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Unfolding Convex Polyhedra via Quasigeodesic Star Unfoldings*

Jin-ichi Itoh[†] Joseph O’Rourke[‡] Costin Vilcu[§]

December 12, 2008

Abstract

We extend the notion of a star unfolding to be based on a simple quasigeodesic loop Q rather than on a point. This gives a new general method to unfold the surface of any convex polyhedron \mathcal{P} to a simple, planar polygon: shortest paths from all vertices of \mathcal{P} to Q are cut, and all but one segment of Q is cut.

1 Introduction

There are two general methods known to unfold the surface \mathcal{P} of any convex polyhedron to a simple polygon in the plane: the source unfolding and the star unfolding. Both unfoldings are with respect to a point $x \in \mathcal{P}$. Here we define a third general method: the star unfolding with respect to a simple closed “quasigeodesic loop” Q on \mathcal{P} . In a companion paper [IOV08], we will extend the analysis to the source unfolding with respect to a wider class of curves Q .

The *point source unfolding* cuts the *cut locus* of the point x : the closure of set of all those points y to which there is more than one shortest path on \mathcal{P} from x . Alternatively, the cut locus is the set of all extremities (different from x) of maximal (with respect to inclusion) shortest paths starting at x . The notion of cut locus was introduced by Poincaré [Poi05] in 1905, and since then has gained an important place in global Riemannian geometry; see, e.g., [Kob67] or [Sak96]. The point source unfolding has been studied for polyhedral surfaces since [SS86] (where the cut locus is called the “ridge tree”). The *point star unfolding* cuts the shortest paths from x to every vertex of \mathcal{P} . The idea

*A preliminary version of this work appeared in [IOV07a, IOV07b].

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goes back to Alexandrov [Ale50, p. 181];¹ that it unfolds \mathcal{P} to a simple polygon was established in [AO92].

In this paper we extend the star unfolding to be based on a simple closed polygonal curve Q with particular properties, rather than based a single point. This unfolds any convex polyhedron to a simple polygon, answering a question raised in [DO07, p. 307].

The curves Q for which our star unfolding works are quasigeodesics, which we now define.

Geodesics & Quasigeodesics. A *geodesic* is a locally shortest path on a smooth surface. A quasigeodesic is a generalization that extends the notion to nondifferentiable, and in particular, to polyhedral surfaces. Let Γ be any directed curve on a convex surface \mathcal{P} , and $p \in \Gamma$ be any point in the relative interior of Γ , i.e., not an endpoint. Let $L(p)$ be the total face angle incident to the left side of p , and $R(p)$ the angle to the right side. If Γ is a geodesic, then $L(p)=R(p) = \pi$. A *quasigeodesic* Γ loosens this condition to $L(p) \leq \pi$ and $R(p) \leq \pi$, again for all p interior to Γ [AZ67, p. 16] [Pog73, p. 28]. So a quasigeodesic Γ has π total face angle incident to each side at all nonvertex points (just like a geodesic), and has $\leq \pi$ angle to each side where Γ passes through a polyhedron vertex. (Geodesics can never pass through vertices.) A *simple closed geodesic* is non-self-intersecting (*simple*) closed curve that is a geodesic, and a *simple closed quasigeodesic* is a simple closed curve on \mathcal{P} that is quasigeodesic throughout its length. As all curves we consider must be simple, we will henceforth drop that prefix. Pogorelov showed that any convex polyhedron \mathcal{P} has at least three closed quasigeodesics [Pog49], extending the celebrated earlier result of Lyusternik-Schnirelmann showing that the same holds for geodesics on differentiable convex surfaces.

A *geodesic loop* is a closed curve that is geodesic everywhere except possibly at one point, and similarly a *quasigeodesic loop* is quasigeodesic except possibly at one point x , the *loop point*, at which the angle conditions on $L(x)$ and $R(x)$ may be violated—one may be $>\pi$. Quasigeodesic loops encompass closed geodesics and quasigeodesics, as well as geodesic loops.

Although it is known that every \mathcal{P} must have at least three closed quasigeodesics, there is no algorithm known that will find a simple closed quasigeodesic in polynomial time: Open Problem 24.2 [DO07, p. 374]. Fortunately it is in general easy to find quasigeodesic loops on a given \mathcal{P} : start at any nonvertex point p , and extend a geodesic from p in opposite directions, following each branch until they meet at x . If no vertices are encountered, we have a geodesic loop; if vertices are encountered, maintaining the angle conditions through the vertices (which is always possible, e.g., by bisecting the surface angle) will result in a quasigeodesic loop.

An exception to this ease of finding a geodesic loop could occur on an *isosceles tetrahedron*: a tetrahedron whose four faces are congruent triangles, or, equivalently, one at which the total face angle incident to each vertex is π . It is

¹ It is called the “Alexandrov unfolding” in [MP08].

proved in [IV08b] that a convex surface possesses a *simple quasigeodesic line*—a non-self-intersecting quasigeodesic infinite in both directions—if and only if the surface is an isosceles tetrahedron. So, excepting this case, the procedure described above will produce a quasigeodesic loop.

Discrete Curvature. The discrete *curvature* $\omega(p)$ at any point $p \in \mathcal{P}$, is the *angle deficit* or *gap*: 2π minus the sum of the face angles incident to p . The curvature is only nonzero at vertices of \mathcal{P} ; at each vertex it is positive because \mathcal{P} is convex. By the Gauss-Bonnet theorem, a closed geodesic partitions the curvature into 2π in each “hemisphere” of P . For quasigeodesics that pass through vertices, the curvature in each half is $\leq 2\pi$. The curvature in each half defined by a quasigeodesic loop depends on the angle at the exceptional loop point.

