2018

The Dihedral Genus of a Knot

Patricia Cahn
Smith College, pcahn@smith.edu

Alexandra Kjuchukova
University of Wisconsin-Madison

THE DIHEDRAL GENUS OF A KNOT

PATRICIA CAHN AND ALEXANDRA KJUCHUKOVA

Abstract. Let $K \subset S^3$ be a Fox p-colored knot and assume K bounds a locally flat surface $S \subset B^4$ over which the given p-coloring extends. This coloring of S induces a dihedral branched cover $X \to S^4$. Its branching set is a closed surface embedded in S^4 locally flatly away from one singularity whose link is K. When S is homotopy ribbon and X a definite four-manifold, a condition relating the signature of X and the Murasugi signature of K guarantees that S in fact realizes the four-genus of K. We exhibit an infinite family of knots K_m with this property, each with a colored surface of minimal genus m. As a consequence, we classify the signatures of manifolds X which arise as dihedral covers of S^4 in the above sense.

1. Introduction

The Slice-Ribbon Conjecture of Fox [6] asks whether every smoothly slice knot in S^3 bounds a ribbon disk in the four-ball. The analogous question can be asked in the topological category, namely: does every topologically slice knot bound a locally flat homotopy ribbon disk in B^4? Recall that a properly embedded surface with boundary $F' \subset B^4$ is *homotopy ribbon* if the fundamental group of its complement is generated by meridians of $\partial F'$ in S^3. Ribbon disks are easily seen to be homotopy ribbon whereas homotopy ribbon disks need not be smooth.

For knots of higher genus, the generalized topological Slice-Ribbon Conjecture asks whether the topological four-genus of a knot is always realized by a homotopy ribbon surface in B^4. When a knot K admits Fox p-colorings, we approach this problem by studying locally flat, oriented surfaces $F' \subset B^4$ with $\partial F' = K$ over which some p-coloring of K extends, in the sense defined in Section 2.1. The minimal genus of such a surface, when one exists, we call the *p-dihedral genus of K*.

When K is slice and p square-free, it is classically known that the colored surface F' for K can always be chosen to be a disk. That is, p-dihedral genus and classical four-genus coincide for slice knots. Furthermore, the topological Slice-Ribbon Conjecture is true for p-colorable slice knots if and only if the minimal p-dihedral genus for these knots can always be realized by homotopy ribbon surfaces. With this in mind, given a square-free integer p and a p-colorable knot K, we ask:

Question 1. Is the four-genus of K equal to its p-dihedral genus?

Question 2. Is the p-dihedral genus of K realized by a homotopy ribbon surface?

When both of these questions are answered in the affirmative for a knot K with respect to some integer p, it follows that the topological four-genus and homotopy ribbon genus of K are equal; that is, the generalized topological Slice-Ribbon Conjecture holds for K. If K is not slice, requiring that it satisfies Questions 1 and 2 is a priori a stronger condition than satisfying the generalized Slice-Ribbon Conjecture; however, the advantage of this point of view is that dihedral genus can be studied using dihedral branched covers.

This work was partially supported by NSF-DMS grant 1821212 to P. Cahn and 1821257 to A. Kjuchukova.

Specifically, our approach is the following. Starting with a branched cover of $f' : X' \to B^4$ branched along a locally flat properly embedded surface F' with $\partial F = K$, we construct a new cover $f : X \to S^4$ by taking the cones of $\partial X'$, S^3 and the map f. The branching set of f is a surface F embedded in S^4 locally flatly except for one singular point whose link is K. When f is a p-fold *irregular dihedral cover* (Definition 1), one obtains a knot invariant $\Xi_p(K)$ from this construction, with $\Xi_p(K) = -\sigma(X)$, by [12]. This invariant is our main tool. As implied by the above, Ξ_p is only defined for knots which arise as singularities in this setting [8]. Such knots are called *p-admissible*, and they are precisely the knots for which the p-dihedral genus, the main subject of this note, is defined. Admissibility of knots is studied in [13].

In the next section, we put side by side the relevant notions of knot four-genus, recall several definitions, and state our main result, Theorem 1. Therein, we give a lower bound on the homotopy-ribbon p-dihedral genus of a colored knot K in terms of the invariant $\Xi_p(K)$. We also give a sufficient condition for when this bound is sharp.

In Theorem 2 and Corollary 3, we construct, for any integer $m \geq 0$, infinite families of knots for which the 3-dihedral genus and the topological four-genus are both equal to m. The basis of this construction are the knots K_m pictured in Figure 1. The various four-genera of these knots are computed with the help of Theorem 1. In particular, for these knots, the lower bound on genus obtained via branched covers is exact and the generalized topological Slice-Ribbon Conjecture holds.

The technique we apply is the following. One can evaluate $\Xi_p(K)$ by realizing K as the only singularity on the branch surface of a dihedral cover of S^4. Each of the knots K_m arises as the only singularity on the branching set of a 3-fold dihedral cover

$$f_m : \#(2m+1)\overline{\mathbb{C}P}^2 \to S^4.$$

The branching set of f_m is the boundary union of the cone on K_m with the surface F'_m realizing the four-genus of K_m. We construct these covering maps explicitly using singular triplane diagrams, a technique introduced in [3]. Equivalently, we construct a family of covers $\#(2m+1)\mathbb{C}P^2 \to S^4$, again with oriented, connected branching sets, with the mirror images of the knots K_m as singularities.

We work in the topological category, except where explicitly stated otherwise. Throughout, F denotes a closed, connected, oriented surface, and F' a connected, oriented surface with boundary. D_p denotes the dihedral group of order $2p$, and p is always assumed odd.

Acknowledgments. The concept of dihedral genus for knots is partially due to Kent Orr; it was conceived while working on [13]. The examples given in Figure 1 and the associated dihedral covers of S^4 were inspired by discussions with Ryan Blair and our work on [1].

2. Dihedral Four-Genus and the Main Theorem

2.1. **Some old and new notions of knot genus.** We study the interplay between the following notions of four-genus for a Fox p-colorable knot $K \subset S^3$. Classically, the *smooth (topological) four-genus* is the minimum genus of a smooth (locally flat) embedded orientable surface in B^4 with boundary K. The *smooth (topological) p-dihedral genus* of a p-colored knot K is, informally, the minimum genus of such a surface F' in B^4 over which the p-coloring of K extends. Precisely, this means that there exists a homomorphism $\bar{\rho}$ which makes the following diagram commute, where ρ is the given p-coloring of K and i_* is the map induced by inclusion:

$$
\begin{tikzcd}
\pi_1(S^3 - K) \arrow[r, "i_*"] \arrow[d, "\rho"] & \pi_1(B^4 - F') \arrow[dl, dashed, "\bar{\rho}"] \\
D_p &
\end{tikzcd}
$$

The p-dihedral genus above is defined for a knot K with a fixed coloring ρ, hence we denote it $g_p(K,\rho)$ in the topological case. We define the p-dihedral genus of a p-colorable knot K to be the minimum p-dihedral genus of K over all p-colorings ρ of K, and denote this by $g_p(K)$ in the topological case. Note that not every p-colored knot K admits a surface F' as above. In [13], we determine a necessary and sufficient condition for the existence of a connected oriented surface that fits into this diagram.

