6-2017

How Low Can You Go? New Bounds on the Biplanar Crossing Number of Low-dimensional Hypercubes

Gregory J. Clark
University of South Carolina - Columbia

Gwen Spencer
Smith College, gspencer@smith.edu

How Low Can You Go?

New Bounds on the Biplanar Crossing Number of Low-dimensional Hypercubes

Gregory J. Clark*, Gwen Spencer†

June 2017

Abstract

In this note we provide an improved upper bound on the biplanar crossing number of the 8-dimensional hypercube. The k-planar crossing number of a graph $cr_k(G)$ is the number of crossings required when every edge of G must be drawn in one of k distinct planes. It was shown in [2] that $cr_2(Q_8) \leq 256$ which we improve to $cr_2(Q_8) \leq 128$. Our approach highlights the relationship between symmetric drawings and the study of k-planar crossing numbers. We conclude with several open questions concerning this relationship.

1 Introduction

The traditional *crossing number* of a graph $G = (V, E)$, denoted by $cr(G)$, is the minimum number of edge crossings required to draw G in the 2-dimensional Euclidean plane. To study printed circuit boards, Owens [4] generalized the question: what is the minimum number of edge crossings required by a drawing that is allowed to carefully divide the edges of G among two different 2-dimensional Euclidean planes? Since then the definition has been extended to $k \geq 2$ planes [2].

Suppose that E is partitioned into k disjoint subsets, $E_1, E_2, ..., E_k$, and let $G_i = (V, E_i)$. Each G_i has some crossing number $cr(G_i)$. Suppose further that G_i will be drawn in the ith plane from a set of k distinct planes. The $k-planar$ *crossing number of* G, denoted $cr_k(G)$ is then the minimum of

$$cr(G_1) + cr(G_2) + ... + cr(G_k)$$

*University of South Carolina. Columbia, SC, USA.

†Smith College. Northampton, MA, USA. *This material is based upon work supported by the National Science Foundation under Grant Number DMS 1641020 while both authors attended the "Beyond Planarity: Crossing Numbers of Graphs" workshop (organized by the American Mathematical Society), and under Grant No. DMS-1440140 while G. Spencer was in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Fall 2017 semester.*

over all partitions of the edge set E.

Trivially, letting $E_1 = E$ shows that $cr_k(G) \leq cr(G)$. The question remains: given the freedom to consider any partition of G's edges among k disjoint planes, how low can we drive the number of required crossings?

A significant challenge in designing a crossing-minimizing k-planar drawing of G is that, even for quite simple G_i, $cr(G_i)$ could be unknown. For example: for Q_4, the 4-dimensional hypercube, it is known that $cr(Q_4) = 8$; however, the exact value of $cr(Q_d)$ is unknown for $d > 4$ [3].

The previous upper bound $cr_2(Q_8) \leq 256$ was given by a construction of Czabarka, Sýkora, Székely, and Vrt́o in [2]. Czabarka et al. give a general construction for an upper bound on $cr_2(Q_d)$ that achieves 256 crossings when $d = 8$. Their approach specifies a bi-planar partition of the edges of Q_8 based on a set of lower-dimensional hypercube subgraphs. Their upper bound is minimized when these hypercube subgraphs are as-uniform-as-possible in size. In particular, for Q_8 their construction specifies sixteen disjoint Q_4 subgraphs in Plane 1 and a further sixteen disjoint Q_4 subgraphs in Plane 2. Recall that $cr(Q_4) = 8$, so drawing each disjoint copy of Q_4 optimally yields

$$cr_2(Q_8) \leq 16 \times 2 \times 8 = 256.$$

We now present our main result which improves on the the best known upper bound of $cr(Q_8)$ by a factor of 2.

Theorem 1 *There exists a 2-planar drawing of the 8-dimensional hypercube with 128 crossings so that* $cr_2(Q_8) \leq 128$.

2 A biplanar drawing of Q_8 with 128 crossings

To prove Theorem 1, we provide a biplanar drawing of Q_8 with 128 crossings. We improve the previous construction by plane-swapping edges to give a net reduction in total edge crossings. Our drawing consists of graphs G_1 and G_2 in Plane 1 and 2 respectively such that $G_1 \cong G_2$ where $cr(G_i) \leq 64$. We found several distinct bi-planar drawings of Q_8 with exactly 128 crossings which satisfy these conditions. For ease of exposition, we present a highly symmetric drawing.