Some Notation. For a quasigeodesic loop Q on \mathcal{P} , $\mathcal{P} \setminus Q$ separates \mathcal{P} into two “halves” P_1 and P_2 . As our main focus is usually on one such half, to ease notation we sometimes use P without a subscript to represent either of P_1 or P_2 when the distinction between them does not matter. Unless otherwise stated, vertices of \mathcal{P} are labeled v_i in arbitrary order. Other notation will be introduced as needed. A glossary of all symbols defined (chronologically) throughout the paper is provided in Appendix 1.

2 Example of Star Unfolding

We start with an example. Figure 1(a) shows a geodesic loop Q on the surface \mathcal{P} of a cube. $L(p)=R(p) = \pi$ at every point p of Q except at x , where $R(x)=\frac{3}{2}\pi$ and $L(x)=\frac{1}{2}\pi$. Note that three cube vertices, $\{v_3, v_6, v_7\}$, are to the left of Q , and the other five to the right. This is consistent with the Gauss-Bonnet theorem, because Q has a total turn of $\frac{1}{2}\pi$, so turn plus enclosed curvature is 2π .

For each vertex $v_i \in \mathcal{P}$, we select a shortest path $\text{sp}(v_i)$ to Q : a geodesic from v_i to a point $v'_i \in Q$ whose length is minimal among all geodesics to Q . In general there could be several shortest paths from v_i to Q ; we use $\text{sp}(v_i)$ to represent an arbitrarily selected one. The point $v'_i \in Q$ is called a *projection* of v_i onto Q , and the path $\text{sp}(v_i) = v_i v'_i$ is called a *segment* on \mathcal{P} . In the example, all the shortest path segments $\text{sp}(v'_i)$ are unique, which is the generic situation.

Algorithm. If we view the star unfolding as an algorithm with input \mathcal{P} and Q , it consists of three main steps:

1. Select shortest paths $\text{sp}(v_i)$ from each $v_i \in \mathcal{P}$ to Q .
2. Cut along $\text{sp}(v_i)$ and flatten each half.
3. Cut along Q , joining the two halves at an uncut segment $s \subset Q$.

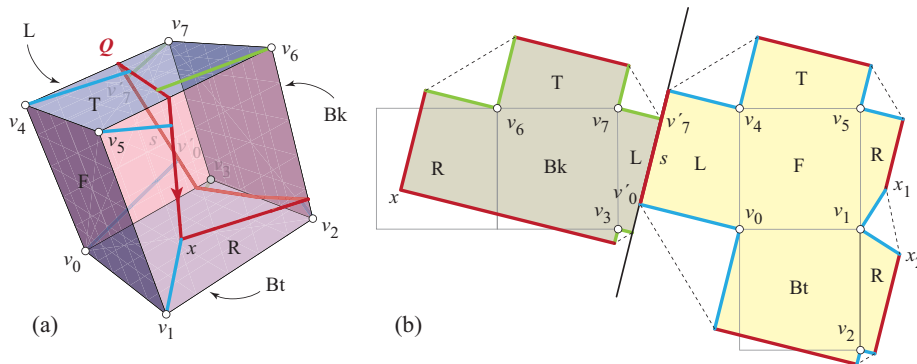


Figure 1: (a) Geodesic loop Q on cube. Shortest paths $\text{sp}(v_i)$ are shown. Faces are labeled $\{F, T, L, R, Bt, Bk\}$. (b) Star unfolding with respect to Q , joined at $s = v'_0v'_7$.

After cutting along $\text{sp}(v_i)$, we conceptually insert an isosceles triangle with apex angle $\omega(v_i)$ at each v_i , which flattens each half. One half (in our example, the left half), is convex, while the other half has several points of nonconvexity, at the images of x . (In our example, only x_1 is nonconvex, when the inserted “curvature triangles” are included.) In the third and final step of the procedure, we select a segment s of Q whose interior contains neither a vertex v_i nor any vertex projection v'_i , such that the extension of s is a supporting line of each half, and cut all of Q except for s . In our example, we choose $s = v'_0v'_7$ (many choices for s work in this example), which leads to non-overlap of the two halves.

We now proceed to detail the three steps of the procedure, this time with proofs. We use a different example to illustrate the main ideas.

3 Shortest Path Cuts

We again use a cube as an illustrative example, but this time with a closed quasigeodesic Q , not a loop: $Q = (v_0, v_5, v_7)$; see Figure 2(a). There is π angle incident to the right at v_5 , and $\frac{1}{2}\pi$ incident to the left; and similarly at v_0 and v_7 . At all other points $p \in Q$, $L(p) = R(p) = \pi$. Thus Q is indeed a quasigeodesic. We will call the left half (including v_2) P_1 , and the right half (including v_4) P_2 . In Figure 2(a), the paths from $\{v_1, v_3, v_6\}$ are uniquely shortest. From v_2 there are three paths tied for shortest, and from v_4 also three are tied.