The *ribbon genus* of K is the minimum genus of a smooth embedded orientable surface F' in B^4 with boundary K, such that F' has only local minima and saddles with respect to the radial height function on B^4. The *smooth (topological) homotopy ribbon genus* of a knot K is the minimum genus of a smooth (locally flat) embedded orientable surface F' in B^4 with boundary K such that $i_* : \pi_1(S^3 - K) \twoheadrightarrow \pi_1(B^4 - F')$, that is, inclusion of the boundary into the surface complement induces a surjection on fundamental groups. Finally, given a p-colorable or p-colored knot, its *ribbon p-dihedral genus* or *smooth (topological) homotopy ribbon p-dihedral genus* are defined in the obvious way. Observe that all notions of dihedral genus refer to surfaces embedded in the four-ball, even though "four" is not among the multitude of qualifiers we inevitably use.

As a straight-forward consequence of the definitions, the following inequalities hold among the *smooth* four-genera of a knot:

$$
\begin{tikzcd}
\text{four-genus} \arrow[r, "\leq"] \arrow[d, "\leq"] & \text{hom. ribbon genus} \arrow[r, "\leq"] \arrow[d, "\leq"] & \text{ribbon genus} \arrow[d, "\leq"] \\
p\text{-dihedral genus} \arrow[r, "\leq"] & p\text{-dihedral hom. ribbon genus} \arrow[r, "\leq"] & p\text{-dihedral ribbon genus}
\end{tikzcd}
$$

Excluding the last column, the inequalities make sense and hold in the topological category too.

2.2. **The Main Theorem.** Denote by $g_4(K)$ the topological 4-genus of a knot K, and by $\mathfrak{g}_p(K,\rho)$ the topological homotopy-ribbon p-dihedral genus of a knot K with coloring ρ. As before, the minimum such genus over all colorings ρ of K is $\mathfrak{g}_p(K)$. Let $\sigma(K)$ be the (Murasugi) signature of K. The invariant Ξ_p discussed in the Introduction can be computed using Equation 3.

Theorem 1. *(A.) The inequality:*

$$
\mathfrak{g}_p(K,\rho) \geq \frac{|\Xi_p(K,\rho)|}{p-1} - \frac{1}{2} \tag{1}
$$

holds whenever $\Xi_p(K,\rho)$ is defined for a knot K with a p-coloring ρ.

(B.) Let K be a p-admissible knot and $F' \subset B^4$ a homotopy ribbon oriented surface for K over which a given p-coloring ρ of K extends. If the associated singular dihedral cover of S^4 branched along $F' \cup_K c(K)$ is a definite manifold, then the inequality (1) is sharp and, in particular, F'

realizes the dihedral genus $\mathfrak{g}_p(K,\rho)$ of K. If, in addition, the equality

$$|\sigma(K)| = \frac{2|\Xi_p(K,\rho)|}{p-1} - 1$$

holds, then the topological four-genus and the topological homotopy ribbon p-dihedral genus of K coincide and equal $\frac{|\sigma(K)|}{2}$, so the generalized topological Slice-Ribbon Conjecture holds for K.

Remark. When a knot K has multiple p-colorings for which the invariant Ξ_p is defined, we can replace $|\Xi_p(K,\rho)|$ in Equation 1 by its minimum value $|\Xi_p(K)|$ among all p-colorings of K. By Theorem 1, we obtain:

$$\mathfrak{g}_p(K) \geq \frac{|\Xi_p(K)|}{p-1} - \frac{1}{2}. \tag{2}$$

Theorem 2. *For every integer $m \geq 0$, there exists a knot K_m and corresponding 3-coloring ρ_m, such that:*

$$g_4(K_m) = \mathfrak{g}_p(K_m) = \frac{|\Xi_3(K_m,\rho_m)|}{2} - \frac{1}{2} = m.$$

That is, the inequality (2) is sharp for these knots and computes their p-dihedral genus as well as their topological four-genus. The generalized Slice-Ribbon Conjecture holds for these knots.

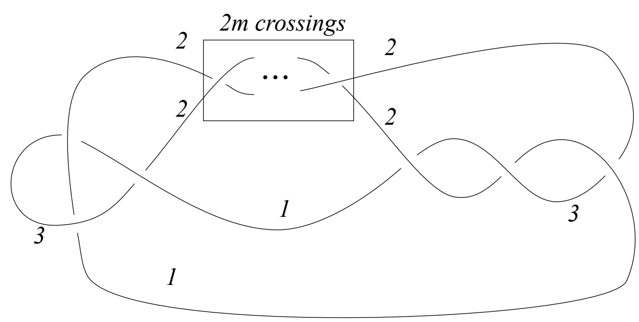

FIGURE 1. The knot K_m, $m \geq 0$, and its 3-coloring. We have $K_0 = 6_1$, $K_1 = 8_{11}$, $K_2 = 10_{21}$ and $K_3 = 12a723$.

Corollary 3. *For any integer $m \geq 0$, there exist infinite families of knots whose 3-dihedral genus and topological four-genus are both equal to m.*

2.3. **Singular dihedral covers of S^4 and the invariant Ξ_p.** Let F', a surface with boundary, be properly embedded in B^4 and assume that the embedding is locally flat. Given a branched cover of manifolds with boundary $f' : X' \to B^4$, one constructs what we call a *singular* branched cover of S^4 by coning off $\partial X'$, ∂B^4 and the map f'. The resulting covering map, $f : X \to S^4$, has total space $X := X' \cup_{\partial X'} c(\partial X')$, where $c(\partial X')$ denotes the cone on $\partial X'$. The branching set is a closed surface $F := F' \cup_K c(K)$ embedded in S^4 with a singularity (the cone point) whose link is K. Remark that the space X constructed in this way is a manifold if and only if $\partial X' \cong S^3$. We will compute $\sigma(X)$, the signature of X, from which we will obtain a lower bound on $g(F')$ when F' is homotopy ribbon. We restrict our attention to the case where $\partial F'$ is connected, i.e. K is a knot. Moreover, the covering spaces we consider arise from homomorphisms of $\pi_1(B^4 - F')$, respectively $\pi_1(S^3 - K)$, onto D_p – see Definition 1. Under these assumptions, the signature $\sigma(X)$ of the manifold X constructed above can be computed by a formula given in [12] and is an invariant of K, together with the associated homomorphism $\rho : \pi_1(S^3 - K) \twoheadrightarrow D_p$. This invariant, properly denoted $\Xi_p(K, \rho)$ but quite often denoted $\Xi_p(K)$ in practice, is the main tool used in this paper. The definition of $\Xi_p(K, \rho)$ and the signature formula for a singular dihedral cover of S^4 are recalled in Equations 3 and 4, respectively.