We define a *depleted n-dimensional hypercube* to be a graph whose vertex set is $V(Q_n)$ and will refer to such graphs as *depleted n-cubes*. We will make use of depleted 5-cubes. To this end we introduce the following partition $V(Q_4) := C_1 \sqcup C_2$ where

$$C_1 := \{0000, 1000, 0010, 1010, 0011, 1011, 0001, 1001\}$$
$$C_2 := \{0111, 1111, 0101, 1101, 0100, 1100, 0110, 1110\}.$$

For ease of notation, we denote $\hat{c} \in C_1$ and $\check{c} \in C_2$. Moreover, we let $b \in \{0, 1\}$ represent the usual binary-bit. Maintaining the notation of [2] we refer to each node of Q_8 by a length-8 binary string from $\{0, 1\}^8$. Given two binary strings s_1 and s_2 we write $s_1 s_2$, or $s_1 - s_2$ for readability, to be the usual string concatenation.

In our construction, each plane contains 512 edges, and furthermore, G_1 and G_2 are isomorphic. For exposition, suppose that we initially have a Plane 0 which contains all the edges and vertices of Q_8. Further suppose that there exist Planes 1 and 2 which each initially contain the vertices of Q_8 and no edges. We move every edge from Plane 0 to either Plane 1 or Plane 2 to create our biplanar partition. In the following table, we describe explicitly the 512 edges we add to Plane 1.

Consider the set of pairs

$$P_1 := \{(0000, 1000), (0010, 1010), (0011, 1011), (0001, 1001)\} \subset \binom{C_1}{2}.$$

For $(\hat{c}_1, \hat{c}_2) \in P_1$ define the *depleted 5-cube of Type 1*, denoted $D_1(\hat{c}_1, \hat{c}_2)$, according to Table 1.

$E(D_1(\hat{c}_1, \hat{c}_2))$ for $\hat{c} \in (\hat{c}_1, \hat{c}_2) \in P_1$			
$(\hat{c} - b000, \hat{c} - b001)$	$(\hat{c} - b000, \hat{c} - b100)$	$(\hat{c} - b100, \hat{c} - b101)$	$(\hat{c} - b001, \hat{c} - b101)$
$(\hat{c} - b010, \hat{c} - b011)$	$(\hat{c} - b010, \hat{c} - b110)$	$(\hat{c} - b110, \hat{c} - b111)$	$(\hat{c} - b011, \hat{c} - b111)$
$(\hat{c} - b000, \hat{c} - b010)$	$(\hat{c} - b001, \hat{c} - b011)$	$(\hat{c} - b100, \hat{c} - b110)$	$(\hat{c} - b101, \hat{c} - b111)$
$(\hat{c} - 0101, \hat{c} - 1101)$	$(\hat{c} - 0111, \hat{c} - 1111)$	$(\hat{c} - 0110, \hat{c} - 1110)$	$(\hat{c} - 0100, \hat{c} - 1100)$
$(\hat{c}_1 - 0000, \hat{c}_2 - 0000)$	$(\hat{c}_1 - 0100, \hat{c}_2 - 0100)$	$(\hat{c}_1 - 1100, \hat{c}_2 - 1100)$	$(\hat{c}_1 - 1000, \hat{c}_2 - 1000)$
$(\hat{c}_1 - 1001, \hat{c}_2 - 1001)$	$(\hat{c}_1 - 1101, \hat{c}_2 - 1101)$	$(\hat{c}_1 - 0101, \hat{c}_2 - 0101)$	$(\hat{c}_1 - 0001, \hat{c}_2 - 0001)$

Table 1: Table of the 64 edges of *depleted 5-cubes of Type 1*.

The four *depleted 5-cubes of Type 1* are vertex disjoint (from the form of pairs in P_1). We present an eight-crossing drawing of a *depleted 5-cube of Type 1* in Figure 2, which proves the following claim.

Claim 1 $cr(D_1(\hat{c}_1, \hat{c}_2)) \leq 8$.