A central fact that enables our construction is this key lemma from [IIV07, Cor. 1], slightly modified for our circumstances:

Lemma 1 *Let Q be a quasigeodesic on a convex surface \mathcal{P} , and p any point of \mathcal{P} not on Q . Then for any choice of $\text{sp}(p) = pp'$, this is the unique shortest path from p to p' and it is orthogonal to Q if p' is in the relative interior of Q .*

In our situation, the orthogonality condition is only guaranteed to hold when

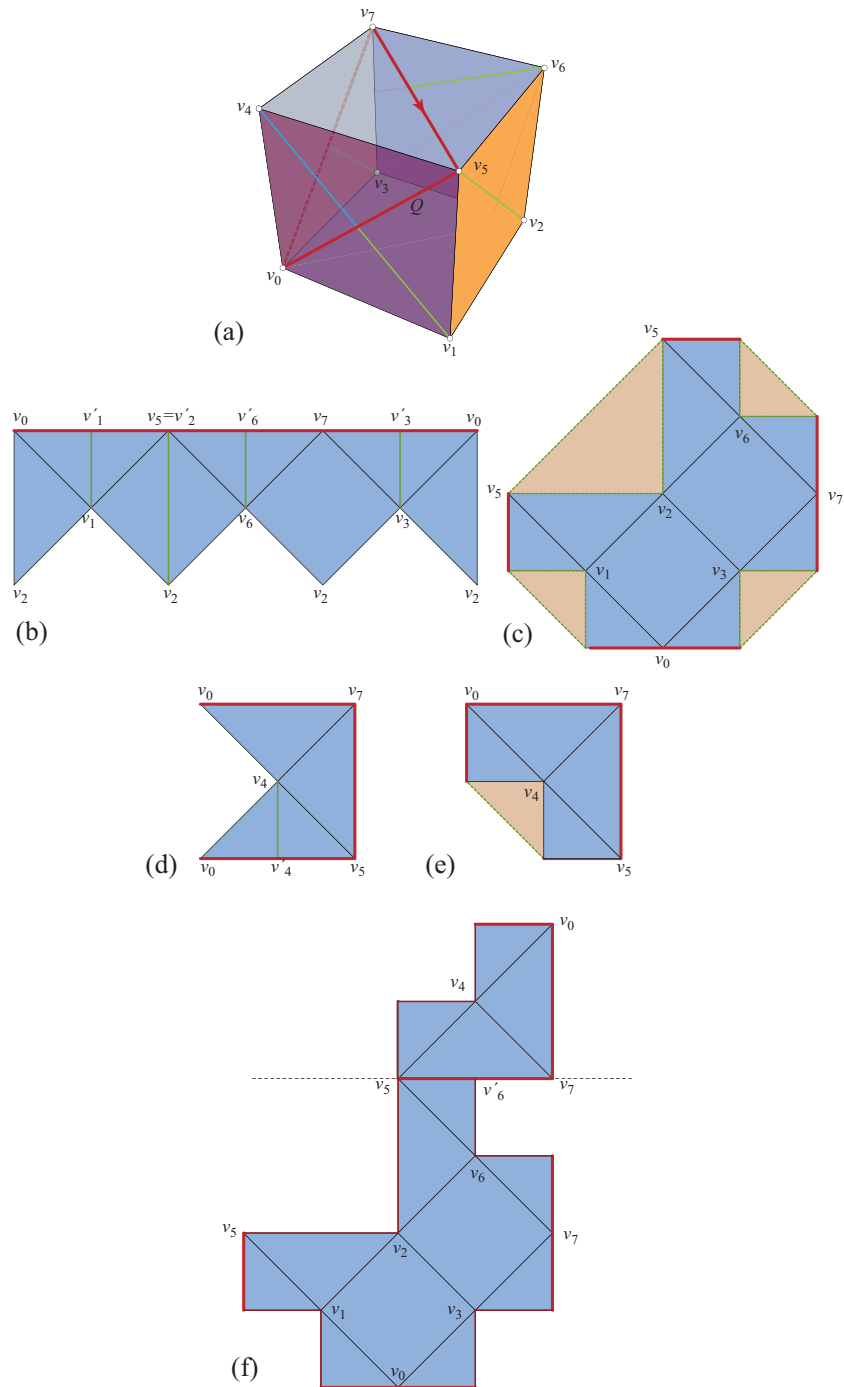


Figure 2: (a) Cube and quasigeodesic $Q = (v_0, v_5, v_7)$. Shortest paths $sp(v_i)$ as indicated. (b,c) Flattening the left half by insertion of curvature triangles along the shortest paths $sp(v_i) = v_i v'_i$. (d,e) Flattening the right half. (f) Two halves joined at $s = v_5 v'_6$.

p' is not the exceptional loop point x of a quasigeodesic loop. In the example of Figure 2(a), there is no exceptional point, so all projections are orthogonal to Q . Note that this lemma does not say that the shortest path from p to Q is unique—which we know is not always true—but that, among those that are tied for shortest, each is the unique shortest path between its two endpoints.

A second fact we need concerning these shortest paths is that they are disjoint, excepting those arriving at the exceptional loop point, in which case they share precisely that point. The reader who accepts this basic fact is invited to skip beyond the proof.

Lemma 2 *Any two shortest paths $\text{sp}(v_1)$ and $\text{sp}(v_2)$, not incident to the loop point x , are disjoint, for distinct vertices $v_1, v_2 \in P$.*

Proof: Suppose for contradiction that at least one point u is shared: $u \in \text{sp}(v_1) \cap \text{sp}(v_2)$. We consider four cases: one shortest path is a subset of the other, the shortest paths cross, the shortest paths touch at an interior point but do not cross, or their endpoints coincide.

