus$  represents the concatenation operator. Assume now that  $A$  is empty. Let  $c$  be at the intersection between the line supporting  $ba'$  and the vertical line through  $a$  (see Figure 2b). We discuss two cases, depending on whether  $\triangle abc$  is empty of points or not.

**Case 1:**  $\triangle abc$  is empty of points. Let  $ad \in P_1(a)$ . We show that  $P_4(d)$  cannot contain an edge crossing  $ab$ . Assume the opposite, and let  $st \in P_4(d)$  cross  $ab$ . Since  $\triangle abc$  is empty,  $s$  must lie

above  $bc$  and  $t$  below  $ab$ , therefore  $|st|_\infty \geq |y(s) - y(t)| > |y(s) - y(b)| = |sb|_\infty$ , contradicting the fact that  $st \in Y_4^\infty$ . It follows that  $P_4(d)$  and  $P_2(b)$  must meet in a point  $i \in P_4(d) \cap P_2(b)$  (see Figure 2b). Now note that  $|P_4(d, i) \oplus P_2(b, i)|_\infty \leq |x(d) - x(b)| + |y(d) - y(b)| < 2|ab|_\infty$ . Thus we have that

$$|sp(a, b)|_\infty \leq |ad \oplus P_4(d, i) \oplus P_2(b, i)|_\infty < |ab|_\infty + 2|ab|_\infty = 3|ab|_\infty.$$

**Case 2:**  $\triangle abc$  is nonempty. In this case, we seek a short path from  $a$  to  $b$  that does not cross to the underside of  $ab$ . This is to avoid oscillating paths that cross  $ab$  arbitrarily many times. Let  $r$  be the rightmost point that lies inside  $\triangle abc$ . Arguments similar to the ones used in Case 1 show that  $P_3(r)$  cannot cross  $ab$  and therefore it must meet  $P_1(a)$  in a point  $i$ . Then  $P_{ar} = P_1(a, i) \oplus P_3(r, i)$  is a path in  $Y_4^\infty$  of length

$$|P_{ar}|_\infty < |x(a) - x(r)| + |y(a) - y(r)| < |ab|_\infty + 2|ab|_\infty = 3|ab|_\infty. \quad (1)$$

The term  $2|ab|_\infty$  in the inequality above represents the fact that  $|y(a) - y(r)| \leq |y(a) - y(c)| \leq 2|ab|_\infty$ . Consider first the simpler situation in which  $P_2(b)$  meets  $P_{ar}$  in a point  $j \in P_2(b) \cap P_{ar}$  (see Figure 2c). Let  $P_{ar}(a, j)$  be the subpath of  $P_{ar}$  extending between  $a$  and  $j$ . Then  $P_{ar}(a, j) \oplus P_2(b, j)$  is a path in  $Y_4^\infty$  from  $a$  to  $b$ , therefore

$$|sp(a, b)|_\infty \leq |P_{ar}(a, j) \oplus P_2(b, j)|_\infty < 2|y(j) - y(a)| + |ab|_\infty \leq 5|ab|_\infty.$$

Consider now the case when  $P_2(b)$  does not intersect  $P_{ar}$ . We argue that, in this case,  $Q_1(r)$  may not be empty. Assume the opposite. Then no edge  $st \in P_2(b)$  may cross  $Q_1(r)$ . This is because, for any such edge,  $|sr|_\infty < |st|_\infty$ , contradicting  $st \in Y_4^\infty$ . This implies that  $P_2(b)$  intersects  $P_{ar}$ , again a contradiction to our assumption.

We have established that  $Q_1(r)$  is nonempty. Let  $rd \in P_1(r)$ . The fact that  $P_2(b)$  does not intersect  $P_{ar}$  implies that  $d$  lies to the left of  $b$ . The fact that  $r$  is the rightmost point in  $\triangle abc$  implies that  $d$  lies outside  $\triangle abc$  (see Figure 2d). It also implies that  $P_4(d)$  shares no points with  $\triangle abc$ . This along with arguments similar to the ones used in case 1 show that  $P_4(d)$  and  $P_2(b)$  meet in a point  $j \in P_4(d) \cap P_2(b)$ . Thus we have found a path

$$P_{ab} = P_1(a, i) \oplus P_3(r, i) \oplus rd \oplus P_4(d, j) \oplus P_2(b, j) \quad (2)$$

extending from  $a$  to  $b$  in  $Y_4^\infty$ . If  $|rd|_\infty = |x(d) - x(r)|$ , then  $|rd|_\infty < |x(b) - x(a)| = |ab|_\infty$ , and the path  $P_{ab}$  has length

$$|P_{ab}|_\infty \leq 2|y(d) - y(a)| + |ab|_\infty < 7|ab|_\infty. \quad (3)$$

In the above, we used the fact that  $|y(d) - y(a)| = |y(d) - y(r)| + |y(r) - y(a)| < |ab|_\infty + 2|ab|_\infty$ . Suppose now that

$$|rd|_\infty = |y(d) - y(r)|. \quad (4)$$

In this case, it is unclear whether the path  $P_{ab}$  defined by (2) is short, since  $rd$  can be arbitrarily long compared to  $ab$ . Let  $e$  be the clockwise neighbor of  $d$  along the path  $P_{ab}$  ( $e$  and  $b$  may coincide). Then  $e$  lies below  $d$ , and either  $de \in P_4(d)$ , or  $ed \in P_2(e)$  (or both).

1. If  $e$  lies above  $r$ , or at the same level as  $r$  (i.e.,  $e \in Q_1(r)$ , as in Figure 2d), then

$$|y(e) - y(r)| < |y(d) - y(r)| \quad (5)$$

Since  $rd \in P_1(r)$  and  $e$  is in the same quadrant of  $r$  as  $d$ , we have  $|rd|_\infty \leq |re|_\infty$ . This along with inequalities (4) and (5) implies  $|re|_\infty > |y(e) - y(r)|$ , which in turn implies  $|re|_\infty = |x(e) - x(r)| \leq |ab|_\infty$ , and so  $|rd|_\infty \leq |ab|_\infty$ . Then inequality (3) applies here as well, showing that  $|P_{ab}|_\infty < 7|ab|_\infty$ .

2. If  $e$  lies below  $r$  (as in Figure 2e), then

$$|ed|_\infty \geq |y(d) - y(e)| \geq |y(d) - y(r)| = |rd|_\infty. \quad (6)$$

Assume first that  $ed \in P_2(e)$ , or  $|ed|_\infty = |x(e) - x(d)|$ . In either case,

$$|ed|_\infty \leq |er|_\infty < 2|ab|_\infty.$$

This along with inequality (6) shows that  $|rd|_\infty < 2|ab|_\infty$ . Substituting this upper bound in (2), we get

$$|P_{ab}|_\infty \leq 2|y(d) - y(a)| + 2|ab|_\infty < 8|ab|_\infty.$$

Assume now that  $ed \notin P_2(e)$ , and  $|ed|_\infty = |y(e) - y(d)|$ . Then  $ee' \in P_2(e)$  cannot go above  $d$  (otherwise  $|ed|_\infty < |ee'|_\infty$ , contradicting  $ee' \in P_2(e)$ ). This along with the fact  $de \in P_4(d)$  implies that  $P_2(e)$  intersects  $P_{ar}$  in a point  $k$ . Redefine

$$P_{ab} = P_{ar}(a, k) \oplus P_2(e, k) \oplus P_4(e, j) \oplus P_2(b, j)$$

Then  $P_{ab}$  is a path in  $Y_4^\infty$  from  $a$  to  $b$  of length

$$|P_{ab}| \leq 2|y(r) - y(a)| + |ab|_\infty \leq 5|ab|_\infty.$$

We have established that  $|sp(a, b)|_\infty \leq |P_{ab}|_\infty < 8|ab|_\infty$ . This concludes the proof.  $\blacksquare$

This theorem will be employed in Section 4.

### 3 $Y_4$ : in the $L_2$ Metric

In this section we establish basic properties of  $Y_4$ . The ultimate goal of this section is to show that, if two edges in  $Y_4$  cross, there is a short path between their endpoints (Lemma 8). We begin with a few definitions.

Let  $Q(a, b)$  denote the infinite quadrant with origin at  $a$  that contains  $b$ . For a pair of nodes  $a, b \in V$ , define recursively a directed path  $\mathcal{P}(a \rightarrow b)$  from  $a$  to  $b$  in  $Y_4$  as follows. If  $a = b$ , then  $\mathcal{P}(a \rightarrow b) = \text{null}$ . If  $a \neq b$ , there must exist  $\vec{ac} \in Y_4$  that lies in  $Q(a, b)$ . In this case, define

$$\mathcal{P}(a \rightarrow b) = \vec{ac} \oplus \mathcal{P}(c \rightarrow b).$$

Recall that  $\oplus$  represents the concatenation operator. This definition is illustrated in Figure 3a. Fischer et al. [3] show that  $\mathcal{P}(a \rightarrow b)$  is well defined and lies entirely inside the square centered at  $b$  whose boundary contains  $a$ .