Definition 1. Let Y be a manifold and $B \subset Y$ a codimension-two submanifold with the property that there exists a surjection $\varphi : \pi_1(Y - B) \twoheadrightarrow D_p$. Denote by $\mathring{X}$ the covering space of $Y - B$ corresponding to the conjugacy class of subgroups $\varphi^{-1}(\mathbb{Z}/2\mathbb{Z})$ in $\pi_1(Y - B)$, where $\mathbb{Z}/2\mathbb{Z} \subset D_p$ is any reflection subgroup. The completion of $\mathring{X}$ to a branched cover $f : X \to Y$ is called the *irregular dihedral* p-fold cover of Y branched along B.

In this paper, Y will be one of S^3, B^4 or S^4. One reason to consider irregular dihedral covers is that there are many infinite families of knots $K \subset S^3$ whose irregular dihedral covers are homeomorphic to S^3. As noted earlier, this guarantees that the construction of a singular branched cover $f : X \to S^4$ with K as a singularity yields a total space X that is again a manifold. We call knots K with this property *strongly p-admissible*. This set-up allows us to study invariants of K using four-dimensional techniques as well as to construct manifolds that are singular branched covers of S^4 starting with appropriately chosen knots. Criteria for admissibility of singularities are outlined in [3] where we also use the invariant $\Xi_p(K)$ to give a homotopy ribbon obstruction for a *strongly* p-admissible slice knot K. A generalization of the ribbon obstruction derived from Ξ_p to all p-admissible knots appears in [8].

We conclude this section by reviewing the formula for computing the invariant Ξ_p given in [12]. Let p be an odd integer and K a p-admissible knot. Let V be a Seifert surface for K and $\beta \subset V'$ any mod p characteristic knot for K (as defined in [5]), corresponding to a given p-coloring of K, ρ. Denote by L_V the symmetrized linking form for V and by σ_{ζ^i} the Tristram-Levine ζ^i-signature, where ζ is a primitive p^{th} root of unity. Finally, let $W(K, \beta)$ be the cobordism constructed in [5] between the p-fold cyclic cover of S^3 branched along β and the p-fold dihedral cover of S^3 branched along K and determined by ρ. By Theorem 1.4 of [12],

$$\Xi_p(K, \rho) = \frac{p^2 - 1}{6} L_V(\beta, \beta) + \sigma(W(K, \beta)) + \sum_{i=1}^{p-1} \sigma_{\zeta^i}(\beta). \tag{3}$$

This formula allows $\Xi_p(K)$ to be evaluated directly from a p-colored diagram of K, without reference to any four-dimensional construction. An explicit algorithm for performing this computation is outlined in [4]. Note also that when a knot K is realized as the only singularity on an embedded surface $F \subset S^4$ and moreover this surface is equipped with a Fox p-colored singular triplane diagram, [3] gives a method for computing $\Xi_p(K)$ from this data, via the signature of the associated cover of S^4. This technique is reviewed and applied in Section 3 below.

Finally, we recall the formula relating $\Xi_p(K, \rho)$ to the signature of a singular dihedral branched cover X of S^4. The branching set F is an embedded surface, locally flat away from one singularity of type K. The induced coloring of F is an extension of ρ.

$$\Xi_p(K) = -\frac{p-1}{4} e(F) - \sigma(X). \tag{4}$$

Here, $e(F)$ denotes the self-intersection number of F. This is a special case of Kjuchukova's signature formula for singular dihedral covers over an arbitrary base [12, Theorem 1.4]. Note that, when F is orientable, Equation 4 reduces to $\Xi_p(K) = -\sigma(X)$. In this paper we focus on covers with orientable branching sets because the signatures of these covers can be understood entirely in terms of the Ξ invariants of their singularities and vice-versa. We note that it is possible to realize *all* connected sums $\#n\mathbb{C}P^2$ as 3-fold dihedral covers of S^4 with one knot singularity on a connected, embedded branching set, if one allows the branching set to be non-orientable [1]. By contrast, we see in Corollary 6 that orientability of the branching set, together with a single singular point, imply that the signature of such a cover is odd.

3. Knots with Equal Topological and Dihedral Genus

We construct a family of 3-fold dihedral covers of S^4 which realize the knots K_m given in Figure 1 as singularities on the branching sets. This construction allows us to compute the values of $\Xi_3(K_m)$ using trisection techniques introduced in [2] and reviewed below. As a corollary, we obtain Theorem 5, which establishes the range of the invariant Ξ_3.

Proposition 4. *Each knot K_m in Figure 1 arises as the only singularity on a 3-fold dihedral branched cover $f_m : \#(2m+1)\overline{\mathbb{CP}}^2 \to S^4$ whose branching set F_m is an embedded oriented surface of genus m. Similarly, each knot $\bar{K}_m$ arises as the only singularity on a 3-fold dihedral branched cover $\bar{f}_m : \#(2m+1)\mathbb{CP}^2 \to S^4$, also with an embedded oriented branching set of genus m.*

Remark. By deleting a small neighborhood of the singularity on the branching set in S^4, one obtains an oriented, 3-colored surface in $F'_m \subset B^4$ with $\partial F'_m = K_m$. In Section 4, we prove that the genus of F'_m is minimal, that is, equal to $g_4(K_m)$. Moreover, by construction, each surface F'_m is ribbon.

Before proving Proposition 4, we informally review trisections of four-manifolds [7], tri-plane diagrams [14], and singular tri-plane diagrams [3].

Given a smooth, oriented, 4-manifold X, a (g, k_1, k_2, k_3)-trisection of X is a decomposition of $X = X_1 \cup X_2 \cup X_3$ into three 4-handlebodies with boundary, such that

- $X_i \cong \natural k_i(B^3 \times S^1)$
- $X_1 \cap X_2 \cap X_3 \cong \Sigma_g$ is a closed, oriented surface of genus g
- $Y_{ij} = \partial(X_i \cup X_j) \cong \#k_i(S^2 \times S^1)$
- $\Sigma_g \subset Y_{ij}$ is a Heegaard surface for Y_{ij}.

