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A 2-CHAIN CAN INTERLOCK WITH A k-CHAIN

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ABSTRACT. One of the open problems posed in [3] is: what is the minimal number k such that an open, flexible k-chain can interlock with a flexible 2-chain? In this paper, we establish the assumption behind this problem, that there is indeed some k that achieves interlocking. We prove that a flexible 2-chain can interlock with a flexible, open 16-chain.

1. INTRODUCTION

A *polygonal chain* (or just *chain*) is a linkage of rigid bars (line segments, edges) connected at their endpoints (joints, vertices), which forms a simple path (an *open chain*) or a simple cycle (a *closed chain*). A *folding* of a chain is any reconfiguration obtained by moving the vertices so that the lengths of edges are preserved and the edges do not intersect or pass through one another. The vertices act as universal joints, so these are *flexible chains*. If a collection of chains cannot be separated by foldings, the chains are said to be *interlocked*.

Interlocking of polygonal chains was studied in [4, 3], establishing a number of results regarding which collection of chains can and cannot interlock. One of the open problems posed in [3] asked for the minimal k such that a flexible open k-chain can interlock with a flexible 2-chain. An unmentioned assumption behind this open problem is that there is some k that achieves interlocking. It is this question we address here, showing that $k = 16$ suffices.

It was conjectured in [3] that the minimal k satisfies $6 \le k \le 11$. This conjecture was based on a construction of an 11-chain that likely does interlock with a 2-chain. We employ some ideas from this construction in the example described here, but for a 16-chain. Our main contribution is a proof that $k = 16$ suffices. It appears that using more bars makes it easier to obtain a formal proof of interlockedness.

Results from [3] include:

(1) Two open 3-chains cannot interlock.
(2) No collection of 2-chains can interlock.
(3) A flexible open 3-chain can interlock with a flexible open 4-chain.

This third result is crucial to the construction we present, which establishes our main theorem, that a 2-chain can interlock a 16-chain (Theorem 1 below.)

2. Idea of Proof

We first sketch the main idea of the proof. If we could build a rigid trapezoid with small rings at its four vertices (T_1, T_2, T_3, T_4), this could interlock with a 2-chain, as illustrated in Figure 1(a). For then pulling vertex v of the 2-chain away from the trapezoid would necessarily diminish the half apex angle α, and pushing v down toward the trapezoid would increase α. But the only slack provided for α is that determined by the diameter of the rings. We make as our subgoal, then, building such a trapezoid.
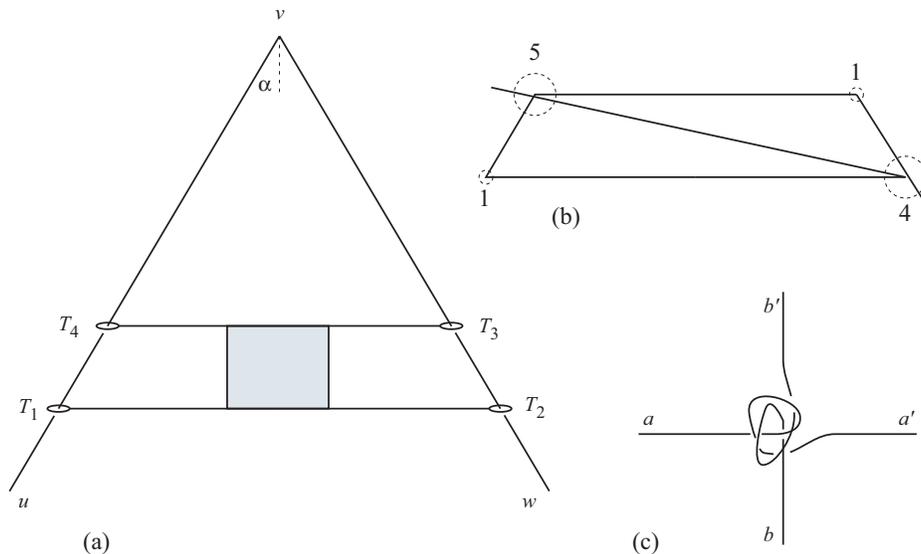

FIGURE 1. (a) A rigid trapezoid with rings would interlock with a 2-chain; (b) An open chain that simulates a rigid trapezoid; (b) Fixing a crossing of aa' with bb'.