We similarly define $D_2(\check{c}_1, \check{c}_2)$, the *Depleted 5-cube of Type 2*, according to Table 2 given

$$P_2 := \{(0111, 1111), (0101, 1101), (0100, 1100), (0110, 1110)\} \subset \binom{C_2}{2}.$$

Again, the four *depleted 5-cubes of Type 2* are vertex disjoint. An eight-crossing drawing of a *depleted 5-cube of Type 2* is given in Figure 2, which proves the following claim.

Claim 2 $cr(D_2(\check{c}_1, \check{c}_2)) \leq 8$.

Each *depleted 5-cube* has 64 edges, so Plane 1 contains 512 edges. Further, no *depleted 5-cube of Type 1* shares a vertex with a *depleted 5-cube of Type 2*. This follows from the form of the pairs in P_1 and P_2 and the form of the edge sets described in Tables 1 and 2. Thus, these 512 edges can be drawn in Plane 1 with at most 64 crossings.

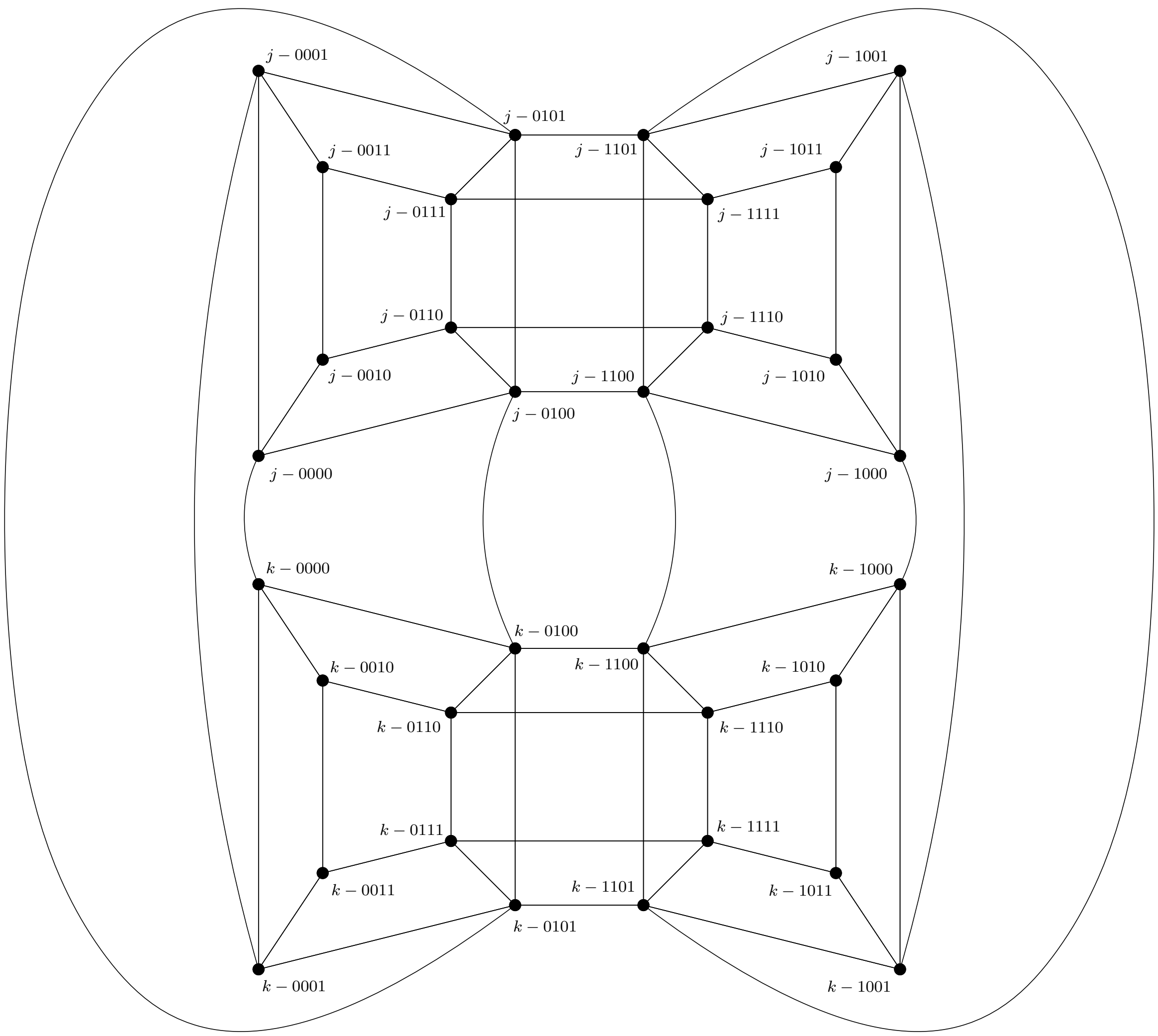

Figure 1: A drawing of $D_1(\hat{c}_1, \hat{c}_2)$ for $(\hat{c}_1, \hat{c}_2) \in P_1$ with eight crossings.

Remark 1 *Plane 2 contains all the edges of Q_8 which are not in Plane 1. Moreover, $G_1 \cong G_2$.*

We now provide a more illuminating description of the edges of Plane 2. The edges in Plane 2 have a symmetric representation in terms of the edges in Plane 1. Let $\rho : E(Q_8) \to E(Q_8)$ such that

$$\rho((v_p v_s, u_p u_s)) = (v_s v_p, u_s u_p)$$

$E(D_2(\check{c}_1, \check{c}_2))$ for $\check{c} \in (\check{c}_1, \check{c}_2) \in P_2$.			
$(\check{c} - b000, \check{c} - b001)$	$(\check{c} - b000, \check{c} - b100)$	$(\check{c} - b100, \check{c} - b101)$	$(\check{c} - b001, \check{c} - b101)$
$(\check{c} - b010, \check{c} - b011)$	$(\check{c} - b010, \check{c} - b110)$	$(\check{c} - b110, \check{c} - b111)$	$(\check{c} - b011, \check{c} - b111)$
$(\check{c} - b000, \check{c} - b010)$	$(\check{c} - b001, \check{c} - b011)$	$(\check{c} - b100, \check{c} - b110)$	$(\check{c} - b101, \check{c} - b111)$