Every embedded surface $\Sigma \subset S^4$ can be described combinatorially by a $(b; c_1, c_2, c_3)$-*tri-plane diagram* [14]. This is a set of three b-strand trivial tangles (A, B, C), such that each boundary union of tangles $A \cup \overline{B}$, $B \cup \overline{C}$, and $C \cup \overline{A}$ is a c_i-component unlink, for $i = 1, 2, 3$ respectively. Here $\overline{T}$ denotes the mirror image of T. To obtain Σ from (A, B, C), one views each of $A \cup \overline{B}$, $B \cup \overline{C}$, and $C \cup \overline{A}$ as unlinks in bridge position in the spokes Y_{12}, Y_{23}, and Y_{31} of the standard genus-0 trisection of S^4, glues D_i disks to the components of each of these unlinks, and pushes these disks into the X_i to obtain an embedded surface.

The authors introduce *singular tri-plane diagrams* in [3]. A $(b; 1, c_2, c_3)$ singular tri-plane diagram is a triple of b-strand trivial tangles (A, B, C). As above $B \cup \overline{C}$ and $C \cup \overline{A}$ are c_2- and c_3-component unlinks. $A \cup \overline{B}$ is a knot K. To build a surface with one singularity of type $K = \overline{K}$, one again views each of $A \cup \overline{B}$, $B \cup \overline{C}$, and $C \cup \overline{A}$ as unlinks in bridge position in the three spokes Y_{12}, Y_{23}, and Y_{31} of the standard genus-0 trisection of S^4 and glues D_2 and D_3 disks to the components of each

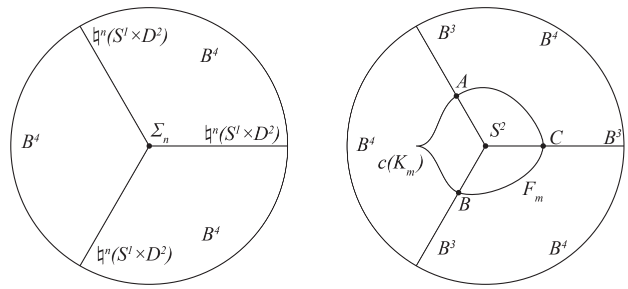

FIGURE 2. A $(n; 0, 0, 0)$-trisection of $\#n\overline{\mathbb{C}P}^2$, obtained as branched cover of S^4 over a trisected surface F_m with one singularity K_m.

of the two unlinks. Rather than glue disks to $A \cup \overline{B}$, one attaches the cone on K. Note that by interchanging the order of the tangles A and B, one obtains a surface with singularity $\overline{K}$.

Proof of Proposition 4. We will construct the surface F_m and will give its Fox colorings using a colored (singular) tri-plane diagram. From this information, we will produce a trisection of the dihedral cover of S^4 determined by this coloring. We will identify this cover as $\#n\overline{\mathbb{CP}}^2$, where $n = 2m + 1$.

The colored tri-plane diagram (A_n, B_n, C_n) for F_m, where $m = (n-1)/2$, is shown in Figure 3. The union $A_n \cup \overline{B}_n$ is the knot $\overline{K}_m$, while $B_n \cup \overline{C}_n$ and $C_n \cup \overline{A}_n$ are each 2-component unlinks; see Figure 4 for a verification of this fact when $n = 3$. A tri-plane diagram with b bridges and c_i components in each link diagram has Euler characteristic $c_1 + c_2 + c_3 - b$; hence, the surface F_m with singularity K_m has Euler characteristic $3 - n$ and genus $m = (n-1)/2$ since F_m is connected and orientable.

The fact that F_m is orientable requires a careful check. Consider the cell structure on F_m corresponding to its tri-plane structure. To show that F_m is orientable, we show that it is possible to coherently orient the faces of this cell structure so that each edge (a bridge in one of the three tangles A_n, B_n, or C_n) inherits two different orientations from the two faces adjacent to it. This is shown in Figure 4 in the case $m = 1$ (or $n = 3$).

An Euler characteristic computation shows that the 3-fold dihedral branched cover of the bridge sphere S^2, branched along the $2(n+2)$ endpoints of the bridges, is a surface Σ_n of genus n. We now show this 3-colored tri-plane diagram (A_n, B_n, C_n) gives rise to a genus n trisection of $\#n\overline{\mathbb{C}P}^2$ with central surface Σ_n following a method explained in [3]. The branching set F_m is orientable and has one singularity of type K_m, so it will follow from Equation 4 that $\Xi_3(K_m) = -\sigma(\#_n\overline{\mathbb{C}P}^2) = n$.

If a properly embedded b-strand tangle $(T, \partial T) \subset (B^3, S^2)$ with arcs $t_1, t_2, \dots t_b$ is trivial, then by definition there exists a collection of disjoint arcs $d_1, d_2, \dots, d_b$ in S^2 such that the boundary unions $t_i \cup d_i$ bound a collection of disjoint disks in B_3. We refer to the d_i as *disk bottoms.* The existence of such a collection of disks is equivalent to the arcs of T being simultaneously isotopic to a collection of disjoint arcs (the d_i) in S^2.

To determine the trisection diagram, we must first find the disk bottoms for the three tangles A_n, B_n and C_n, then lift them from the bridge sphere S^2 to its irregular dihedral cover Σ_n. The curves in the trisection diagram are formed by certain lifts of these disk bottoms; we identify these lifts later.