We can construct a trapezoid with four links, and rigidify it with two crossing diagonal links. In fact, only one diagonal is necessary to rigidify a trapezoid in the plane, but clearly a single diagonal leaves the freedom to fold along that diagonal in 3D. This freedom will be removed by the interlocked 2-chain, however, so a single diagonal suffices. To create this rigidified trapezoid with a single open chain, we need to employ 5 links, as shown in Figure 1(b). But this will only be rigid if the links that meet at the two vertices incident to the diagonal are truly "pinned" there. In general we want to take one subchain aa' and pin its crossing with another subchain bb' to some small region of space. See Figure 1(c) for the idea.

This pinning can be achieved by the "3/4-tangle" interlocking from [3], result (3) above. So the idea is replace the two critical crossings with a small copy of this configuration. This can be accomplished with 7 links per 3/4-tangle, but sharing with the incident incoming and outgoing trapezoid links potentially reduces the number of

links needed per tangle. We have achieved 5 links at one tangle and 4 at the other. The other two vertices of the trapezoid need to simulate the rings in Figure 1(a), and this can be accomplished with one extra link per vertex. Together with the 5 links for the main trapezoid skeleton, we employ a total of $5 + (5 + 4 + 1 + 1) = 16$ links.

3. A 2-CHAIN CAN INTERLOCK AN OPEN 16-CHAIN

3.1. **Open flexible 3- and 4-chains can interlock.** It was proved that open flexible 3- and 4-chains can interlock in [3]. The construction, which we call a *3/4-tangle*, is repeated in Figure 2.
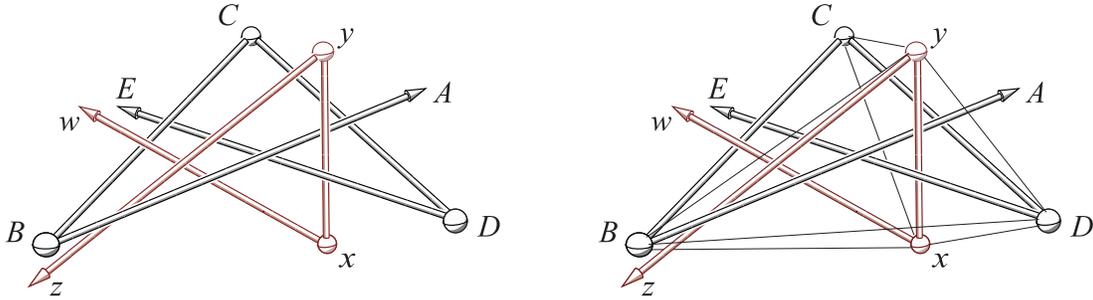

FIGURE 2. Fig. 6 from [3].

It was proved in Theorem 11 of [3] that the convex hull $CH(B, C, D, x, y)$ of the joints B, C, D, x, and y does not change.

We first establish bounds on how far the vertices of the construction can move. Let $BC = CD = xy = 1$ unit, and end bars $AB = DE = xw = yz = 3$ units.

Lemma 1. *Let P be the midpoint of xy. Then in any folding of the interlocked 3- and 4- chain: (1) The distance between P and the endpoints w, z of the 3-chain can be no more than 3.5 units, (2) The distance between P and joints B, C, D, x, and y can be no more than 2.5 units, and (3) The distance between P and the endpoints A, E of the 4-chain can be no more than 5.5 units.*

Proof. (1) Since P is the midpoint of bar xy, x and y are exactly 0.5 units away from P. The joints w, x and P form a triangle, by the triangle inequality $Pw < Px + xw = 0.5 + 3 = 3.5$ units; similarly, $Pz < 3.5$.