$(\check{c} - 0011, \check{c} - 1011)$	$(\check{c} - 0001, \check{c} - 1001)$	$(\check{c} - 0000, \check{c} - 1000)$	$(\check{c} - 0010, \check{c} - 1010)$
$(\check{c}_1 - 0110, \check{c}_2 - 0110)$	$(\check{c}_1 - 0111, \check{c}_2 - 0111)$	$(\check{c}_1 - 0011, \check{c}_2 - 0011)$	$(\check{c}_1 - 1011, \check{c}_2 - 1011)$
$(\check{c}_1 - 1111, \check{c}_2 - 1111)$	$(\check{c}_1 - 1110, \check{c}_2 - 1110)$	$(\check{c}_1 - 1010, \check{c}_2 - 1010)$	$(\check{c}_1 - 0010, \check{c}_2 - 0010)$

Table 2: Table of 64 edges of *depleted 5-cubes of Type 2.*

where v_p is a prefix string of length four, $v_1v_2v_3v_4$, and v_s is a suffix string of length four, $v_5v_6v_7v_8$ that together define vertex $v = v_1v_2 \ldots v_8$. Indeed ρ captures the symmetric relationship between edges in Plane 1 and the edges in Plane 2. Assuming an ordering on the vertices of Q_8 one can check that ρ is indeed a bijection. As an example, in Table 1 we assign edge ($\hat{c}$b-000, $\hat{c}$b-001) to Plane 1. So we send

$$\rho((\hat{c}b - 000, \hat{c}b - 001)) = (b000 - \hat{c}, b001 - \hat{c})$$

to Plane 2. If we let $\mathcal{P}_i$ be the set of edges partitioned into Plane i then $\mathcal{P}_2 = \rho(\mathcal{P}_1)$. Moreover, the drawings provided in Figures 1 and 2 for *depleted 5-cubes of Type* 1 (or *Type 2*, resp.) are also drawings of their images under ρ. It follows that, for the edge partition we describe, each plane can be drawn with at most 64 crossings implying that $cr_2(Q_8) \leq 128$ as desired.

A natural next step in this research is to determine whether or not this bound is sharp. The authors believe this to be the case; however, such a proof remains elusive. Alas, we leave the reader with the following conjecture.