For any node  $a \in V$ , let  $D(a, r)$  denote the open disk centered at  $a$  of radius  $r$ , and let  $\partial D(a, r)$  denote the boundary of  $D(a, r)$ . Let  $D[a, r] = D(a, r) \cup \partial D(a, r)$ . For any path  $P$  and any pair of nodes  $a$  and  $b$  on  $P$ , let  $P[a, b]$  denote the subpath of  $P$  that starts at  $a$  and ends at  $b$ . Let  $R(a, b)$  denote the closed rectangle with diagonal  $ab$ .

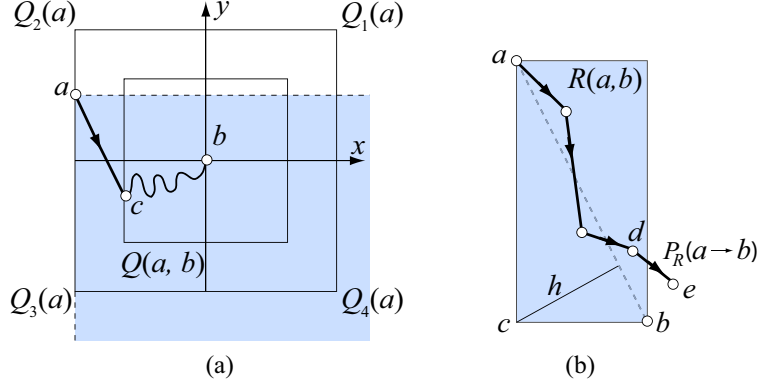


Figure 3: Definitions. (a)  $Q(a, b)$  and  $\mathcal{P}(a \rightarrow b)$ . (b)  $\mathcal{P}_R(a \rightarrow b)$ .

For a fixed pair of nodes  $a, b \in V$ , define a path  $\mathcal{P}_R(a \rightarrow b)$  as follows. Let  $e \in V$  be the first node along  $\mathcal{P}(a \rightarrow b)$  that is not strictly interior to  $R(a, b)$ . Then  $\mathcal{P}_R(a \rightarrow b)$  is the subpath of  $\mathcal{P}(a \rightarrow b)$  that extends between  $a$  and  $e$ . In other words,  $\mathcal{P}_R(a \rightarrow b)$  is the path that follows the  $Y_4$  edges pointing towards  $b$ , truncated as soon as it leaves the rectangle with diagonal  $ab$ , or as it reaches  $b$ . Formally,

$$\mathcal{P}_R(a \rightarrow b) = \mathcal{P}(a \rightarrow b)[a, e]$$

This definition is illustrated in Figure 3b.

Our proofs will make use of the following two propositions.

**Proposition 1** *The sum of the lengths of crossing diagonals of a nondegenerate (necessarily convex) quadrilateral  $abcd$  is strictly greater than the sum of the lengths of either pair of opposite sides:*

$$\begin{aligned} |ac| + |bd| &> |ab| + |cd| \\ |ac| + |bd| &> |bc| + |da| \end{aligned}$$

This can be proved by partitioning the diagonals into two pieces each at their intersection point, and then applying the triangle inequality twice.

**Proposition 2** *For any triangle  $\triangle abc$ , the following inequalities hold:*

$$|ac|^2 \begin{cases} < |ab|^2 + |bc|^2, & \text{if } \angle abc < \pi/2 \\ = |ab|^2 + |bc|^2, & \text{if } \angle abc = \pi/2 \\ > |ab|^2 + |bc|^2, & \text{if } \angle abc > \pi/2 \end{cases}$$

This proposition follows immediately from the Law of Cosines applied to triangle  $\triangle abc$ .

**Lemma 2** *For each pair of nodes  $a, b \in V$ ,*

$$|\mathcal{P}_R(a \rightarrow b)| \leq |ab|\sqrt{2} \tag{7}$$

*Furthermore, each edge of  $\mathcal{P}_R(a \rightarrow b)$  is no longer than  $|ab|$ .*



**Proof.** Let  $c$  be one of the two corners of  $R(a, b)$ , other than  $a$  and  $b$ . Let  $\vec{de} \in \mathcal{P}_R(a \rightarrow b)$  be the last edge on  $\mathcal{P}_R(a \rightarrow b)$ , which necessarily intersects  $\partial R(a, b)$  (note that it is possible that  $e = b$ ). Refer to Figure 3b. Then  $|de| \leq |db|$ , otherwise  $\vec{de}$  could not be in  $Y_4$ . Since  $db$  lies in the rectangle with diagonal  $ab$ , we have that  $|db| \leq |ab|$ , and similarly for each edge on  $\mathcal{P}_R(a \rightarrow b)$ . This establishes the latter claim of the lemma. For the first claim of the lemma, let

$$p = \mathcal{P}_R(a \rightarrow b)[a, d] \oplus db$$

Since  $|de| \leq |db|$ , we have that  $|\mathcal{P}_R(a \rightarrow b)| \leq |p|$ . Since  $p$  lies entirely inside  $R(a, b)$  and consists of edges pointing towards  $b$ , we have that  $p$  is an  $xy$ -monotone path. It follows that  $|p| \leq |ac| + |cb|$ . We now show that  $|ac| + |cb| \leq |ab|\sqrt{2}$ , thus establishing the first claim of the lemma.

Let  $x = |ac|$  and  $y = |cb|$ . Then the inequality  $|ac| + |cb| \leq |ab|\sqrt{2}$  can be written as  $x + y \leq \sqrt{2x^2 + 2y^2}$ , which is equivalent to  $(x - y)^2 \geq 0$ . This latter inequality obviously holds, completing the proof of the lemma.  $\blacksquare$

**Lemma 3** *Let  $a, b, c, d \in V$  be four disjoint nodes such that  $\vec{ab}, \vec{cd} \in Y_4$ ,  $b \in Q_i(a)$  and  $d \in Q_i(c)$ , for some  $i \in \{1, 2, 3, 4\}$ . Then  $ab$  and  $cd$  cannot cross each other.*

**Proof.** We may assume without loss of generality that  $i = 1$  and  $c$  is to the left of  $a$ . The proof is by contradiction. Assume that  $ab$  and  $cd$  cross each other. Let  $j$  be the intersection point between  $ab$  and  $cd$  (see Figure 4a). Since  $j \in Q_1(a) \cap Q_1(c)$ , it follows that  $d \in Q_1(a)$  and  $b \in Q_1(c)$ . Thus  $|ab| \leq |ad|$ , because otherwise,  $\vec{ab}$  cannot be in  $Y_4$ . By Proposition 1 applied to the quadrilateral  $adb$ ,

$$|ad| + |cb| < |ab| + |cd|$$

This along with the fact that  $|ab| \leq |ad|$  implies that  $|cb| < |cd|$ , contradicting the fact that  $\vec{cd} \in Y_4$ .  $\blacksquare$

The next four lemmas (4–8) each concern a pair of crossing  $Y_4$  edges, culminating (in Lemma 8) in the conclusion that there is a short path in  $Y_4$  between a pair of endpoints of those edges.

**Lemma 4** *Let  $a, b, c$  and  $d$  be four disjoint nodes in  $V$  such that  $\vec{ab}, \vec{cd} \in Y_4$ , and  $ab$  crosses  $cd$ . Then the following are true: (i) the ratio between the shortest side and the longer diagonal of the quadrilateral  $acbd$  is no greater than  $1/\sqrt{2}$ , and (ii) the shortest side of the quadrilateral  $acbd$  is strictly shorter than either diagonal.*

**Proof.** The first part of the lemma is a well-known fact that holds for any quadrilateral (see [5], for instance). For the second part of the lemma, let  $ab$  be the shorter of the diagonals of  $acbd$ , and assume without loss of generality that  $\vec{ab} \in Q_1(a)$ . Imagine two disks  $D_a = D(a, |ab|)$  and  $D_b = D(b, |ab|)$ , as in Figure 4b. If either  $c$  or  $d$  belongs to  $D_a \cup D_b$ , then the lemma follows: a shortest quadrilateral edge is shorter than  $|ab|$ .