(2) We now prove that the distance between P and the joints B, C, D, x, and y can be no more than 2.5 units. In the convex hull $CH(B, C, D, x, y)$, bar xy pierces $\triangle BCD$, where B and D can be imagined to be connected by a rubber band, then $BD < BC + CD = 2$. We observe that: (i) any two points inside $\triangle BCD$ or on the boundary BC, CD, BD are less than 2 units apart, and (ii) the distance between the midpoint P and the plane determined by B, C, D must be less than 0.5 units. From the fact that bar xy pierces $\triangle BCD$, the distance between P and any point on or inside $\triangle BCD$ is less than $2 + 0.5 = 2.5$ units. Since P is the midpoint of bar xy, x

and y are exactly 0.5 units away from P. Therefore, P and the joints B, C, D, x, and y can be no more than 2.5 units as claimed.

(3) Finally, by the triangle inequality $PA < PB + AB < 2.5 + 3 = 5.5$ units; similarly, $PE < 5.5$. $\qquad\square$

For $\epsilon > 0$, choosing $BC = CD = xy = \frac{1}{6}\epsilon$, and end bars $Ab = DE = xw = yz = \frac{1}{2}\epsilon$ yields the following:

Corollary 1. *In the above interlocked 3- and 4-chains, let P be the midpoint of xy, then all joints B, C, D, x, y and endpoints A, E, w, z stay inside the ϵ-ball centered at P.*

3.2. **A 2-chain can interlock an open 16-chain.** Take two 3/4-tangles, where all joints and end points of the pair stay within an ϵ-ball centered at the midpoint of the middle link of the 3-chain. Position the tangles as two of the "vertices" of a trapezoid with the links arranged as shown in Figure 3. This design follows Figure 1(b) in spirit, but varies the connections at the diagonal endpoints to increase link sharing. The lower right vertex achieves maximum sharing, in that all three incident trapezoid edges are shared with links of the 3/4-tangle. The upper left vertex shares two incident links. We extend the first and last links of the trapezoid chain to be very long so that the end vertices of the chain are well exterior to any of the ϵ-balls.

3.2.1. *2-chain Through Trapezoid Jag Corners.* Call the simple structure at the other two corners *jag loops*. These corners also can be assured to remain in an ϵ-ball simply by making the extra link length ϵ. Thus we have that all corners of the trapezoid stay within ϵ-balls.

We first argue that the jag loop "grips" the 2-chain link through it, under the assumption of near rigidity of the trapezoid. Let (u, v, w) be the 2-link chain, and let (a, b, c, d) be the vertices constituting a 1-link jag at a corner of the trapezoid. The short link of the jag is bc. The near-rigidity of the trapezoid permits us to take ab to be roughly horizontal (the base of the trapezoid) and cd to be roughly at angle θ with respect to the base (the angle at a base corner of the trapezoid). The link uv is nearly parallel to de, and is woven through the jag as illustrated in Figure 4. The words "roughly" and "nearly" here are intended as shorthand for "approaches, as $\epsilon \to 0$."

Lemma 2. *The plane containing $\triangle abc$ continues to separate v above from u below (where "above" is determined by the counterclockwise ordering of a, b, c) under all nonintersecting foldings of the chains.*