1. $\text{sp}(v_2) \subset \text{sp}(v_1)$. Then $\text{sp}(v_1)$ contains a vertex v_2 in its interior, which violates a property of shortest paths [SS86, Lem. 4.1].
2. $\text{sp}(v_1)$ and $\text{sp}(v_2)$ cross properly at u . It must be that $|uv'_1| = |uv'_2|$, otherwise both paths would follow whichever tail is shorter. But now it is possible to shortcut the path in the vicinity of u via σ as shown in Figure 3(a), and the path (v_1, σ, v'_2) is shorter than $\text{sp}(v_1)$.
3. $\text{sp}(v_1)$ and $\text{sp}(v_2)$ touch at u but do not cross properly there. Then there is a shortcut σ to one side (the side with angle $< \pi$), as shown in Figure 3(b).
4. $v'_1 = v'_2$. Then from Lemma 1, we know the two paths are orthogonal to the quasigeodesic Q . If we are not in the previous case, then it must be that there is an angle $\alpha > 0$ separating the paths in a neighborhood of the common endpoint; see Figure 3(c). Then Q has more than π angle to one side at this point, violating the definition of a quasigeodesic. Note that it is here we use the assumption that the paths are not incident to the loop point x .

□

This lemma ensures that the cuttings along $\text{sp}(v_i)$ do not interfere with one another.

4 Flattening the Halves.

The next step is to flatten each chosen half P_1 and P_2 (independently) by suturing in “curvature triangles” along each $\text{sp}(v)$ path. Let P be one of P_1 or P_2 . The basic idea goes back to Alexandrov [Ale05][p. 241, Fig. 103], and was used also in [IV08a]. Let ℓ be the length $|\text{sp}(v)|$ of a shortest path, and let $\omega = \omega(v) > 0$ be the curvature at v . We glue into $\text{sp}(v) = vv'$ the isosceles

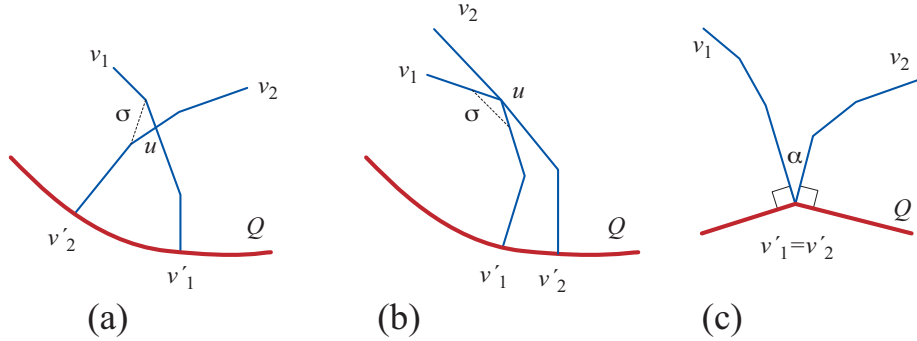


Figure 3: Lemma 2: (a) paths cross; (b) paths touch at an interior point; (c) paths meet at endpoint.

curvature triangle Δ with apex angle ω gluing to v , and incident sides of length ℓ gluing along the cut vv' . This is illustrated in Figure 2, where we show the faces incident to Q in a planar development in (b) and (d), and after gluing in the triangles in (c) and (e). We display this in the plane for convenience of presentation; the triangle insertion should be viewed as operations on the manifolds P_1 and P_2 , each independently.

This procedure only works if $\omega < \pi$, for ω becomes the apex of the inserted triangle Δ . If $\omega \geq \pi$, we glue in two triangles of apex angle $\omega/2$, both with their apexes at v .² Slightly abusing notation, we use Δ to represent these two triangles together. In fact we must have $\omega < 2\pi$ for any vertex v (else there would be no face angle at v), so $\omega/2 < \pi$ and this insertion is indeed well defined.

We should remark that an alternative method of handling $\omega \geq \pi$ would be to simply not glue in anything to the vertex v with $\omega(v) \geq \pi$, in which case we still obtain the lemma below leading to the exact same unfolding.

Now, because ω is the curvature (angle deficit) at v , gluing in Δ there flattens v to have total incident angle 2π . Thus v disappears as a vertex from P (and two new vertices are created along Q).

Let P^Δ be the new manifold with boundary after insertion of all curvature triangles into P . We want to claim that a planar development of P^Δ does not overlap. This is straightforward for a closed quasigeodesic, but requires some argument for a quasigeodesic loop.

Lemma 3 *For each half P of \mathcal{P} , P^Δ is a planar, simple (non-overlapping) polygon.*

Proof: P^Δ is clearly a topological disk: P is, and the insertions of Δ 's maintains it a disk. At every interior point of P^Δ , the curvature is zero by construction. So the interior is flat.

Let ω_Q be the total curvature enclosed within Q on P , and τ_Q the total turn of Q , i.e., the turn of ∂P . The Gauss-Bonnet Theorem yields $\tau_Q + \omega_Q = 2\pi$.

² One can view this as having two vertices with half the curvature collocated at v .

This is precisely the total turn of ∂P^Δ , because that boundary turns τ_Q , plus a total of ω_Q for all the inserted curvature triangles. So indeed the boundary of P^Δ turns just as much as it should if it is a planar polygon. It remains to establish that it is a simple polygon.

There are two cases to consider, depending on whether Q is a closed quasi-geodesic, or a closed quasi-geodesic loop.