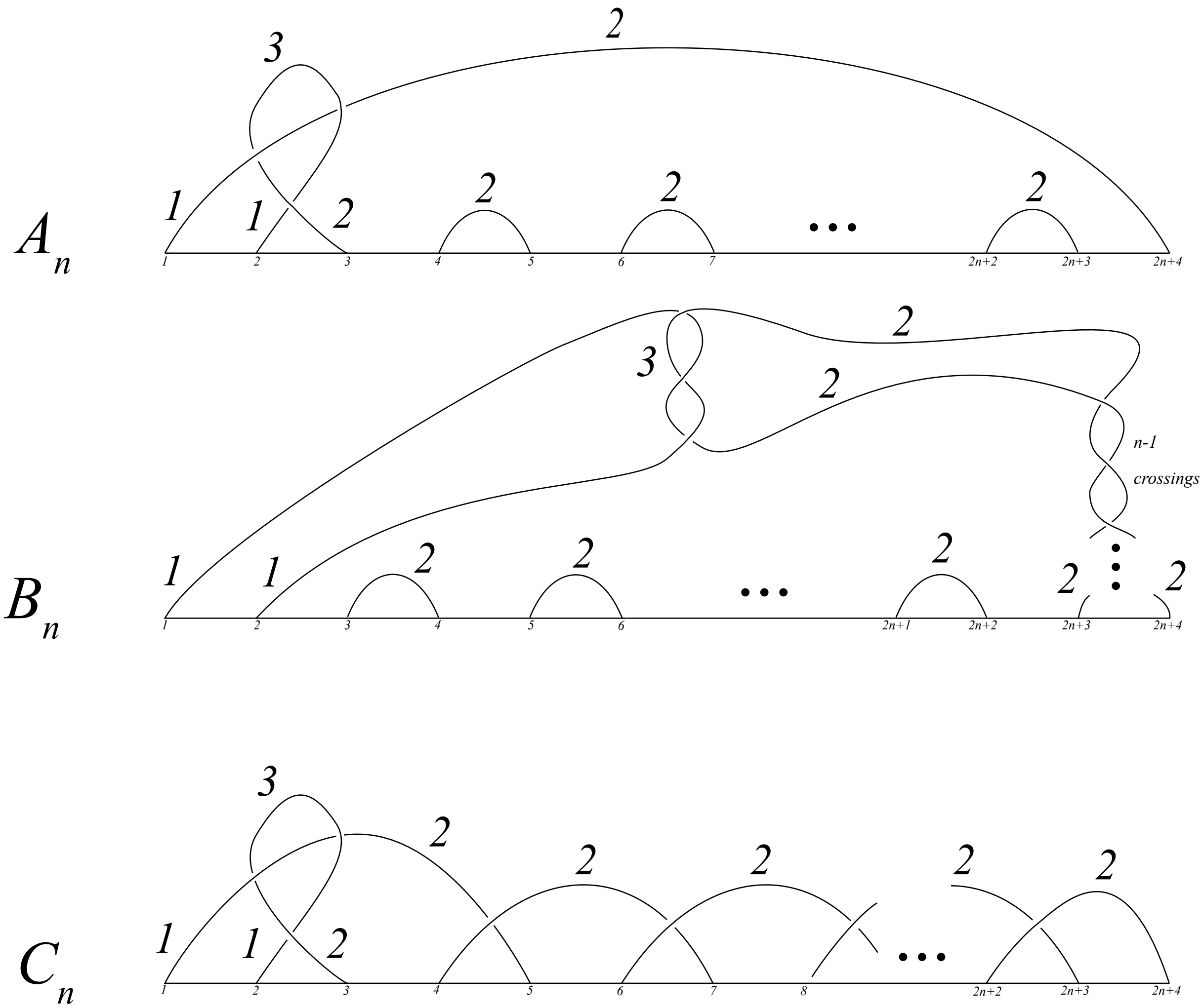

FIGURE 3. A colored tri-plane diagram corresponding to a branched covering $\#n\overline{\mathbb{C}P}^2 \to S^4$, in the case where n is odd. There is one singularity K_m on the branching set, where $m = (n-1)/2$. By reversing the roles of A_n and B_n, one obtains a branched covering $\#n\mathbb{C}P^2 \to S^4$ with singularity $\overline{K}_m$.

The disk bottoms for each tangle A_n, B_n, and C_n are depicted in Figure 5, in the case $n = 3$. In Figure 6, we draw just three of disk bottoms for each of A_n (blue), B_n (red), and C_n (green) on the same copy of S^2.

Next we lift the disk bottoms from the bridge sphere to Σ_n. We use a construction of Σ_n due to Hilden [10]. This construction assumes that the meridians of two branch points, denoted a and b in Figure 6, map to the transposition (23) (equivalently, are colored '1'), and the remaining $2n$ branch points map to the transposition (13) (equivalently, are colored '2'). One first constructs the 6-fold *regular* dihedral cover of S^2 branched along $2(n+2)$ points determined by this coloring. Figure 7 illustrates the case $n = 3$. The resulting surface has genus $3n + 1$. The 3-fold irregular dihedral cover Σ_n is obtained from this regular one by an involution. This involution can be visualized as

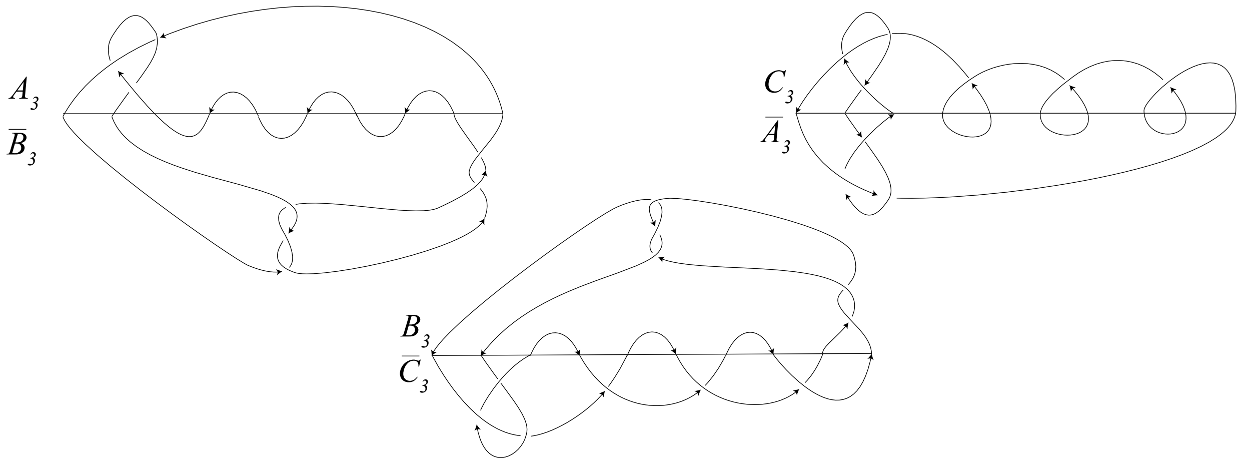

FIGURE 4. The links $A_3 \cup \overline{B}_3$, $B_3 \cup \overline{C}_3$, and $C_3 \cup \overline{A}_3$. Note that $A_3 \cup \overline{B}_3$ is the knot $\overline{K}_1$.

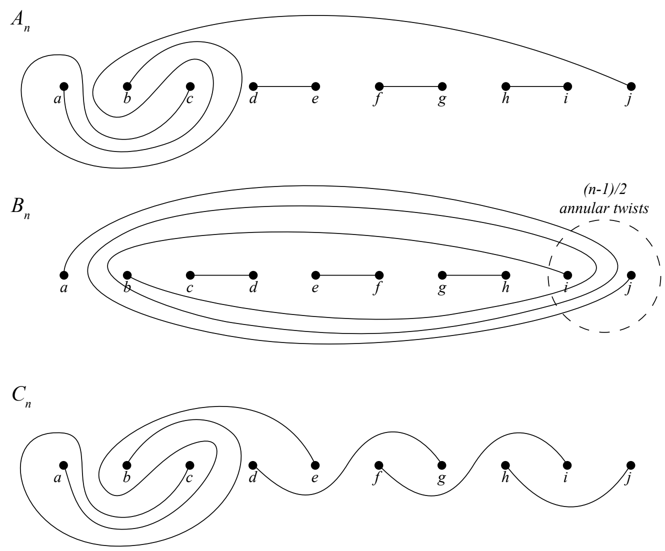

FIGURE 5. Disk bottoms for the tri-plane diagram (A_n, B_n, C_n) when $n = 3$.

the 180° rotation about the vertical axis, as shown in Figure 7. The resulting surface is shown in Figure 8.