So suppose that neither  $c$  nor  $d$  lies in  $D_a \cup D_b$ . In this case, we use the fact that  $cd$  crosses  $ab$  to show that  $\vec{cd}$  cannot be an edge in  $Y_4$ . Define the following regions (see Figure 4b):

$$\begin{aligned} R_1 &= (Q_1(a) \cap Q_2(b)) \setminus (D_a \cup D_b) \\ R_2 &= (Q_2(a) \cap Q_3(b)) \setminus (D_a \cup D_b) \\ R_3 &= (Q_4(a) \cap Q_3(b)) \setminus (D_a \cup D_b) \\ R_4 &= (Q_1(a) \cap Q_4(b)) \setminus (D_a \cup D_b) \end{aligned}$$

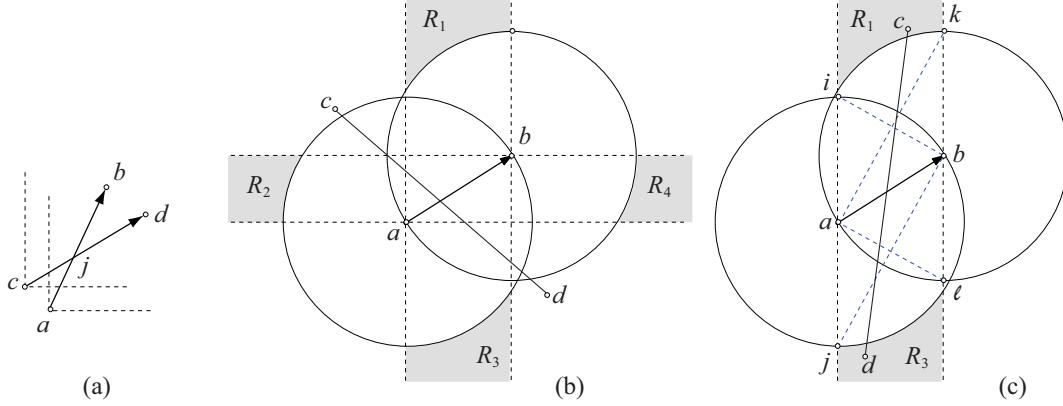


Figure 4: (a) Lemma 3. (b) Lemma 4:  $c \notin R_1 \cup R_2 \cup R_3 \cup R_4$  (c) Lemma 4:  $c \in R_1$ .

If the node  $c$  is not inside any of the regions  $R_i$ , for  $i = \{1, 2, 3, 4\}$ , then the nodes  $a$  and  $b$  are in the same quadrant of  $c$  as  $d$ . In this case, note that either  $\angle cad > \pi/2$  or  $\angle cbd > \pi/2$ , which implies that either  $|ca|$  or  $|cb|$  is strictly smaller than  $|cd|$ . These together show that  $\vec{cd} \notin Y_4$ .

So assume that  $c$  is in  $R_i$  for some  $i \in \{1, 2, 3, 4\}$ . In this situation, the node  $d$  must lie in the region  $R_j$ , with  $j = (i + 2) \bmod 4$  (with the understanding that  $R_0 = R_4$ ), because otherwise, (i)  $a$  and  $d$  are in the same quadrant of  $c$  and  $|ca| < |cd|$  or (ii)  $b$  and  $d$  are in the same quadrant of  $c$  and  $|cb| < |cd|$ . Either case contradicts the fact  $\vec{cd} \in Y_4$ . Consider now the case  $c \in R_1$  and  $d \in R_3$ ; the other cases are treated similarly. Let  $i$  and  $j$  be the intersection points between  $D_a$  and the vertical line through  $a$ . Similarly, let  $k$  and  $\ell$  be the intersection points between  $D_b$  and the vertical line through  $b$  (see Figure 4c). Since  $ij$  is a diameter of  $D_a$ , we have that  $\angle ibj = \pi/2$  and similarly  $\angle kal = \pi/2$ . Also note that  $\angle cbd \geq \angle ibj = \pi/2$ , meaning that  $|cd| > |cb|$ . Similarly,  $\angle cad \geq \angle kal = \pi/2$ , meaning that  $|cd| > |ca|$ . These along with the fact that at least one of  $a$  and  $b$  is in the same quadrant for  $c$  as  $d$ , imply that  $\vec{cd} \notin Y_4$ . This completes the proof.  $\blacksquare$

**Lemma 5** Let  $a, b, c, d$  be four distinct nodes in  $V$ , with  $c \in Q_1(a)$ , such that

- (a)  $\vec{ab} \in Q_1(a)$  and  $\vec{cd} \in Q_2(c)$  are two edges in  $Y_4$  that cross each other.
- (b)  $ad$  is a shortest side of the quadrilateral  $acbd$ .

Then  $\mathcal{P}_R(a \rightarrow d)$  and  $\mathcal{P}_R(d \rightarrow a)$  have a nonempty intersection.

**Proof.** The proof consists of two parts showing that the following claims hold: (i)  $d \in Q_2(a)$  and (ii)  $\mathcal{P}_R(d \rightarrow a)$  does not cross  $ab$ .

Before we prove these two claims, let us argue that they are sufficient to prove the lemma. Lemma 3 and (i) imply that  $\mathcal{P}_R(a \rightarrow d)$  cannot cross  $cd$ . As a result,  $\mathcal{P}_R(a \rightarrow d)$  intersects the left side of the rectangle  $R(d, a)$ . Consider the last edge  $\vec{xy}$  of the path  $\mathcal{P}_R(d \rightarrow a)$ . If this edge crosses the right side of  $R(a, d)$ , then (ii) implies that  $y$  is in the wedge bounded by  $ab$  and the upwards vertical ray starting at  $a$ ; this implies that  $|ay| < |ab|$ , contradicting the fact that  $\vec{ab}$  is an edge in  $Y_4$ . Therefore,  $\vec{xy}$  intersects the bottom side of  $R(d, a)$ , and the lemma follows (see Figure 5b).

To prove the first claim (i), we observe that the assumptions in the lemma imply that  $d \in Q_1(a) \cup Q_2(a)$ . Therefore, it suffices to prove that  $d$  is not in  $Q_1(a)$ . Assume to the contrary that

$d \in Q_1(a)$ . Since  $c \in Q_1(a)$ , it must be that  $b \in Q_2(c)$ ; otherwise,  $\angle acb \geq \pi/2$ , which implies  $|ab| > |ac|$ , contradicting the fact that  $\vec{ab} \in Y_4$ . Let  $i$  and  $j$  be the intersection points between  $cd$  and  $\partial D(a, |ab|)$ , where  $i$  is to the left of  $j$ . Since  $\angle dbc \geq \angle ibj > \pi/2$ , we have  $|cb| < |cd|$ . This, together with the fact that  $b$  and  $d$  are in the same quadrant  $Q_2(c)$ , contradicts the assumption that  $\vec{cd}$  is an edge in  $Y_4$ . This completes the proof of claim (i).

Next we prove claim (ii) by contradiction. Thus, we assume that there is an edge  $\vec{xy}$  on the path  $\mathcal{P}_R(d \rightarrow a)$  that crosses  $ab$ . Then necessarily  $x \in R(a, d)$  and  $y \in Q_1(a) \cup Q_4(a)$ . If  $y \in Q_4(a)$ , then  $\angle xay > \pi/2$ , meaning that  $|xy| > |xa|$ , a contradiction to the fact that  $\vec{xy} \in Y_4$ . Thus, it must be that  $y \in Q_1(a)$ , as in Figure 5a. This implies that  $|ab| \leq |ay|$ , because  $\vec{ab} \in Y_4$ .

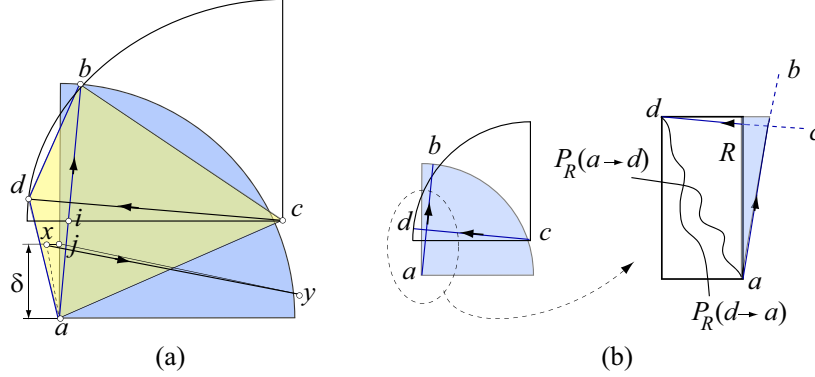


Figure 5: (a) Lemma 5:  $xy \in \mathcal{P}_R(d \rightarrow a)$  cannot cross  $ab$ .

The contradiction to our assumption that  $\vec{xy}$  crosses  $ab$  will be obtained by proving that  $|xy| > |xa|$ . Indeed, this inequality contradicts the fact that  $\vec{xy} \in Y_4$ .

Let  $\delta$  be the distance from  $x$  to the horizontal line through  $a$ . Our intermediate goal is to show that

$$\delta \leq |ab|/\sqrt{2}. \quad (8)$$

We claim that  $\angle acb < \pi/2$ . Indeed, if this is not the case, then  $|ac| < |ab|$ , contradicting the fact that  $\vec{ab}$  is an edge in  $Y_4$ . By a similar argument, and using the fact that  $\vec{cd}$  is an edge in  $Y_4$ , we obtain the inequality  $\angle cbd < \pi/2$ . We now consider two cases, depending on the relative lengths of  $ac$  and  $cb$ .