1. Q is a closed quasi-geodesic. In this case we show that the boundary ∂P^Δ is convex. This follows from the orthogonality of $\text{sp}(v)$ guaranteed by Lemma 1, as the base angle of the inserted triangle(s) is $\pi/2 - \omega/2$ for $\omega < \pi$, or $\pi/2 - \omega/4$ for $\omega \geq \pi$ (see Figure 4; $\omega = \omega(v)$), so the new angle is smaller than π by $\omega/2$ or $\omega/4$. Thus P^Δ is a planar convex polygon,

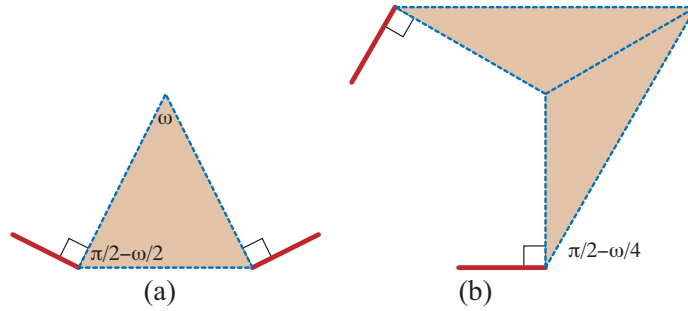


Figure 4: Lemma 3, Case 1: (a) $\omega < \pi$; (b) $\omega \geq \pi$.

and therefore simple. See Figure 2(c,e) for examples.

2. Q is a closed quasi-geodesic loop, with loop point x . If P is the half of \mathcal{P} in which the angle at x is $< \pi$, then the argument above applies. So assume P is the half in which the angle β at x exceeds π . We consider two subcases.

- (a) No vertex of P projects to x .

Then after insertion of the curvature triangles, P^Δ is a topological disk whose boundary is locally convex at all points except at x , whose internal angle is $\beta > \pi$. We now argue that a planar development of such a domain is non-overlapping. Let R_1 and R_2 be rays from x along the two edges of P^Δ incident to x ; see Figure 5. The boundary ∂P^Δ of P^Δ must be exterior to the cone delimited by R_1 and R_2 in a neighborhood of those rays, because the boundary turns convexly at each boundary vertex. So now we have a convex curve, a subset of ∂P^Δ , leaving R_1 and returning to R_2 . For the purposes of contradiction, assume this curve self-intersects. Then it must “spiral,” enclosing a point z of winding number ≥ 2 . We noted above that the total turn of ∂P^Δ is 2π . Thus the total turn of the convex portion of ∂P^Δ , i.e., $\partial P^\Delta \setminus \{x\}$, exceeds 2π by the amount $0 < \beta - \pi$ needed

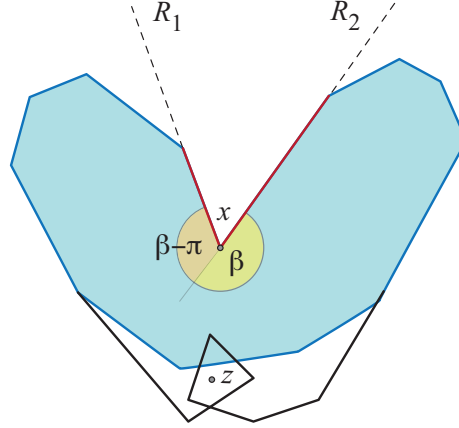


Figure 5: Lemma 3, Case 2(a): P^Δ has one point x of local nonconvexity. Here $\beta = 305^\circ$. The spiral depicted encloses a point z of winding number 2.

to close the shape at x . But $\beta - \pi < \pi$, so the convex curve turns at most $2\pi + (\beta - \pi) < 3\pi$. However, the point z must “see” a turn of $\geq 4\pi$ to have winding number ≥ 2 . Therefore, ∂P^Δ does not self-intersect, and P^Δ is a simple polygon.

(b) One or more vertices of P project to x .

Let v_1, v_2, \dots, v_k be the vertices that project to x in circular order, as illustrated in Figure 6(a). Let x_1 and x_2 be the extreme images of x in a planar development of P^Δ after insertion of all curvature triangles, i.e., incident to the planar image of R_1 and of R_2 respectively; see Figure 6(b). View the curve ∂P^Δ as composed of two pieces: $C_{1,2}$, the curve counterclockwise from x_1 to x_2 , and $C_{2,1}$, the complementary curve counterclockwise from x_2 to x_1 ; $\partial P^\Delta = C_{1,2} \cup C_{2,1}$. $C_{1,2}$ is locally convex everywhere, but $C_{2,1}$ may be nonconvex, as illustrated in Figure 6(b). Now we partition the remaining argument into three parts.

- i. $C_{1,2}$ does not self-intersect. The convex portion does not self-intersect for the same reason we just established in the case above: the curve would have to spiral and turn $\geq 4\pi$, but that total turn angle is not available.
- ii. $C_{2,1}$ does not self-intersect. It clearly cannot if $k=1$, so we henceforth assume $k \geq 2$.

This portion of ∂P^Δ is composed entirely of bases of curvature triangles. For this portion to self-intersect, it must turn at least π . We now compute the total turn $\tau_{2,1}$ and show this leads to a contradiction. Let α_1 be the angle $R_1 x_1 v_1$ and α_2 the angle $R_2 x_2 v_k$. It must be that $\alpha_i \geq \pi/2$ because $v_i x$ is a shortest path. Let β_i be the angle at x on P between $v_i x$ and $v_{i+1} x$. See

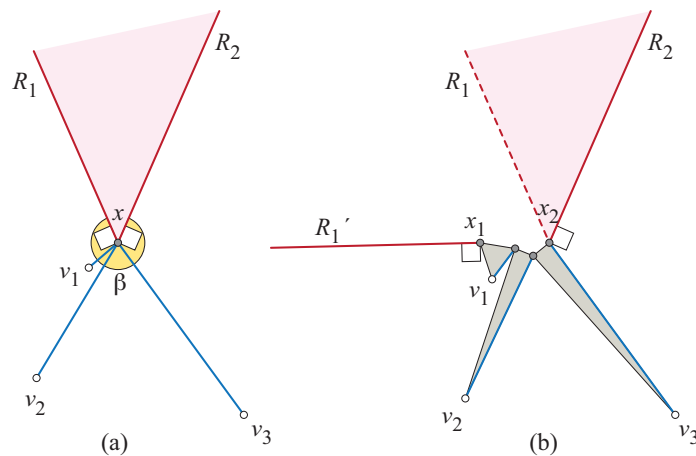


Figure 6: Lemma 3, Case 2(b): (a) Several vertices project to x . (b) After insertion of curvature triangles, with R_2 held fixed. R_1' is the planar image of R_1 .