Each disk bottom has three lifts to Σ_n, two of which fit together to form a closed curve. Not all of these closed curves are guaranteed to be essential curves on Σ_n; reasons for this are discussed in [3]. However, we may choose $n-2$ disk bottoms for each tangle (A_n, B_n, C_n) whose lifts are essential. These lifts are shown in Figure 8, again in the case $n = 3$.

The resulting curves form a trisection diagram for $\#n\overline{\mathbb{C}P}^2$. Moreover, the standard trisection of S^4, branched along F_m, lifts to a $(n; 0, 0, 0)$-trisection of $\#n\overline{\mathbb{C}P}^2$. This can be found by analyzing the lifts of the three pieces of the trisection of (S^4, F_m); for details see Theorem 8 of [3]. □

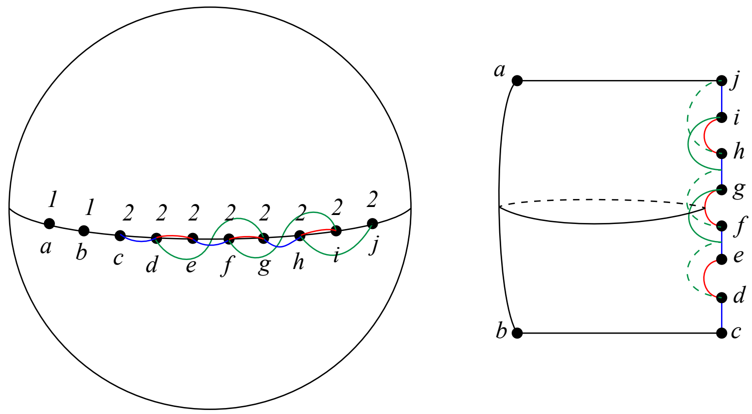

FIGURE 6. Disk bottoms for the tri-plane diagram (A_n, B_n, C_n) when $n = 3$, drawn on the bridge sphere.

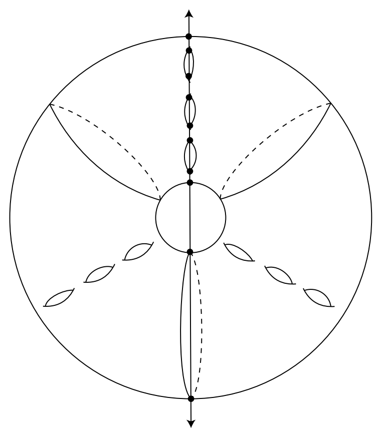

FIGURE 7. A 3-fold regular dihedral cover R of S^2 branched along 10 points; the irregular cover is the quotient R by $180°$ rotation about the vertical axis.

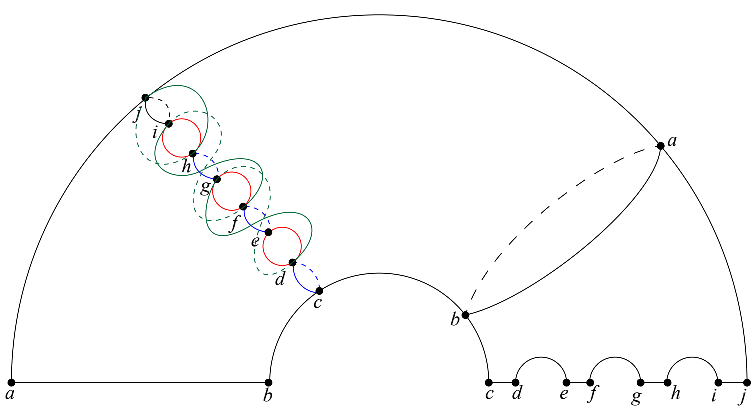

FIGURE 8. Lifts of disk bottoms to the 3-fold dihedral cover of S^2, for the tri-plane diagram (A_n, B_n, C_n), when $n = 3$.

We use the above construction to establish the range of the invariant Ξ_3.

Theorem 5. *There exists an admissible singularity K and a 3-coloring ρ of K such that $\Xi_3(K,\rho) = n$ if and only if $n \in 2\mathbb{Z}+1$.*

Remark. In the proof of Theorem 5, we establish that $\Xi_p(K,\rho)$ is odd for *any* p-coloring ρ of an admissible singularity K, without the assumption that $p = 3$. Realizability of all odd integers by Ξ_p for $p \neq 3$ is open.

Corollary 6. *Let $f : X \to S^4$ be a p-fold dihedral cover whose branching set F is an oriented surface embedded in S^4 locally flatly away from one cone singularity of type K. Then, $\sigma(X)$ is odd.*

Proof of Theorem 5. We have given a construction realizing each of the knots K_m as the only singularity on a branched cover $\#(2m+1)\overline{\mathbb{CP}}^2 \to S^4$ whose branching set is oriented. By Equation 4, it follows that $\Xi_3(K_m) = -\sigma(\#(2m+1)\overline{\mathbb{CP}}^2) = 2m+1$, where $m \geq 0$. Note also that $\Xi_p(\overline{K}_m) = -\Xi_p(K_m)$ as proved in [3], where $\overline{K}$ denotes the mirror image of K and, of course, K is p-admissible if and only if $\overline{K}$ is. This proves that all odd integers are contained in the range of the invariant Ξ_3 on 3-admissible knots.

Conversely, we will verify that for any p-coloring ρ of any p-admissible singularity K, the integer $\Xi_p(K,\rho)$ is odd. We use Equation 3. Since p is odd, $p^2 \equiv 1 \mod 4$, so $\dfrac{p^2-1}{6}$ is even. As shown in [12], $\sigma(W(K,\beta))$ is the signature of an odd-dimensional nonsingular matrix, and hence is odd. Each of the σ_{ζ^i} are even. It follows that $\Xi_p(K)$ is odd. □

Remark. The knot K_m is has bridge number 2, showing that two-bridge knots realize the full range of Ξ_p when $p = 3$. This answers a question posed in [11]. It is not known whether the full range of Ξ_p is realized by two-bridge knots when $p \neq 3$. It would be of interest to establish that it is "sufficient" to consider two-bridge knots when constructing singular dihedral covers of four-manifolds since p-admissibility is particularly easy to detect for two-bridge singularities [11].