Proof. We argue that uv continues to properly pierce $\triangle abc$ under all foldings, from which it follows that the initial separating property is maintained. The overall structure of the trapezoid prevents uv from moving directly through $\triangle abc$: neither v nor u can get close to the triangle. So the only way the piercing could end is if uv passes through a side of $\triangle abc$. Two of these sides—ab and bc—are links, and avoiding intersection prevents passage through those. Thus uv would have to pass through ac,
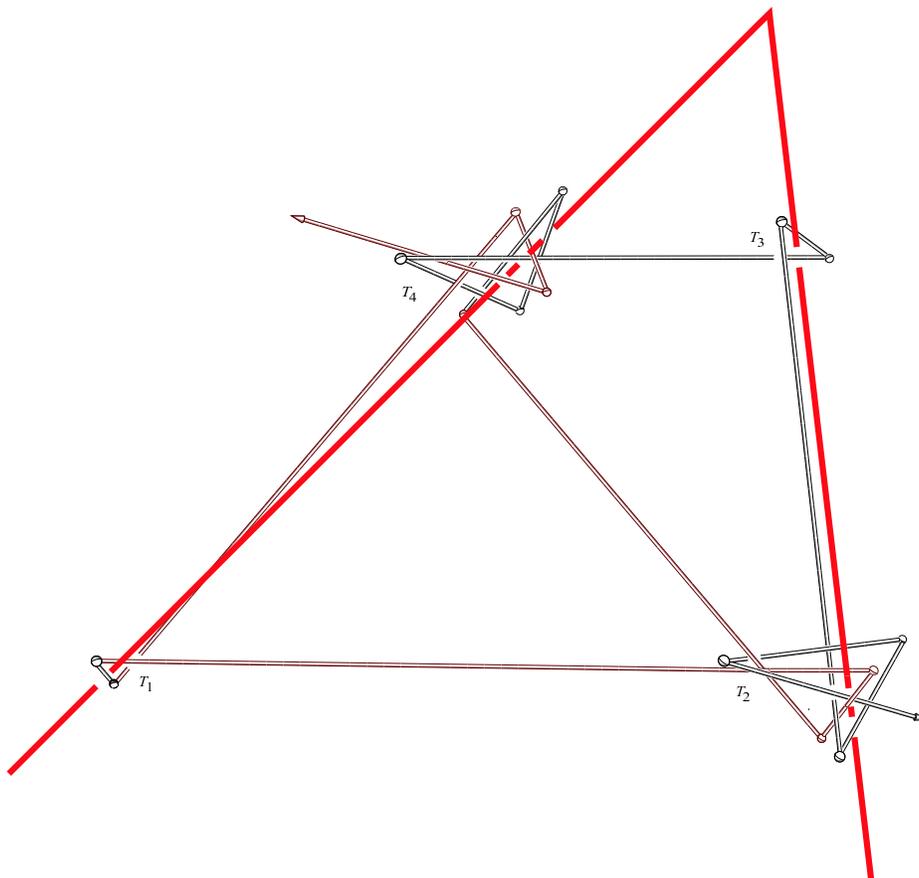
FIGURE 3. An open 16-chain forming a nearly rigid trapezoid.

which is not a link. However, to do this, we now argue it would have to pass through
the link cd.

The gap between ab and cd is at most $|bc| = \epsilon$. uv must pass through this gap
to "escape" and pass through the segment ac. Because $|uv| \gg \epsilon$, uv must turn
"sideways" to pass through it. More precisely, let Q be a plane parallel to ab and cd
and midway between them, i.e., Q passes through the midpoint of the gap. uv must
align to lie nearly in Q to pass through the gap. Because uv is on the "wrong side"
of ab, there are only two ways uv can reach Q: either to align roughly parallel to ab,
or to align roughly parallel to cd. In either case, it would then be possible to pass
uv through the gap, by keeping it close to the long link to which it is nearly parallel.
However, the first alignment places uv at an angle near θ with respect to cd; but it
must be nearly parallel to cd. The second alignment requires flipping uv around so
that u is above v in the view shown in the figure, in order to get on the other side
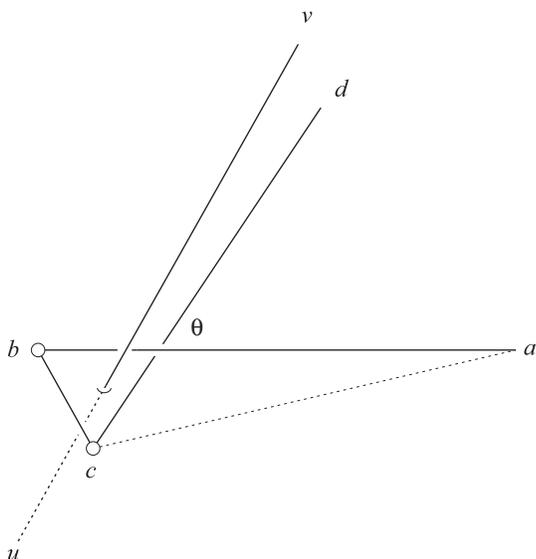of ab. But this then makes uv approximately antiparallel to cd, rather than nearly

FIGURE 4. A 1-link "jag."

parallel as it must be. Thus the only escape route is impossible, and uv maintains its piercing of $\triangle abc$. □

Corollary 2. *The 1-link jag interlocks with uv, under the constraints imposed by the nearly rigid trapezoid.*