Conjecture 1 $cr_2(Q_8) = 128$.

3 Lower Bounds on *structurally-symmetric* k-planar crossing numbers for Hypercubes

Notably, our bi-planar drawing of Q_8 satisfies $G_1 \cong G_2$. This is a rather special property and is termed *self-complementary* in [2]. It could be the case that there exists a non-isomorphic partition of $E(Q_8)$ which admits strictly fewer crossings. Yet, we wonder whether demanding that the G_i be isomorphic truly forces a suboptimal number of crossings for k-planar drawings. In particular, such symmetry would be expected when considering highly symmetric graphs like hyper-cubes.

To formalize this question, we introduce the following generalization of self-complementary edge partitions.

Definition 1 *For a finite graph $G = (V, E)$, let P denote an edge-partition $E = (E_1, E_2, ..., E_k)$ and define $G_i = (V, E_i)$ for all i. If for all pairs $(r, s) \in [k] \times [k]$ we have $G_r \cong G_s$, then P is a k-structurally-symmetric partition of G.*

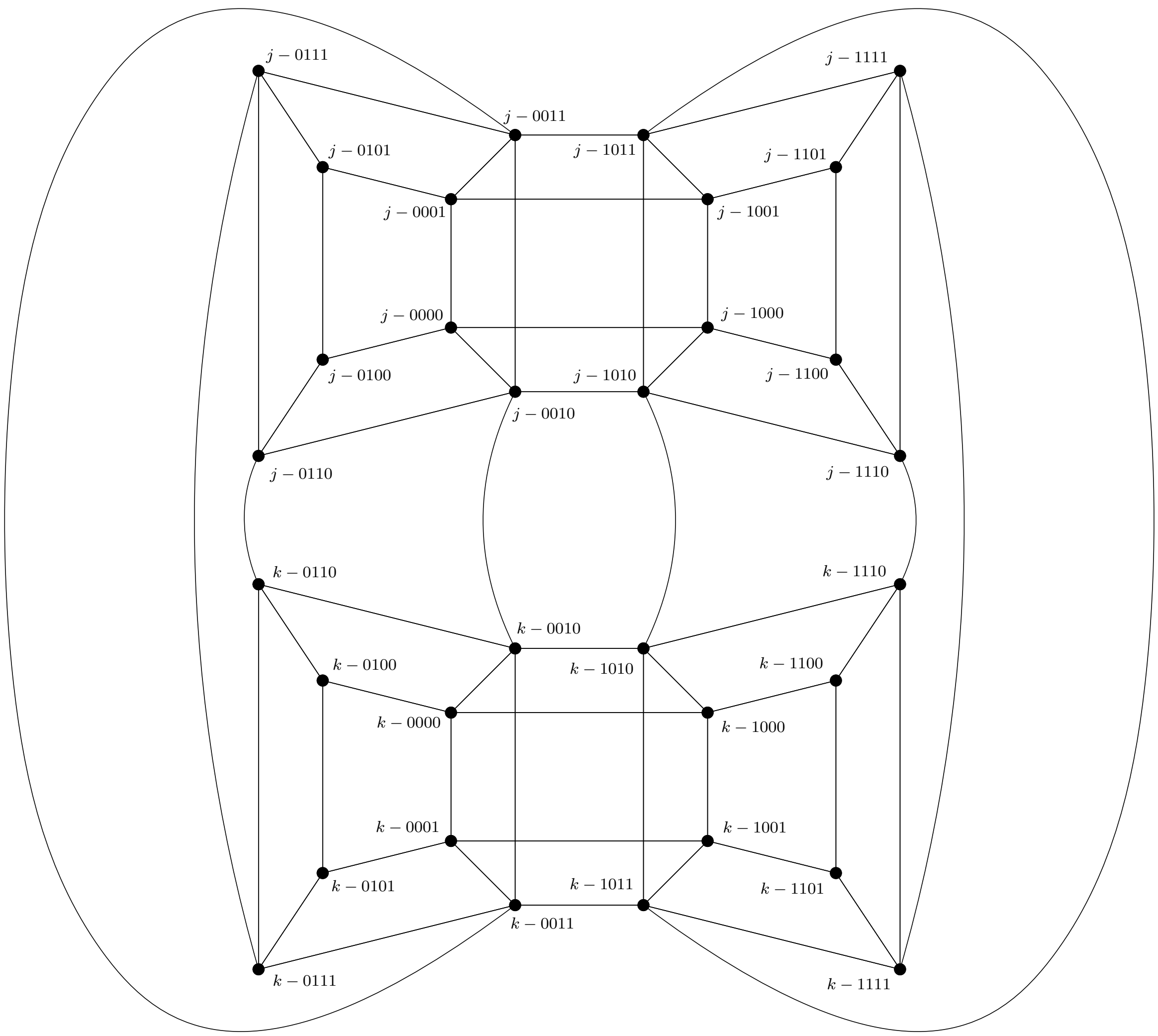

Figure 2: A drawing of $D_2(\check{c}_1, \check{c}_2)$ for $(\check{c}_1, \check{c}_2) \in P_2$ with eight crossings.

Trivially, when $|E|$ is not a multiple of k, no *k-structurally-symmetric partition* of E exists.

Definition 2 *If there exists a k-structurally-symmetric partition for G that can be drawn with $cr_k(G)$ crossings then we say that the graph G is k-structurally-symmetric.*

It is unclear whether graphs exist for which any *k-structurally-symmetric* *partition* of E forces a sub-optimal k-planar drawing (which requires strictly more than $cr_k(G)$ crossings).

In particular, we leave the reader with the following question.

Question 1 *Is the d-dimensional hypercube 2-structurally-symmetric?*

This question motivates the following definition.

Definition 3 *Let $cr_{kss}(G)$ denote the minimum number of crossings required among all k-structurally symmetric partitions of G. We call cr_{kss}* the k-structurally-symmetric crossing number of G.