Proof of Corollary 6. By the proof of Theorem 5, $\Xi_p(K)$ is odd. Since F is oriented, by Equation 4, $\sigma(X) = -\Xi_p(K)$. □

4. Proof of the Main Theorem

Proof of Theorem 1. (A.) Let K be p-admissible, and let F' be topologically locally flat homotopy ribbon surface for K of genus $\mathfrak{g}_p(K,\rho)$. Let $\bar{\rho} : \pi_1(B^4 - F') \to D_p$ be the homomorphism extending the coloring $\rho : \pi_1(S^3 - K) \to D_p$. Let U be the unbranched irregular dihedral cover of $S^3 - K$ corresponding to ρ, and $\hat{U}$ the induced branched cover. Let F be the singular surface which is the boundary union of F' and the cone on K. Let Z be the unbranched irregular dihedral cover of $B^4 - F'$ corresponding to $\bar{\rho}$, and $\hat{Z}$ the induced branched cover. Let Y be the dihedral cover of S^4 with branching set B. We know by [12] that:

$$\chi(Y) = 2p - \frac{p-1}{2}\chi(B) - \frac{p-1}{2}.$$

We will show that Y is simply-connected. Consider the commutative diagram below. All maps in the diagram are either induced by inclusions or by covering maps. Clearly p_* and q_* are injective, as they are induced by covering maps, and ι_{U*} and ι_{Z*} are surjective, as they are induced by inclusions of unbranched covering spaces into their branched counterparts. The homomorphisms ρ and $\bar{\rho}$ are surjective by definition. Finally, since F' is a homotopy-ribbon surface for K, i_* is surjective.

We first show that j_* is surjective as well. Consider an element $\gamma \in \pi_1(Z)$. Since i_* is surjective, there exists an element $\delta \in \pi_1(S^3 - K)$ such that $i_*(\delta) = q_*(\gamma)$. We have that $\bar{\rho} \circ q_*(\gamma) \in \mathbb{Z}/2\mathbb{Z} \subset D_p$, the reflection subgroup which determines the cover Z of $B^4 - F'$. By commutativity of the lower triangle, $\rho(\delta) = \bar{\rho} \circ q_*(\gamma) \in \mathbb{Z}/2$, so $\delta \in \text{Im } p_*$. Take $\tilde{\delta} \in \pi_1(U)$ such that $p_*(\tilde{\delta}) = \delta$. Consider $q_* \circ j_*(\tilde{\delta})$, which by commutativity is equal to $i_* \circ p_*(\tilde{\delta})$. Now $q_* \circ j_*(\tilde{\delta}) = i_*(\delta) = q_*(\gamma)$. By injectivity of q_*, we have $j_*(\tilde{\delta}) = \gamma$, so j_* is indeed surjective.

Next we observe that, since j_* and ι_{Z*} are both surjective, ι_* is surjective. But $\hat{U} \cong S^3$ so $\pi_1(\hat{U})$ is trivial. It follows that $\pi_1(\hat{Z})$ is trivial, so Y is simply-connected.

$$\begin{tikzcd}
\pi_1(\hat{U}) \arrow[r, "\iota_*"] & \pi_1(\hat{Z}) \\
\pi_1(U) \arrow[u, "\iota_{U*}"] \arrow[r, "j_*"] \arrow[d, "p_*"] & \pi_1(Z) \arrow[u, "\iota_{Z*}"'] \arrow[d, "q_*"] \\
\pi_1(S^3 - K) \arrow[r, "i_*"] \arrow[d, "\rho"] & \pi_1(B^4 - F') \arrow[dl, "\bar{\rho}"] \\
D_p &
\end{tikzcd}$$

Since Y is simply-connected, Poincaré Duality implies that $\chi(Y) = 2 + \text{rk } H_2(Y)$. Furthermore since F is orientable, $|\sigma(Y)| = |\Xi_p(K)|$, and $|\sigma(Y)| \leq \chi(Y) - 2$. Combining with the formula for $\chi(Y)$ above gives the Inequality (1).

(B.) If Y is definite, then rk $H_2(Y) = |\sigma(Y)|$, so $|\Xi_p(K)| = |\sigma(Y)| = \chi(Y) - 2$. Again, substituting in the above formula for $\chi(Y)$ gives the desired equality.

Now consider the signature $\sigma(K)$ of the knot singularity and assume $|\sigma(K)| = 2\mathfrak{g}_p(K)$. Murasugi's signature bound [15, Theorem 9.1] states that $g_4(K) \geq |\sigma(K)|/2$. Thus, we have $g_4(K) \geq \mathfrak{g}_p(K)$. But $g_4(K) \leq \mathfrak{g}_p(K)$ in general, so $g_4(K) = \mathfrak{g}_p(K)$. □

Proof of Corollary 2. By *(B.)* of Theorem 1, it suffices to show that

(1) Each K_m is the boundary of a homotopy-ribbon surface F'_m such that $\mathfrak{g}_3(K) = g(F'_m)$, and

(2) The signature $\sigma(K_m)$ satisfies the equality
$$|\sigma(K)| = \frac{2|\Xi_p(K, \rho)|}{p-1} - 1$$
for $p = 3$.

We first address (1). Surfaces F'_m realizing the lower bound on dihedral homotopy-ribbon genus for the knots K_m are constructed in the proof of Proposition 4: we have shown $g(F'_m) = m$ and $|\Xi_3(K_m)| = 2m + 1$, so

$$\frac{|\Xi_p(K_m)|}{p-1} - \frac{1}{2} = m.$$

We note that, since the knots K_m are two-bridge, each of them has a unique 3-coloring (up to permuting the colors), so there is no distinction between $\mathfrak{g}_p(K_m, \rho)$ and $\mathfrak{g}_p(K_m)$. By construction, the surface $F_m' \subset B^4$ obtained by deleting a small neighborhood of the singularity K_m is ribbon since $A_m \cup \overline{B}_m$ only bounds the cone on K_m, while the unlinks $B_m \cup \overline{C}_m$ and $C_m \cup \overline{A}_m$ bound standard unknotted disks in B^4.

We now address (2). We will compute the signature $\sigma(K_m)$, and show it is equal to $2m = \frac{2|\Xi_p(K)|}{p-1} - 1$.

The signature of K can be computed using the Goeritz matrix $G(K)$, the matrix of a quadratic form associated to a knot diagram via a checkerboard coloring, and hence a (not necessarily orientable) spanning surface; this technique was introduced by Gordon and Litherland [9]. The advantage of this technique is that the dimension of the Goeritz matrix associated to a projection of a knot may be much smaller than the dimension of the corresponding Seifert matrix; indeed, the dimension of $G(K_m)$ is 4 for all m.

Gordon and Litherland proved that the signature of a knot is equal the signature of the Goeritz matrix of a diagram of the knot plus a certain correction term: $\sigma(K) = \sigma(G(K)) - \mu$. We start by computing the Goeritz matrix $G(K_m)$ and its signature.