1. Assume first that  $|ac| > |cb|$ . If  $\angle cad \geq \pi/2$ , then  $|cd| \geq |ac| > |cb|$ , contradicting the fact that  $\vec{cd}$  is an edge in  $Y_4$  (recall that  $b$  and  $d$  are in the same quadrant of  $c$ ). Therefore, we have  $\angle cad < \pi/2$ . Thus far we have established that three angles of the convex quadrilateral  $acbd$  are acute. It follows that the fourth one ( $\angle adb$ ) is obtuse. Proposition 2 applied to  $\triangle adb$  tells us that

$$|ab|^2 > |ad|^2 + |db|^2 \geq 2|ad|^2,$$

where the latter inequality follows from the assumption that  $ad$  is a shortest side of  $acbd$  (and, therefore,  $|db| \geq |ad|$ ). Thus, we have that  $|ad| \leq |ab|/\sqrt{2}$ . This along with the fact that  $x \in R(a, d)$  implies inequality (8).

2. Assume now that  $|ac| \leq |cb|$ . Let  $i$  be the intersection point between  $ab$  and the horizontal line through  $c$  (refer to Figure 5a). Note that  $\angle aic \geq \pi/2$  and  $\angle bic \leq \pi/2$  (these two angles

sum to  $\pi$ ). This along with Proposition 2 applied to triangle  $\triangle aic$  shows that

$$|ac|^2 \geq |ai|^2 + |ic|^2.$$

Similarly, Proposition 2 applied to triangle  $\triangle bic$  shows that

$$|bc|^2 \leq |bi|^2 + |ic|^2.$$

The two inequalities above along with our assumption that  $|ac| \leq |cb|$  imply that  $|ai| \leq |bi|$ , which in turn implies that  $|ai| \leq |ab|/2$ , because  $|ai| + |ib| = |ab|$ . Since  $x$  is below  $i$  (otherwise,  $|cx| < |cd|$ , contradicting the fact that  $\vec{cd}$  is an edge in  $Y_4$ ), we have  $\delta \leq |ai|$ . It follows that  $\delta \leq |ab|/2$ .

Finally we derive a contradiction using the now established inequality (8). Let  $j$  be the orthogonal projection of  $x$  onto the vertical line through  $a$  (thus  $|aj| = \delta$ ). Note that  $\angle ajy < \pi/2$ , because  $y \in Q_4(x)$ . By Proposition 2 applied to  $\triangle ajy$ , we have

$$|ay|^2 < |aj|^2 + |jy|^2 = \delta^2 + |jy|^2.$$

Since  $y$  and  $b$  are in the same quadrant of  $a$ , and since  $\vec{ab} \in Y_4$ , we have that  $|ab| \leq |ay|$ . This along with the inequality above and (8) implies that  $|jy| \geq |ab|/\sqrt{2} \geq \delta$ . By Proposition 2 applied to  $\triangle xjy$ , we have  $|xy|^2 > |xj|^2 + |jy|^2 \geq |xj|^2 + \delta^2 = |xj|^2 + |ja|^2 = |xa|^2$ . It follows that  $|xy| > |xa|$ , contradicting our assumption that  $\vec{xy} \in Y_4$ .  $\blacksquare$

**Lemma 6** *Let  $a, b, c, d$  be four distinct nodes in  $V$ , with  $c \in Q_1(a)$ , such that*

- (a)  $\vec{ab} \in Q_1(a)$  and  $\vec{cd} \in Q_3(c)$  are two edges in  $Y_4$  that cross each other.
- (b)  $ad$  is a shortest side of the quadrilateral  $acbd$ .

Then  $\mathcal{P}_R(d \rightarrow a)$  does not cross  $ab$ .

**Proof.** We first show that  $d \notin Q_3(a)$ . Assume the opposite. Since  $c \in Q_1(a)$  and  $d \in Q_3(a)$ , we have that  $\angle cad > \pi/2$ . This implies that  $|ca| < |cd|$ , which along with the fact that  $a, d \in Q_3(c)$  contradict the fact that  $\vec{cd} \in Y_4$ . Also note that  $d \notin Q_1(a)$ , since in that case  $ab$  and  $cd$  could not intersect. In the following we discuss the case  $d \in Q_2(a)$ ; the case  $d \in Q_4(a)$  is symmetric.

A first observation is that  $c$  must lie below  $b$ ; otherwise  $|cb| < |cd|$  (since  $\angle cbd > \pi/2$ ), which would contradict the fact that  $\vec{cd} \in Y_4$ . We now prove by contradiction that there is no edge in  $\mathcal{P}_R(d \rightarrow a)$  crossing  $ab$ . Assume the contrary, and let  $\vec{xy} \in \mathcal{P}_R(d \rightarrow a)$  be such an edge. Then necessarily  $x \in R(a, d)$  and  $\vec{xy} \in Q_4(x)$ . Note that  $y$  cannot lie below  $a$ ; otherwise  $|xa| < |xy|$  (since  $\angle xay > \pi/2$ ), which would contradict the fact that  $\vec{xy} \in Y_4$ . Also  $y$  must lie outside  $D(c, |cd|) \cap Q(c, d)$ , otherwise  $\vec{cd}$  could not be in  $Y_4$ . These together show that  $y$  sits to the right of  $c$ . See Figure 6(a). Then the following inequalities regarding the quadrilateral  $xyab$  must hold:

- (i)  $|by| > |bc|$ , due to the fact that  $\angle bcy > \pi/2$ .
- (ii)  $|bx| \geq |bd|$  ( $|bx| = |bd|$  if  $x$  and  $d$  coincide). If  $x$  and  $d$  are distinct, the inequality  $|bx| > |bd|$  follows from the fact that  $|cx| \geq |cd|$  (since  $x$  is outside  $D(c, |cd|)$ ), and Proposition 1 applied to the quadrilateral  $xcdb$ :

$$|bd| + |cx| < |bx| + |cd|$$

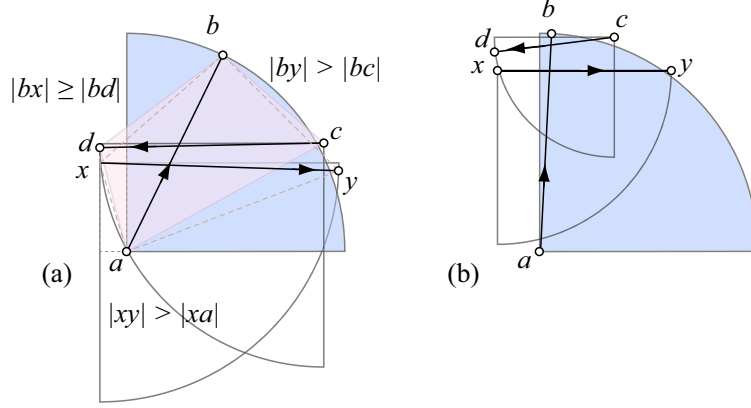


Figure 6: Lemma 6: (a)  $\mathcal{P}_R(d \rightarrow a)$  does not cross  $ab$ . (b) If  $ad$  is not the shortest side of  $acbd$ , the lemma conclusion might not hold.

Inequalities (i) and (ii) show that  $by$  and  $bx$  are longer than sides of the quadrilateral  $acbd$ , and so they must be longer than the shortest side of  $acbd$ , which by assumption (b) of the lemma is  $ad$ :  $\min\{|bx|, |by|\} \geq |ad| \geq |ax|$  (this latter inequality follows from the fact that  $x \in R(d, a)$ ). Also note that  $|ab| \leq |ay|$ , since  $\vec{ab} \in Y_4$  and  $y$  lies in the same quadrant of  $a$  as  $b$ . The fact that both diagonals of  $xyab$  are in  $Y_4$  enables us to apply Lemma 4(ii) to conclude that  $ay$  is not a shortest side of the quadrilateral  $xyab$ . Thus  $xa$  is a shortest side of the quadrilateral  $xyab$ , and we can use Lemma 4(ii) to claim that

$$|xa| < \min\{|xy|, |ab|\} \leq |xy|.$$

This contradicts our assumption that  $\vec{xy} \in Y_4$ . ■

Figure 6(b) shows that the claim of the lemma might be false without assumption (b). The next lemma relies on all of Lemmas 2–6.

**Lemma 7** *Let  $a, b, c, d \in V$  be four distinct nodes such that  $\vec{ab} \in Y_4$  crosses  $\vec{cd} \in Y_4$ , and let  $xy$  be a shortest side of the quadrilateral  $abcd$ . Then there exist two paths  $\mathcal{P}_x$  and  $\mathcal{P}_y$  in  $Y_4$ , where  $\mathcal{P}_x$  has  $x$  as an endpoint and  $\mathcal{P}_y$  has  $y$  as an endpoint, with the following properties:*

- (a)  $\mathcal{P}_x$  and  $\mathcal{P}_y$  have a nonempty intersection.
- (b)  $|\mathcal{P}_x| + |\mathcal{P}_y| \leq 3\sqrt{2}|xy|$ .
- (c) Each edge on  $\mathcal{P}_x \cup \mathcal{P}_y$  is no longer than  $|xy|$ .