Trivially, $cr_{kss}(G) \geq cr_k(G)$. So, *k-structurally symmetric graphs* are precisely those graphs G that have $cr_k(G) = cr_{kss}(G)$. We conclude by presenting the reader questions concerning k-structurally-symmetric crossing numbers.

Question 2 *Characterize the set of all k-structurally-symmetric graphs. To this end, what structural properties ensure that a graph is k-structurally-symmteric or otherwise?*

Question 3 *Provide a graph for which the difference between $cr_{kss}(G)$ and $cr_k(G)$ is large (or even > 0). Further, is there an infinite family $(G_n)_{n\geq 1}$ such that $G_n \subseteq G_{n+1}$ and $(cr_{kss}(G_n) - cr_k(G_n))_{n\geq 1} \uparrow \infty$?*

4 Acknowledgements

This material is based upon work that started at the Mathematics Research Communities workshop "Beyond Planarity: Crossing Numbers of Graphs", organized by the American Mathematical Society, with the support of the National Science Foundation under Grant Number DMS 1641020.

We would like to extend our thanks to the organizers of the workshop for their commitment to engendering academic growth in young career mathematicians. We are particularly thankful for László Székely and his exemplary mentoring which made this project possible.

References

[1] A. Aggarwal, M. Klawe, P. Shor, Multi-layer grid embeddings for VLSI, Algorithmica 6 (1991) 129-151.

[2] É. Czabarka, O. Sýkora, L.A. Székely, I. Vrt́o, Biplanar crossing numbers: a survey of results and problems, in: Györi, E., Katona, G.O.H., Lovász, L. (Eds.), More Sets, Graphs and Numbers, Bolyai Society Mathematical Studies, vol. 15, Springer, Berlin, 2006, pp.57-77.

[3] Faria, L., de Figueiredo, C. M. H., On the Eggleton and Guy conjectured upper bound for the crossing number of the n-cube, *Mathematica Slovaca* **50** (2000), 271-287.

[4] A. Owens, On the biplanar crossing number, IEEE Trans. Circuit Theory **18** (1971) 277-280.

[5] J. Pach, L. A. Székely, Cs. D. Tóth, G. Tóth, Note on k-planar crossing numbers, to appear in *Computational Geometry: Theory and Applications* Special Issue in Memoriam Ferran Hurtado.

[6] F. Shahrokhi, O. Sýkora, L. A. Székely and I. Vrťo, k-planar crossing numbers, *Discrete Appl. Math.* **155** (2007), 1106-1115.