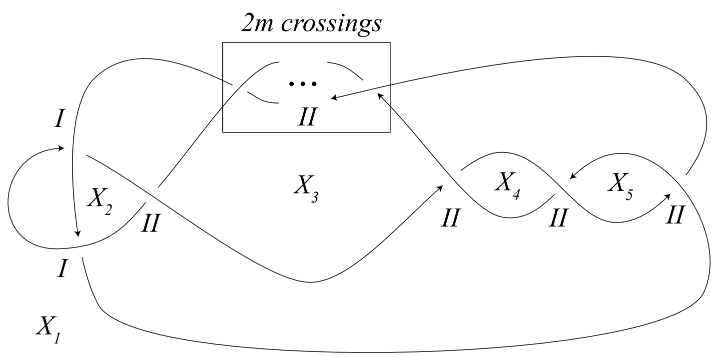

FIGURE 9. The "white" regions of a checkerboard coloring of K_m, labelled $X_1, X_2, \ldots X_5$.

One first computes the *unreduced* Goeritz matrix. To do this, one chooses a checkerboard coloring of the knot diagram, and labels the "white" regions $X_1, \ldots, X_k$. Such a labelling for the K_m is shown in Figure 9. The entries g_{ij} of the unreduced Goeritz matrix are computed as follows:

$$g_{ij} = \begin{cases} -\sum \eta(c) & i \neq j \text{ and } c \text{ a double point incident to regions } X_i \text{ and } X_j \\ -\sum_{s \in \{1,\ldots,k\}\setminus\{i\}} g_{is} & i = j \end{cases}.$$

The signs $\eta(c)$ are computed as in Figure 10; shaded area corresponds to "black" regions of the checkerboard coloring.

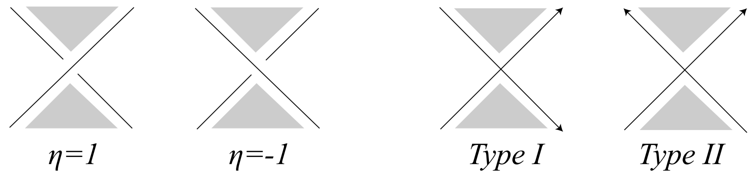

FIGURE 10. Incidence numbers η and Type I and II crossings.

The unreduced Goeritz matrix of K_m is

$$G'(K_m) = \begin{pmatrix} -2m-3 & -2 & -2m & 0 & -1 \\ -2 & -3 & -1 & 0 & 0 \\ -2m & -1 & -2m-2 & -1 & 0 \\ 0 & 0 & -1 & -2 & -1 \\ -1 & 0 & 0 & -1 & -2 \end{pmatrix}.$$

The Goeritz matrix $G(K_m)$ is obtained by deleting the first row and column of $G'(K_m)$. The characteristic polynomial of this matrix is

$$p_{G(K_m)}(\lambda) = (\lambda+3)(\lambda(\lambda+3)^2 + 2(\lambda+1)(\lambda+3)m + 3).$$

Hence $\lambda = -3$ is an eigenvalue. In addition, since $m \geq 0$, it is straightforward to verify that any root of the cubic factor must be negative (if λ is nonnegative, the cubic, as written above, is a sum of three nonnegative terms). Hence, $\sigma(G(K_m)) = -4$.

The correction term $\mu(K)$ in Gordon and Litherland's formula for $\sigma(K)$ is computed as follows. Each crossing c of K can be classified as type I or type II, as shown in Figure 10. Let $\mu(K) = \sum_c \eta(c)$ where the sum is taken over all type II crossings.

The knot K_m has $4+2m$ type II crossings, each of negative sign; see Figure 9. Hence $\sigma(K_m) = -4 + (4+2m) = 2m$. $\square$

Proof of Corollary 3. Let γ be any ribbon knot and let $D \subset B^4$ be a ribbon disk with $\partial D = \gamma$. The knot $K_m \# \gamma$ has 3-dihedral genus and topological four-genus equal to m. It is clear that the smooth and topological four-genera of $K_m \# \gamma$ are both equal to m since the knot is smoothly concordant to K_m. Next, remark that the given 3-coloring ρ_m of K_m induces a 3-coloring ρ_γ of $K_m \# \gamma$ which restricts trivially to γ. Moreover, since ρ_m extends over F'_m, ρ_γ extends over the ribbon surface $F' \# D$. Therefore, the ribbon 3-dihedral genus of $K_m \# \gamma$ is at most m. Since g_4 is a lower bound for the topological 3-dihedral genus, which in turn is a lower bound for the ribbon 3-dihedral genus, it follows that these genera are equal, as claimed. $\square$

Patricia Cahn
Smith College
pcahn@smith.edu

Alexandra Kjuchukova
Max Planck Institute for Mathematics–Bonn
University of Wisconsin–Madison
sashka@mpim-bonn.mpg.de

References

1. Ryan Blair, Patricia Cahn, and Alexandra Kjuchukova, *Simply-connected manifolds as 3-fold covers of S^4*, In preparation.
2. Patricia Cahn and Alexandra Kjuchukova, *Linking numbers in three-manifolds*, arXiv preprint arXiv:1611.10330 (2016).
3. ———, *Singular branched covers of four-manifolds*, arXiv preprint arXiv:1710.11562 (2017).
4. ———, *Computing ribbon obstructions for colored knots*, arXiv preprint arXiv: 1812.09553 (2018).
5. Sylvain Cappell and Julius Shaneson, *Linking numbers in branched covers*, Contemporary Mathematics **35** (1984), 165–179.
6. Ralph Fox, *Some problems in knot theory*, Topology of 3-manifolds and related topics **3** (1962), 168–176.
7. David Gay and Robion Kirby, *Trisecting 4-manifolds*, Geometry and Topology **20** (2016), no. 6, 3097–3132.
8. Christian Geske, Alexandra Kjuchukova, and Julius L. Shaneson, *Signatures of branched covers in dimension four and a ribbon obstruction for Fox-colorable knots*, In preparation.
9. Cameron Gordon and Richard Litherland, *On the signature of a link*, Inventiones mathematicae **47** (1978), 53–70.
10. Hugh Hilden, *Every closed orientable 3-manifold is a 3-fold branched covering space of S^3*, Bulletin of the American Mathematical Society **80** (1974), no. 6, 1243–1244.
11. Alexandra Kjuchukova, *On the classification of irregular dihedral branched covers of four-manifolds*, PhD Thesis, University of Pennsylvania, 2015.
12. ———, *Dihedral branched covers of four-manifolds*, Advances in Mathematics **332** (2018), 1–33.
13. Alexandra Kjuchukova and Kent Orr, *Admissible singularities on dihedral covers between four-manifolds*, In preparation.
14. Jeffrey Meier and Alexander Zupan, *Bridge trisections of knotted surfaces in S^4*, Trans. Amer. Math. Soc. **369** (2017), 7343–7386.
15. Kunio Murasugi, *On a certain numerical invariant of link types*, Transactions of the American Mathematical Society **117** (1965), 387–422.