**Proof.** Assume without loss of generality that  $b \in Q_1(a)$ . We discuss the following exhaustive cases:

1.  $c \in Q_1(a)$ , and  $d \in Q_1(c)$ . In this case,  $ab$  and  $cd$  cannot cross each other (by Lemma 3), so this case is finished.
2.  $c \in Q_1(a)$ , and  $d \in Q_2(c)$ , as in Figure 7a. Since  $ab$  crosses  $cd$ ,  $b \in Q_2(c)$ . Since  $\vec{ab} \in Y_4$ ,  $|ab| \leq |ac|$ . Since  $\vec{cd} \in Y_4$ ,  $|cd| \leq |cb|$ . These along with Lemma 4 imply that  $ad$  and  $db$  are the only candidates for a shortest edge of  $acbd$ .

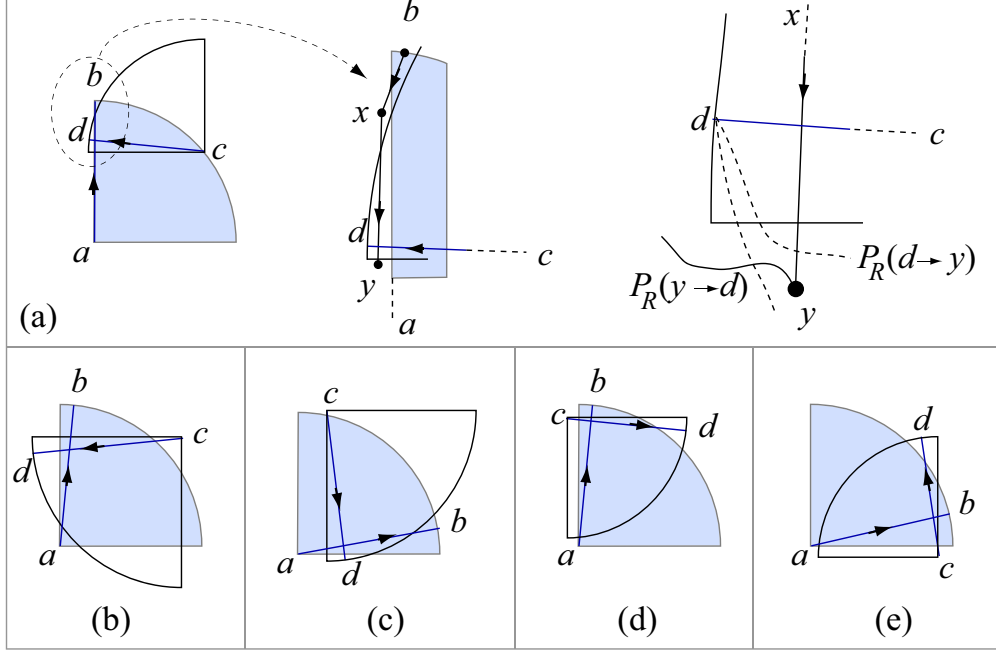


Figure 7: Lemma 7: (a)  $c \in Q_1(a)$ , and  $d \in Q_2(c)$  (b)  $c \in Q_1(a)$ , and  $d \in Q_3(c)$  (c)  $c \in Q_2(a)$  (d)  $c \in Q_4(a)$ .

Assume first that  $ad$  is a shortest edge of  $acbd$ . By Lemma 3,  $\mathcal{P}_a = \mathcal{P}_R(a \rightarrow d)$  does not cross  $cd$ . It follows from Lemma 5 that  $\mathcal{P}_a$  and  $\mathcal{P}_d = \mathcal{P}_R(d \rightarrow a)$  have a nonempty intersection. Furthermore, by Lemma 2,  $|\mathcal{P}_a| \leq |ad|\sqrt{2}$  and  $|\mathcal{P}_d| \leq |ad|\sqrt{2}$ , and no edge on these paths is longer than  $|ad|$ , proving the lemma true for this case.

Consider now the case when  $db$  is a shortest edge of  $acbd$  (see Figure 7a). Note that  $d$  is below  $b$  (otherwise,  $d \in Q_2(c)$  and  $|cd| > |cb|$ ) and, therefore,  $b \in Q_1(d)$ . By Lemma 3,  $\mathcal{P}_d = \mathcal{P}_R(d \rightarrow b)$  does not cross  $ab$ . If  $\mathcal{P}_b = \mathcal{P}_R(b \rightarrow d)$  does not cross  $cd$ , then  $\mathcal{P}_b$  and  $\mathcal{P}_d$  have a nonempty intersection, proving the lemma true for this case. Otherwise, there exists  $\vec{xy} \in \mathcal{P}_R(b \rightarrow d)$  that crosses  $cd$  (see Figure 7a). Define

$$\begin{aligned}\mathcal{P}_b &= \mathcal{P}_R(b \rightarrow d) \oplus \mathcal{P}_R(y \rightarrow d) \\ \mathcal{P}_d &= \mathcal{P}_R(d \rightarrow y)\end{aligned}$$

By Lemma 3,  $\mathcal{P}_R(y \rightarrow d)$  does not cross  $cd$ . Then  $\mathcal{P}_b$  and  $\mathcal{P}_d$  must have a nonempty intersection. We now show that  $\mathcal{P}_b$  and  $\mathcal{P}_d$  satisfy conditions (b) and (c) of the lemma. Proposition 1 applied on the quadrilateral  $xdyc$  tells us that

$$|xc| + |yd| < |xy| + |cd|$$

We also have that  $|cx| \geq |cd|$ , since  $\vec{cd} \in Y_4$  and  $x$  is in the same quadrant of  $c$  as  $d$ . This along with the inequality above implies  $|yd| < |xy|$ . Because  $xy \in \mathcal{P}_R(b \rightarrow d)$ , by Lemma 2 we have that  $|xy| \leq |bd|$ , which along with the previous inequality shows that  $|yd| < |bd|$ . This along with Lemma 2 shows that condition (c) of the lemma is satisfied. Furthermore,  $|\mathcal{P}_R(y \rightarrow d)| \leq |yd|\sqrt{2}$  and  $|\mathcal{P}_R(d \rightarrow y)| \leq |yd|\sqrt{2}$ . It follows that  $|\mathcal{P}_b| + |\mathcal{P}_d| \leq 3\sqrt{2}|bd|$ .

3.  $c \in Q_1(a)$ , and  $d \in Q_3(c)$ , as in Figure 7b. Then  $|ac| \geq \max\{ab, cd\}$ , and by Lemma 4  $ac$  is not a shortest edge of  $abcd$ . The case when  $bd$  is a shortest edge of  $abcd$  is settled by Lemmas 3 and 2: Lemma 3 tells us that  $\mathcal{P}_d = \mathcal{P}_R(d \rightarrow b)$  does not cross  $ab$ , and  $\mathcal{P}_b = \mathcal{P}_R(b \rightarrow d)$  does not cross  $cd$ . It follows that  $\mathcal{P}_d$  and  $\mathcal{P}_b$  have a nonempty intersection. Furthermore, Lemma 2 guarantees that  $\mathcal{P}_d$  and  $\mathcal{P}_b$  satisfy conditions (b) and (c) of the lemma.

Consider now the case when  $ad$  is a shortest edge of  $abcd$ ; the case when  $bc$  is shortest is symmetric. By Lemma 6,  $\mathcal{P}_R(d \rightarrow a)$  does not cross  $ab$ . If  $\mathcal{P}_R(a \rightarrow d)$  does not cross  $cd$ , then this case is settled:  $\mathcal{P}_d = \mathcal{P}_R(d \rightarrow a)$  and  $\mathcal{P}_a = \mathcal{P}_R(a \rightarrow d)$  satisfy the three conditions of the lemma. Otherwise, let  $\overrightarrow{xy} \in \mathcal{P}_R(a \rightarrow d)$  be the edge crossing  $cd$ . Arguments similar to the ones used in case 1 above show that

$$\begin{aligned}\mathcal{P}_a &= \mathcal{P}_R(a \rightarrow d) \oplus \mathcal{P}_R(y \rightarrow d) \\ \mathcal{P}_d &= \mathcal{P}_R(d \rightarrow y)\end{aligned}$$

are two paths that satisfy the conditions of the lemma.