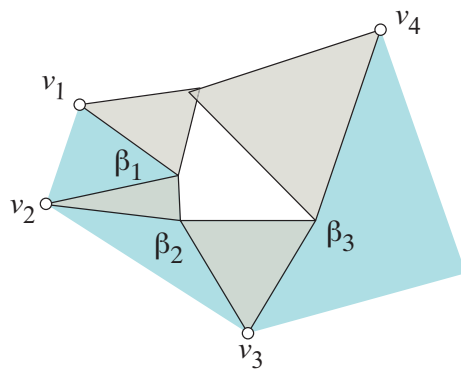


Figure 7: Lemma 3, Case 2(b)ii: The curve $C_{2,1}$ is formed from the bases of curvature triangles. It needs to turn π to self-intersect.

Figure 7. Thus, because $\alpha_1, \alpha_2 \geq \pi/2$, $\alpha_1 + \alpha_2 = \beta - \sum_{i=1}^k \beta_i \geq \pi$, and because $\sum_{i=1}^k \beta_i + \alpha_1 + \alpha_2 = \beta < 2\pi$ we get $\sum_{i=1}^k \beta_i < \pi$. We henceforth drop the limits on the sums, which all run the full appropriate range. The base angle for the curvature triangle incident to v_i is $\frac{1}{2}(\pi - \omega(v_i))$. Thus the turn of the curve at the angle β_i is

$$\left[\frac{1}{2}(\pi - \omega(v_i)) + \frac{1}{2}(\pi - \omega(v_{i+1})) + \beta_i\right] - \pi = \beta_i - \frac{1}{2}[\omega(v_i) + \omega(v_{i+1})].$$

The total turn is therefore

$$\tau_{2,1} = \sum \left\{ \beta_i - \frac{1}{2}[\omega(v_i) + \omega(v_{i+1})] \right\} = \sum \beta_i - \sum \omega(v_i).$$

To insist that $\tau_{2,1} > \pi$ is to say that

$$0 < \sum \omega(v_i) < \sum \beta_i - \pi$$

a contradiction to $\sum \beta_i - \pi < 0$. Thus $\tau_{2,1} \leq \pi$. It is also worth noting that the same turn-angle bound holds for any subchain of $C_{2,1}$ (just by narrowing the sum limits), a fact we will use below.

- iii. $C_{1,2}$ and $C_{2,1}$ do not intersect. For $C_{2,1}$, a potentially nonconvex curve, to intersect $C_{1,2}$, it would have to form a path from x_1 , crossing R'_1 (see Figure 6(b)), and returning to x_2 (or symmetrically, from x_2 cross R_2 and return to x_1). But this requires some subchain of $C_{2,1}$ to turn $> \pi$, contradicting the turn-angle conclusion above. Therefore, ∂P^Δ does not self-cross, and P^Δ is indeed a simple polygon.

□

The above argument would be simpler if it were established that the cone $R_1 x R_2$ in Figure 6(b) is empty in the planar development. We leave this for future work.

5 Joining the Halves

The third and final step of the unfolding procedure selects a *supporting segment* $s \subset Q$ whose relative interior does not contain a projection v' of a vertex. All of Q will be cut except for s . Our choice of s depends on whether Q is a closed quasigeodesic or a quasigeodesic loop:

1. Q is a closed quasigeodesic. Then any s generates a supporting line to a planar development of P_i , $i = 1, 2$, because P_i^Δ is a convex domain. Then joining planar developments of P_1 and P_2 along s places them on opposite sides of the line through s , thus guaranteeing non-overlap. See Figure 2(f), where $s = v_5 v'_6$.

2. Q is a quasigeodesic loop. Let P be the half of \mathcal{P} that contains the angle $\beta > \pi$ at x . Thus P^Δ is potentially nonconvex at points along the chain $C_{2,1}$ from x_2 to x_1 . Let R_1 and R_2 be the rays from x_1 and x_2 respectively, tangent to ∂P^Δ . Let L_1 and L_2 be lines parallel to R_1 and R_2 tangent

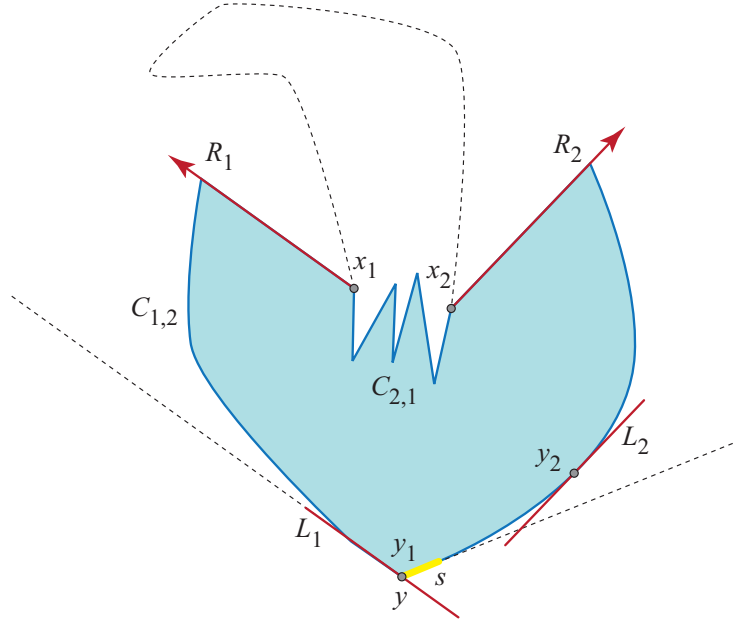


Figure 8: $C_{2,1}$ cannot cross both extensions of edges incident to y . Here $y = y_1$.