4.  $c \in Q_1(a)$ , and  $d \in Q_4(c)$ , as in Figure 7c. Note that a horizontal reflection of Figure 7c, followed by a rotation of  $\pi/2$ , depicts a case identical to case 1, which has already been settled.
5.  $c \in Q_2(a)$ , as in Figure 7d. Note that Figure 7d rotated by  $\pi/2$  depicts a case identical to case 1, which has already been settled.
6.  $c \in Q_3(a)$ . Then it must be that  $d \in Q_1(c)$ , otherwise  $cd$  cannot cross  $ab$ . By Lemma 3 however,  $ab$  and  $cd$  may not cross, unless one of them is not in  $Y_4$ .
7.  $c \in Q_4(a)$ , as in Figure 7e. Note that a vertical reflection of Figure 7e depicts a case identical to case 1, so this case is settled as well.

Having exhausted all cases, we conclude that the lemma holds. ■

We are now ready to establish the main lemma of this section, showing that there is a short path between the endpoints of two intersecting edges in  $Y_4$ .

**Lemma 8** *Let  $a, b, c, d \in V$  be four distinct nodes such that  $\overrightarrow{ab} \in Y_4$  crosses  $\overrightarrow{cd} \in Y_4$ , and let  $xy$  be a shortest side of the quadrilateral  $abcd$ . Then  $Y_4$  contains a path  $p(x, y)$  connecting  $x$  and  $y$ , of length*

$$|p(x, y)| \leq \frac{6}{\sqrt{2}-1} \cdot |xy|.$$

*Furthermore, no edge on  $p(x, y)$  is longer than  $|xy|$ .*

**Proof.** Let  $\mathcal{P}_x$  and  $\mathcal{P}_y$  be the two paths whose existence in  $Y_4$  is guaranteed by Lemma 7. By condition (c) of Lemma 7, no edge on  $\mathcal{P}_x$  and  $\mathcal{P}_y$  is longer than  $|xy|$ . By condition (a) of Lemma 7,  $\mathcal{P}_x$  and  $\mathcal{P}_y$  have a nonempty intersection. If  $\mathcal{P}_x$  and  $\mathcal{P}_y$  share a node  $u \in V$ , then the path

$$p(x, y) = \mathcal{P}_x[x, u] \oplus \mathcal{P}_y[y, u]$$

is a path from  $x$  to  $y$  in  $Y_4$  no longer than  $3\sqrt{2}|xy|$ ; the length restriction follows from guarantee (b) of Lemma 7. Otherwise, let  $\overrightarrow{a'b'} \in \mathcal{P}_x$  and  $\overrightarrow{c'd'} \in \mathcal{P}_y$  be two edges crossing each other. Let  $x'y'$

be a shortest side of the quadrilateral  $a'c'b'd'$ , with  $x' \in \mathcal{P}_x$  and  $y' \in \mathcal{P}_y$ . Lemma 7 tells us that  $|a'b'| \leq |xy|$  and  $|c'd'| \leq |xy|$ . These along with Lemma 4 imply that

$$|x'y'| \leq |xy|/\sqrt{2}. \quad (9)$$

This enables us to derive a recursive formula for computing a path  $p(x, y) \in Y_4$  as follows:

$$p(x, y) = \begin{cases} x, & \text{if } x = y \\ \mathcal{P}_x[x, x'] \oplus \mathcal{P}_y[y, y'] \oplus p(x', y'), & \text{if } x \neq y \end{cases} \quad (10)$$

Next we use induction on the length of  $xy$  to prove the claim of the lemma. The base case corresponds to  $x = y$ , case in which  $p(x, y)$  degenerates to a point and  $|p(x, y)| = 0$ . To prove the inductive step, pick a shortest side  $xy$  of a quadrilateral  $acbd$ , with  $\overrightarrow{ab}, \overrightarrow{cd} \in Y_4$  crossing each other, and assume that the lemma holds for all such sides shorter than  $xy$ . Let  $p(x, y)$  be the path determined recursively as in (10). By the inductive hypothesis, we have that  $p(x', y')$  contains no edges longer than  $|x'y'| \leq |xy|$ , and

$$|p(x', y')| \leq \frac{6}{\sqrt{2}-1} |x'y'| \leq \frac{6}{2-\sqrt{2}} |xy|.$$

This latter inequality follows from (9). This along with Lemma 7 and formula (10) implies

$$|p(x, y)| \leq (3\sqrt{2} + \frac{6}{2-\sqrt{2}}) \cdot |xy| = \frac{6}{\sqrt{2}-1} \cdot |xy|.$$

This completes the proof. ■

## 4 $Y_4^\infty$ and $Y_4$

We prove that every individual edge of  $Y_4^\infty$  is spanned by a short path in  $Y_4$ . This, along with the result of Theorem 1, establishes that  $Y_4$  is a spanner.

Fix an edge  $\overrightarrow{ab} \in Y_4^\infty$ . Define an edge or a path as *short* if its length is within a constant factor of  $|ab|$ . In our proof that  $ab$  is spanned by a short path in  $Y_4$ , we will make use of the following three statements (which will be proved in the appendix).

- S1** If  $ab$  is short, then  $\mathcal{P}_R(a \rightarrow b)$ , and therefore its reverse,  $\mathcal{P}_R^{-1}(a \rightarrow b)$ , are short by Lemma 2.
- S2** If  $ab \in Y_4$  and  $cd \in Y_4$  are short, and if  $ab$  intersects  $cd$ , Lemma 8 shows that then there is a short path between any two of the endpoints of these edges.
- S3** If  $p(a, b)$  and  $p(c, d)$  are short paths that intersect, then there is a short path  $P$  between any two of the endpoints of these paths, by **S2**.

**Lemma 9** *For any edge  $ab \in Y_4^\infty$ , there is a short path  $p(a, b) \in Y_4$  of length*

$$|p(a, b)| \leq (29 + 23\sqrt{2})|ab|.$$



**Proof.** For the sake of clarity, we only prove here that there is a short path  $p(a, b)$ , and defer the calculations of the actual stretch factor of  $p(a, b)$  to the appendix. Assume without loss of generality that  $\vec{ab} \in Y_4^\infty$ , and  $\vec{ab} \in Q_1(a)$ . If  $\vec{ab} \in Y_4$ , then  $p(a, b) = ab$  and the proof is finished. So assume the opposite, and let  $\vec{ac} \in Q_1(a)$  be the edge in  $Y_4$ ; since  $Q_1(a)$  is nonempty,  $\vec{ac}$  exists. Because  $\vec{ac} \in Y_4$  and  $b$  is in the same quadrant of  $a$  as  $c$ , we have that

$$\begin{aligned} |ac| &\leq |ab| & (i) \\ |bc| &\leq |ac|\sqrt{2} & (ii) \end{aligned} \tag{11}$$

Thus both  $ac$  and  $bc$  are short. And this in turn implies that  $\mathcal{P}_R(b \rightarrow c)$  is short by **S1**. We next focus on  $\mathcal{P}_R(b \rightarrow c)$ . Let  $b' \notin R(b, c)$  be the other endpoint of  $\mathcal{P}_R(b \rightarrow c)$ . We distinguish three cases.

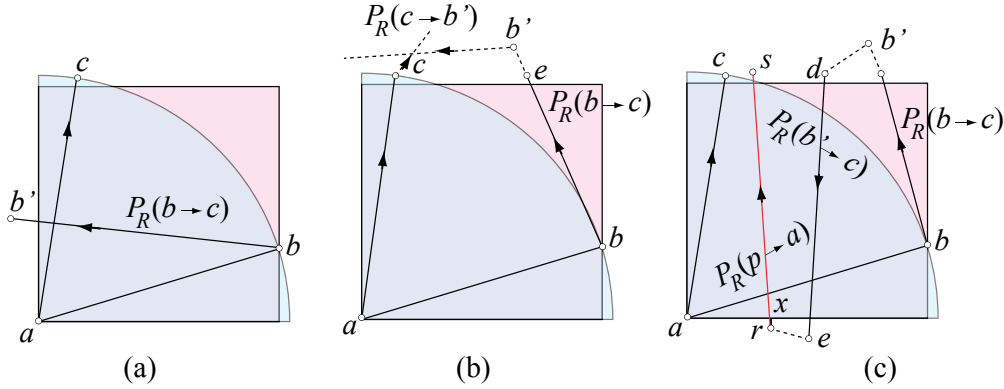


Figure 8: Lemma 9: (a) Case 1:  $\mathcal{P}_R(b \rightarrow c)$  and  $ac$  have a nonempty intersection. (b) Case 2:  $\mathcal{P}_R(b' \rightarrow a)$  and  $ab$  have an empty intersection. (c) Case 3:  $\mathcal{P}_R(b' \rightarrow a)$  and  $ab$  have a non-empty intersection.

**Case 1:**  $\mathcal{P}_R(b \rightarrow c)$  and  $ac$  intersect. Then by **S3** there is a short path  $p(a, b)$  between  $a$  and  $b$ .

**Case 2:**  $\mathcal{P}_R(b \rightarrow c)$  and  $ac$  do not intersect, and  $\mathcal{P}_R(b' \rightarrow a)$  and  $ab$  do not intersect (see Figure 8b). Note that because  $b'$  is the endpoint of the short path  $\mathcal{P}_R(b \rightarrow c)$ , the triangle inequality on  $\triangle abb'$  implies that  $ab'$  is short, and therefore  $\mathcal{P}_R(b' \rightarrow a)$  is short. We consider two cases:

(i)  $\mathcal{P}_R(b' \rightarrow a)$  intersects  $ac$ . Then by **S3** there is a short path  $p(a, b')$ . So

$$p(a, b) = p(a, b') \oplus \mathcal{P}_R^{-1}(b \rightarrow c)$$

is short.

(ii)  $\mathcal{P}_R(b' \rightarrow a)$  does not intersect  $ac$ . Then  $\mathcal{P}_R(c \rightarrow b')$  must intersect  $\mathcal{P}_R(b \rightarrow c) \oplus \mathcal{P}_R(b' \rightarrow a)$ . Next we establish that  $b'c$  is short. Let  $\vec{eb'}$  be the last edge of  $\mathcal{P}_R(b \rightarrow c)$ , and so incident to  $b'$  (note that  $e$  and  $b$  may coincide). Because  $\mathcal{P}_R(b \rightarrow c)$  does not intersect  $ac$ ,  $b'$  and  $c$  are in the same quadrant for  $e$ . It follows that  $|eb'| \leq |ec|$  and  $\angle b'ec < \pi/2$ . These along with Proposition 2 for  $\triangle b'ec$  imply that  $|b'c|^2 < |b'e|^2 + |ec|^2 \leq 2|ec|^2 < 2|bc|^2$  (this latter inequality uses the fact that  $\angle bec > \pi/2$ , which implies that  $|ec| < |bc|$ ). It follows that

$$|b'c| \leq |bc|\sqrt{2} \leq 2|ac| \quad (\text{by (11)ii}) \tag{12}$$

Thus  $b'c$  is short, and by **S1** we have that  $\mathcal{P}_R(c \rightarrow b')$  is short. Since  $\mathcal{P}_R(c \rightarrow b')$  intersects the short path  $\mathcal{P}_R(b \rightarrow c) \oplus \mathcal{P}_R(b' \rightarrow a)$ , there is by **S3** a short path  $p(c, b)$ , and so

$$p(a, b) = ac \oplus p(c, b)$$

is short.

**Case 3:**  $\mathcal{P}_R(b \rightarrow c)$  and  $ac$  do not intersect, and  $\mathcal{P}_R(b' \rightarrow a)$  intersects  $ab$  (see Figure 8c). If  $\mathcal{P}_R(b' \rightarrow a)$  intersects  $ab$  at  $a$ , then  $p(a, b) = \mathcal{P}_R(b \rightarrow c) \oplus \mathcal{P}_R(b' \rightarrow a)$  is short. So assume otherwise, in which case there is an edge  $\vec{de} \in \mathcal{P}_R(b' \rightarrow a)$  that crosses  $ab$ . Then  $d \in Q_1(a)$ ,  $e \in Q_3(a) \cup Q_4(a)$ , and  $e$  and  $a$  are in the same quadrant for  $d$ . Note however that  $e$  cannot lie in  $Q_3(a)$ , since in that case  $\angle dae > \pi/2$ , which would imply  $|de| > |da|$ , which in turn would imply  $\vec{de} \notin Y_4$ . So it must be that  $e \in Q_4(a)$ .

Next we show that  $\mathcal{P}_R(e \rightarrow a)$  does not cross  $ab$ . Assume the opposite, and let  $\vec{rs} \in \mathcal{P}_R(e \rightarrow a)$  cross  $ab$ . Then  $r \in Q_4(a)$ ,  $s \in Q_1(a) \cup Q_2(a)$ , and  $s$  and  $a$  are in the same quadrant for  $r$ . Arguments similar to the ones above show that  $s \notin Q_2(a)$ , so  $s$  must lie in  $Q_1(a)$ . Let  $d$  be the  $L_\infty$  distance from  $a$  to  $b$ . Let  $x$  be the projection of  $r$  on the horizontal line through  $a$ . Then

$$|rs| \geq |rx| + d \geq |rx| + |xa| > |ra| \quad (\text{by the triangle inequality})$$

Because  $a$  and  $s$  are in the same quadrant for  $r$ , the inequality above contradicts  $\vec{rs} \in Y_4$ .

We have established that  $\mathcal{P}_R(e \rightarrow a)$  does not cross  $ab$ . Then  $\mathcal{P}_R(a \rightarrow e)$  must intersect  $\mathcal{P}_R(e \rightarrow a) \oplus de$ . Note that  $de$  is short because it is in the short path  $\mathcal{P}_R(b' \rightarrow a)$ . Thus  $ae$  is short, and so  $\mathcal{P}_R(a \rightarrow e)$  and  $\mathcal{P}_R(e \rightarrow a)$  are short. Thus we have two intersecting short paths, and so by **S3** there is a short path  $p(a, e)$ . Then

$$p(a, b) = p(a, e) \oplus \mathcal{P}_R^{-1}(b' \rightarrow a) \oplus \mathcal{P}_R^{-1}(b \rightarrow c)$$

is short. Calculations deferred to the appendix show that, in each of these cases, the stretch factor for  $p(a, b)$  does not exceed  $29 + 23\sqrt{2}$ .  $\blacksquare$

Our main result follows immediately from Theorem 1 and Lemma 9:

**Theorem 2**  $Y_4$  is a  $t$ -spanner, for  $t \geq 8(29 + 23\sqrt{2})$ .

## 5 Conclusion

Our results settle a long-standing open problem, asking whether  $Y_4$  is a spanner or not. We answer this question positively, and establish a loose stretch factor of  $8(29 + 23\sqrt{2})$ . Experimental results, however, indicate a stretch factor of the order  $1 + \sqrt{2}$ , a factor of 200 smaller. Finding tighter stretch factors for both  $Y_4^\infty$  and  $Y_4$  remain interesting open problems. Establishing whether  $Y_5$  and  $Y_6$  are spanners or not is also open.

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## 6 Appendix

### 6.1 Calculations for the stretch factor of $p(a, b)$ in Lemma 9

We start by computing the stretch factor of the short paths claimed by statements **S2** and **S3**.

**S2** If  $ab \in Y_4$  and  $cd \in Y_4$  are short, and if  $ab$  intersects  $cd$ , then there is a short path  $P$  between any two of the endpoints of these edges, of length

$$|P| \leq |ab| + |cd| + 3(2 + \sqrt{2}) \max\{|ab|, |cd|\} \quad (13)$$

This upper bound can be derived as follows. Let  $xy$  be a shortest side of the quadrilateral  $acbd$ . By Lemma 8,  $Y_4$  contains a path  $p(x, y)$  no longer than  $6(\sqrt{2} + 1)|xy|$ . By Lemma 4,  $|xy| \leq \max\{|ab|, |cd|\}/\sqrt{2}$ . These together with the fact that  $|P| \leq |ab| + |cd| + |p(x, y)|$  yield inequality (13).

**S3** If  $p(a, b)$  and  $p(c, d)$  are short paths that intersect, then there is a short path  $P$  between any two of the endpoints of these paths, of length

$$|P| \leq |p(a, b)| + |p(c, d)| + 3(2 + \sqrt{2}) \max\{|ab|, |cd|\} \quad (14)$$

This follows immediately from **S2** and the fact that no edge on  $p(a, b) \cup p(c, d)$  is longer than  $\max\{|ab|, |cd|\}$  (by Lemma 8).

**Case 1:**  $\mathcal{P}_R(b \rightarrow c)$  and  $ac$  intersect. Then by **S3** we have

$$\begin{aligned} |p(a, b)| &\leq |\mathcal{P}_R(b, c)| + |ac| + 3(2 + \sqrt{2}) \max\{|bc|, |ac|\} \\ &\leq \sqrt{2}|bc| + |ac| + 3(2 + \sqrt{2})\sqrt{2}|ac| && \text{(by (7), (11)ii)} \\ &= 3(3 + 2\sqrt{2})|ac| \leq 3(3 + 2\sqrt{2})|ab| && \text{(by (11)i)} \end{aligned}$$

**Case 2(i):**  $\mathcal{P}_R(b \rightarrow c)$  and  $ac$  do not intersect;  $\mathcal{P}_R(b' \rightarrow a)$  and  $ab$  do not intersect; and  $\mathcal{P}_R(b' \rightarrow a)$  intersects  $ac$ . By **S3**, there is a short path  $p(a, b')$  of length

$$\begin{aligned} |p(a, b')| &\leq |\mathcal{P}_R(b', a)| + |ac| + 3(2 + \sqrt{2}) \max\{|b'a|, |ac|\} \\ &\leq |b'a|\sqrt{2} + |ac| + 3(2 + \sqrt{2}) \max\{|b'a|, |ac|\} \quad (\text{by (7)}) \end{aligned} \quad (15)$$