to ∂P^Δ at y_1 and y_2 respectively. Let y be any vertex between y_1 and y_2 ; y may be y_1 or y_2 , as in Figure 8. Now we claim that one of the two edges of ∂P^Δ incident to y can serve as s . For both these edges to fail to extend to supporting lines, $C_{2,1}$ would have to cross both edge extensions. But, the angle at y is $< \pi$ (because $C_{1,2}$ is convex), so crossing both edge extensions would require $C_{2,1}$ to turn more than π , which we established in Lemma 3 is impossible.

Case 2 is illustrated in Figure 1(b), where this reasoning leads to $s = v'_0 v'_7$.

In either case s extends to a supporting line of both halves, and thus we obtain a non-overlapping placement of the planar developments of P_1 and P_2 .

It should be clear now that this procedure works for any convex polyhedron:

Theorem 4 *Let Q be a quasigeodesic loop on a convex polyhedral surface \mathcal{P} . Cutting shortest paths from every vertex to Q , and cutting all but a supporting segment s of Q as designated above, unfolds \mathcal{P} to a simple planar polygon.*

Figure 9 shows another example, a closed geodesic on a dodecahedron, this time a pure geodesic. The unfolding following the above construction is shown in

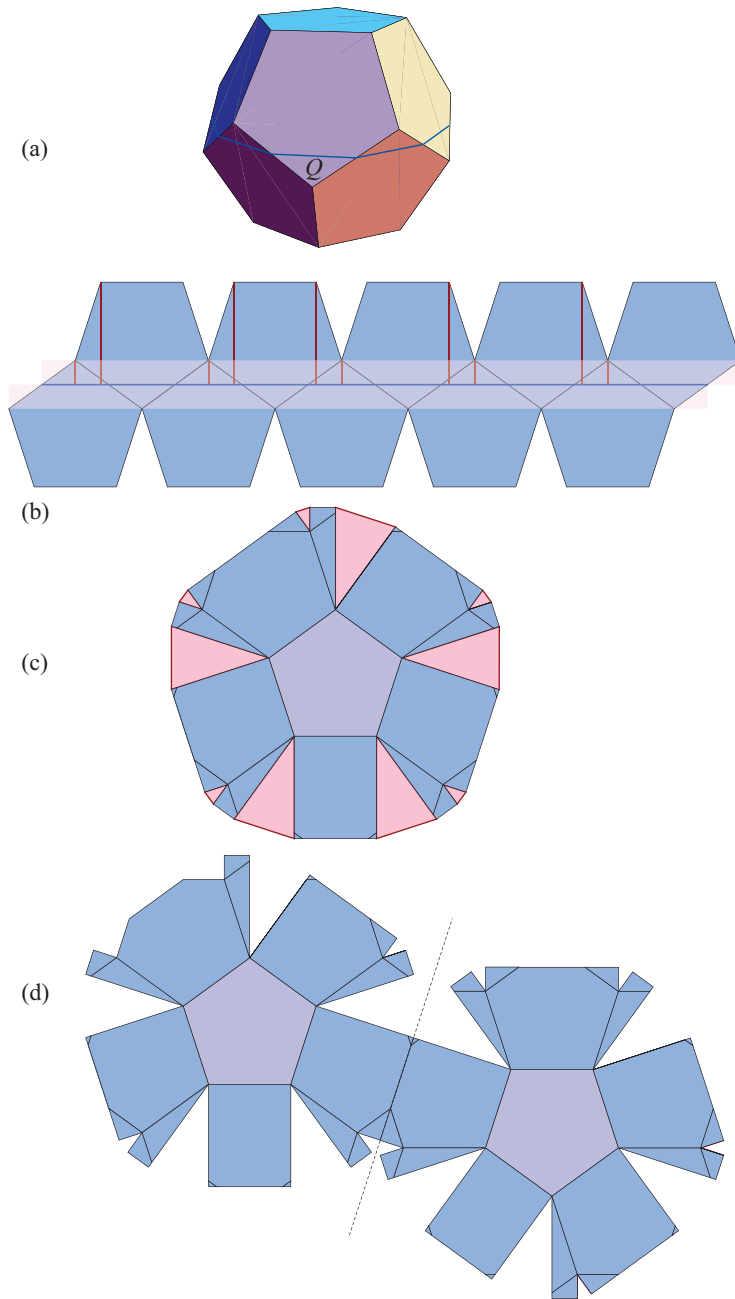


Figure 9: (a) Q here is a geodesic; it includes no vertices, as is evident in the layout (b). The region isometric to a right circular cylinder is highlighted. The convex domain P^Δ from Lemma 3 is shown in (c), and one possible unfolding in (d).

Figure 9(c,d). In this case when Q is a pure, closed geodesic, there is additional structure that can be used for an alternative unfolding. For now Q lives on a region isometric to a right circular cylinder. Figure 9(b) illustrates that the upper and lower rims of the cylinder are loops parallel to Q through the vertices of P at minimum distance to Q (at least one vertex on each side.) In the figure, these shortest distances to the upper rim are the short vertical paths from Q to the five pentagon vertices. Those rim loops are themselves closed quasigeodesics. An alternative unfolding keeps the cylinder between the rim loops intact and attaches the two reduced halves to either side. See Figure 10.

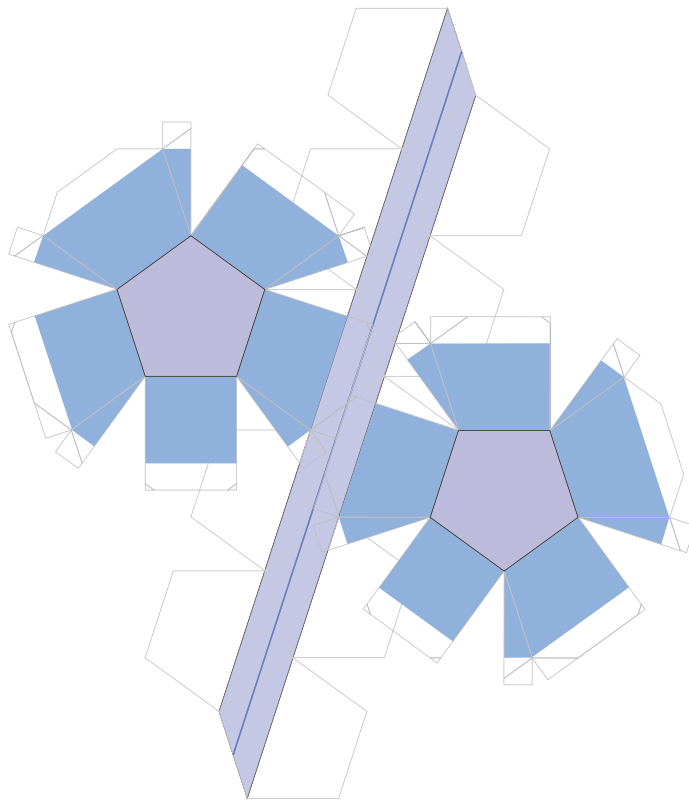


Figure 10: Alternative unfolding of the example in Figure 9. Various construction lines are shaded lightly.

6 Future Work

We have focused on establishing Theorem 4 rather than the algorithmic aspects. Here we sketch preliminary thoughts on computational complexity. Let n be the number of vertices of \mathcal{P} , and let $q = |Q|$ be the number of faces crossed by the

geodesic loop Q . In general q cannot be bound as a function of n . Finally, let $m = n + q$, the total combinatorial complexity of the “input” to the algorithm. Constructing Q from a given point and direction will take $O(q)$ time. Identifying a supporting segment s , and laying out the final unfolding, is proportional to m . The most interesting algorithmic challenge is to find the shortest paths from each vertex v_i to Q . It appears that this can be accomplished efficiently, in $O(m \log m)$ time, by first computing the cut locus of Q . We expect to address this computation in [IOV08].

We do not believe that quasigeodesic loops constitutes the widest class of curves for which the star unfolding leads to non-overlap. In particular, we believe we can extend Theorem 4 to quasigeodesics with two exceptional points, one with angle $>\pi$ to one side, and the other with angle $>\pi$ to the other side. But whether this extension constitutes the widest class of curves for which Theorem 4 holds remains unclear.

If one fixes a nonvertex point $p \in \mathcal{P}$ and a surface direction \vec{u} at p , a quasigeodesic loop can be generated to have direction \vec{u} at p . It might be interesting to study the continuum of star unfoldings generated by spinning \vec{u} around p .

Appendix 1: Symbol Glossary

\mathcal{P}	convex polyhedron
P_1, P_2	the two “halves” $\mathcal{P} \setminus Q$
P	one half, either P_1 or P_2
v_i	vertex of \mathcal{P}
Q	a quasigeodesic loop
x	the exceptional loop point of Q
$L(p), R(p)$	angle incident to left/right side of p on curve
Γ	directed curve
$\text{sp}(v)$	one selected shortest path from v to Q
v'	the projection of v onto Q : $\text{sp}(v) = vv'$
$\omega = \omega(v)$	the curvature at v , 2π minus the incident face angles
Δ	a curvature triangle
ℓ	$ \text{sp}(v) $
∂P	the boundary of P
P^Δ	the manifold P after insertion of all curvature triangles Δ
ω_Q	total curvature inside Q on P
τ_Q	total turn of $Q = \partial P$
∂P^Δ	the boundary of P^Δ
β	angle $> \pi$ at the loop point $x \in Q$
x_1, x_2	extreme images of x in planar development of P^Δ
R_1, R_2	rays along edges incident to x_1, x_2
α_i	α_1 is angle $R_1x_1v_1$ and α_2 is angle $R_2x_2v_k$
β_i	angle at x on P between $v_i x$ and $v_{i+1} x$
$C_{1,2}$	∂P^Δ counterclockwise from x_1 to x_2
$C_{2,1}$	∂P^Δ counterclockwise from x_2 to x_1
$\tau_{2,1}$	total turn of curve $C_{2,1}$ from x_2 to x_1
L_1, L_2	lines parallel to R_1, R_2 tangent to P^Δ
y_1, y_2	tangency points of L_1, L_2
y	a vertex between y_1 and y_2
s	supporting segment
n	number of vertices of \mathcal{P}
q	combinatorial complexity of Q
m	$n + q$
\vec{u}	direction vector through $p \in \mathcal{P}$

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