Next we establish an upper bound on  $|b'a|$ . By the triangle inequality,

$$|ab'| < |ac| + |cb'| \leq 3|ac| \quad (\text{by (12)}) \quad (16)$$

Substituting this inequality in (15) yields

$$|p(a, b')| \leq (19 + 12\sqrt{2})|ac| \quad (17)$$

Thus  $p(a, b) = p(a, b') \oplus \mathcal{P}_R^{-1}(b \rightarrow c)$  is a path in  $Y_4$  of length

$$\begin{aligned} |p(a, b)| &\leq |p(a, b')| + |bc|\sqrt{2} \quad (\text{by (7)}) \\ &\leq |p(a, b')| + 2|ac| \quad (\text{by (11)ii}) \\ &\leq (21 + 12\sqrt{2})|ac| \quad (\text{by (17)}) \\ &\leq (21 + 12\sqrt{2})|ab| \quad (\text{by (11)i}) \end{aligned}$$

**Case 2(ii):**  $\mathcal{P}_R(b \rightarrow c)$  and  $ac$  do not intersect;  $\mathcal{P}_R(b' \rightarrow a)$  and  $ab$  do not intersect; and  $\mathcal{P}_R(b' \rightarrow a)$  does not intersect  $ac$ . Then  $\mathcal{P}_R(c \rightarrow b')$  must intersect  $\mathcal{P}_R(b \rightarrow c) \oplus \mathcal{P}_R(b' \rightarrow a)$ . By **S3** there is a short path  $p(c, b)$  of length

$$\begin{aligned} |p(c, b)| &\leq |\mathcal{P}_R(c \rightarrow b')| + |\mathcal{P}_R(b \rightarrow c)| + |\mathcal{P}_R(b' \rightarrow a)| + 3(2 + \sqrt{2}) \max\{|cb'|, |bc|, |b'a|\} \\ &\leq (|cb'| + |bc| + |b'a|)\sqrt{2} + 3(2 + \sqrt{2}) \max\{|cb'|, |bc|, |b'a|\} \quad (\text{by (7)}) \end{aligned}$$

Inequalities (11)ii, (12) and (16) imply that  $\max\{|cb'|, |bc|, |b'a|\} \leq 3ac$ . Substituting in the above, we get

$$\begin{aligned} |p(c, b)| &\leq (2 + \sqrt{2} + 3)\sqrt{2}|ac| + 9(2 + \sqrt{2})|ac| \\ &\leq (20 + 14\sqrt{2})|ac| \quad (\text{by (11)i}) \end{aligned}$$

Thus  $p(a, b) = ac \oplus p(c, b)$  is a path in  $Y_4$  from  $a$  to  $b$  of length

$$|p(a, b)| \leq (21 + 14\sqrt{2})|ac| \leq (21 + 14\sqrt{2})|ab| \quad (\text{by (11)i})$$

**Case 3:**  $\mathcal{P}_R(b \rightarrow c)$  and  $ac$  do not intersect, and  $\mathcal{P}_R(b' \rightarrow a)$  intersects  $ab$ . If  $\mathcal{P}_R(b' \rightarrow a)$  intersects  $ab$  at  $a$ , then  $p(a, b) = \mathcal{P}_R(b \rightarrow c) \oplus \mathcal{P}_R(b' \rightarrow a)$  is clearly short and does not exceed the spanning ratio of the lemma. Otherwise, there is an edge  $\vec{de} \in \mathcal{P}_R(b' \rightarrow a)$  that crosses  $ab$ , and  $\mathcal{P}_R(a \rightarrow e)$  intersects  $\mathcal{P}_R(e \rightarrow a) \oplus de$  (as established in the proof of Lemma 9). By **S3** there is a short path  $p(a, e)$  of length

$$\begin{aligned} |p(a, e)| &\leq |\mathcal{P}_R(a \rightarrow e)| + |\mathcal{P}_R(e \rightarrow a)| + |de| + 3(2 + \sqrt{2}) \max\{|ae|, |de|\} \\ &\leq 2|ae|\sqrt{2} + |de| + 3(2 + \sqrt{2}) \max\{|ae|, |de|\} \quad (\text{by (7)}) \end{aligned} \quad (18)$$

A loose upper bound on  $|ae|$  can be obtained by employing Proposition 1 to the quadrilateral  $aebd$ :  $|ae| + |bd| < |ab| + |de| < |ab| + |ab'|$ . Substituting the upper bound for  $ab'$  from (16) yields

$$|ae| < |ab| + 3|ac| \leq 4|ab| \quad (19)$$

By Lemma 2,  $|de| \leq |ab'|$  (since  $de \in \mathcal{P}_R(b' \rightarrow a)$ ), which along with (16) implies

$$|de| \leq 3|ab| \quad (20)$$

Substituting (19) and (20) in (18) yields

$$|p(a, e)| \leq (27 + 20\sqrt{2})|ab|$$

Then

$$p(a, b) = p(a, e) \oplus \mathcal{P}_R^{-1}(b' \rightarrow a) \oplus \mathcal{P}_R^{-1}(b \rightarrow c)$$

is a path from  $a$  to  $b$  of length

$$\begin{aligned} |p(a, b)| &\leq |p(a, e)| + |b'a|\sqrt{2} + |bc|\sqrt{2} && \text{(by (7))} \\ &\leq (27 + 20\sqrt{2})|ab| + 3\sqrt{2}|ab| + 2|ab| && \text{(by (16), (11))} \\ &= (29 + 23\sqrt{2})|ab| \end{aligned}$$

## 6.2 $Y_k$ is a Spanner, for $k \geq 7$

**Lemma 10** *Let  $\theta$  be a real number with  $0 < \theta < \pi/3$ , and let*

$$t = \frac{1 + \sqrt{2 - 2\cos\theta}}{2\cos\theta - 1}.$$

*Let  $a, b$ , and  $c$  be three distinct points in the plane such that  $|ac| \leq |ab|$ , let  $\alpha = \angle bac$ , and assume that  $0 \leq \alpha \leq \theta$ . Then*

$$|bc| \leq |ab| - |ac|/t. \quad (21)$$

**Proof.** Refer to Figure 9. By the Law of Cosines, we have

$$|bc|^2 = |ac|^2 + |ab|^2 - 2|ac| \cdot |ab| \cos\alpha.$$

Since  $t > 1$  and  $|ac| \leq |ab|$ , the right-hand side in (21) is positive, so (21) is equivalent to

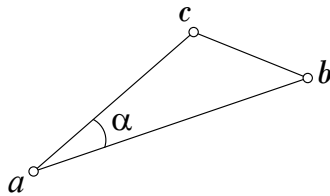


Figure 9: Lemma 1: If  $\alpha < 60$  and  $|ac| \leq |ab|$ , then  $|bc| \leq |ab| - |ac|/t$ .

$$|bc|^2 \leq (|ab| - |ac|/t)^2.$$

Thus, we have to show that

$$|ac|^2 + |ab|^2 - 2|ac| \cdot |ab| \cos \alpha \leq (|ab| - |ac|/t)^2,$$

which simplifies to

$$(1 - 1/t^2) |ac| \leq 2(\cos \alpha - 1/t)|ab|. \quad (22)$$

Since  $|ac| \leq |ab|$  and  $\cos \theta \leq \cos \alpha$ , (22) holds if

$$1 - 1/t^2 \leq 2(\cos \theta - 1/t),$$

which can be rewritten as

$$(2 \cos \theta - 1)t^2 - 2t + 1 \geq 0. \quad (23)$$

By our choice of  $t$ , equality holds in (23). ■

An immediate consequence of Lemma 10 is the following result.

**Theorem 3** *For any  $\theta$  with  $0 < \theta < \pi/3$ , the Yao-graph with cones of angle  $\theta$ , is a  $t$ -spanner for*

$$t = \frac{1 + \sqrt{2 - 2 \cos \theta}}{2 \cos \theta - 1}.$$

**Proof.** The proof of this claim is by induction on the distances defined by the  $\binom{n}{2}$  pairs of nodes. Since  $\theta < \pi/3$ , any closest pair is connected by an edge in the Yao-graph; this proves the basis of the induction. The induction step follows from Lemma 10. ■

What happens to the value of  $t$  from Lemma 10, if  $\theta$  gets close to  $\pi/3$ : Let  $\varepsilon = \cos \theta - 1/2$ , so that  $\varepsilon$  is close to zero. Then

$$\begin{aligned} t &= \frac{1}{2\varepsilon} + \sqrt{\frac{1 - 2\varepsilon}{4\varepsilon^2}} \\ &= \frac{1}{2\varepsilon} + \frac{\sqrt{1 - 2\varepsilon}}{2\varepsilon} \\ &\sim \frac{1}{2\varepsilon} + \frac{1 - \varepsilon}{2\varepsilon} \\ &= -\frac{1}{2} + \frac{1}{\varepsilon} \\ &= -\frac{1}{2} + \frac{1}{\cos \theta - 1/2}. \end{aligned}$$