3.2.2. *2-chain Through Trapezoid Tangle Corners.* Next we argue that the link uv can thread through the corner T_4 of the trapezoid so that it is "gripped" by the 3/4-tangle there. Note that the (T_1, T_4) trapezoid link connects to the 3-chain at T_4, which is itself just a jag loop. But uv cannot thread properly through both jag loops on either end of the (T_1, T_4) link. So instead we thread uv through the 4-chain at T_4.
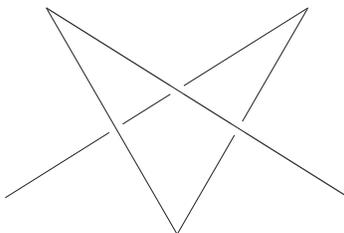

FIGURE 5. A 4-chain, part of a 3/4-tangle, can be viewed as two jag loops.

Now, a 4-chain can be viewed as two jag loops; see Figure 5. Moreover, the 4-chain and 3-chain participating in a 3/4-tangle can be viewed as each lying nearly in planes that are twisted with respect to one another. So we chooose to twist the 4-chain at

T_4 so that uv threads properly through one of its two jag loops. Similarly, the link vw threads from the jag at T_3 through the 3-chain at T_2 (and not through the 4-chain to which (T_4, T_2) is connected).

Applying Corollary 2 to guarantee interlocking yields:

Lemma 3. *The 2-chain links, when threaded as just described, are interlocked with the 3/4-tangles, under the constraints of a nearly rigid trapezoid.*

We should mention that the foregoing argument would be unnecessary if we had instead used a 2-link jags at T_1 and T_3, which would give freedom to position the jag to permit piercing the tangles however desired (and which would lead to an 18-link interlocking chain).

Finally, there is more than enough flexibility in the design to ensure that uv and vw can indeed share the same 2-chain apex v.

3.2.3. *Apex v Cannot Move Far.* Thus the 2-chain (u, v, w) cannot slide free of any of the trapezoid corners unless one of its vertices enters the ϵ-ball containing the corner. We argue below that this cannot occur. We start with a simple preliminary lemma.

Lemma 4. *When ϵ is sufficiently small, a line piercing two disks of radius ϵ can angularly deviate from the line connecting the disk centers at most $\delta \leq 2\epsilon/L$, where L is the distance between the disk centers.*

Proof. Figure 6 illustrates the largest angle δ, $(\frac{1}{2})L \sin \delta = \epsilon$, so $\sin \delta = 2\epsilon/L$, and the claim follows from the fact that $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$. $\qquad \square$

Let the trapezoid have base of length $2B$, side length L, and base angle θ. Let the triangle determined by the trapezoid have height h and half-angle α at the apex, so $\tan \theta = h/B$, or $h = B \tan \theta$. See Figure 6(b). The following lemma captures the key constraint on motion of the 2-link.

Lemma 5. *If the sides of the trapezoid pass through the ϵ-disks illustrated, then the height of the triangle approaches h as $\epsilon \to 0$.*

Proof. h_{min} occurs with a triangle apex angle of $\alpha + \delta$ and a base angle of $\theta - \delta$. Let b be the amount by which B is lengthened. In $\triangle xyz$, $b = \frac{\epsilon}{\sin(\theta - \delta)}$, and in $\triangle xYZ$, $\tan(\theta - \delta) = \frac{h_{min}}{B + b}$. Thus we have that

$$h_{min} = \left(B + \frac{\epsilon}{\sin(\theta - \delta)} \right) \tan(\theta - \delta) = B \tan(\theta - \delta) + \frac{\epsilon}{\cos(\theta - \delta)}.$$

Thus h_{min} is continuous near $\epsilon = 0$. Also, if $\epsilon \to 0$ then $\delta \to 0$ since $\delta \leq \frac{2\epsilon}{L}$. Therefore $\lim_{\epsilon \to 0} h_{min} = B \tan \theta = h$ since $\tan(\theta) = h/B$.

For h_{max} all the signs reverse to yield that $\lim_{\epsilon \to 0} h_{max} = B \tan \theta = h$.

We conclude that the height of the triangle approaches h as ϵ approaches 0 as desired. $\qquad \square$
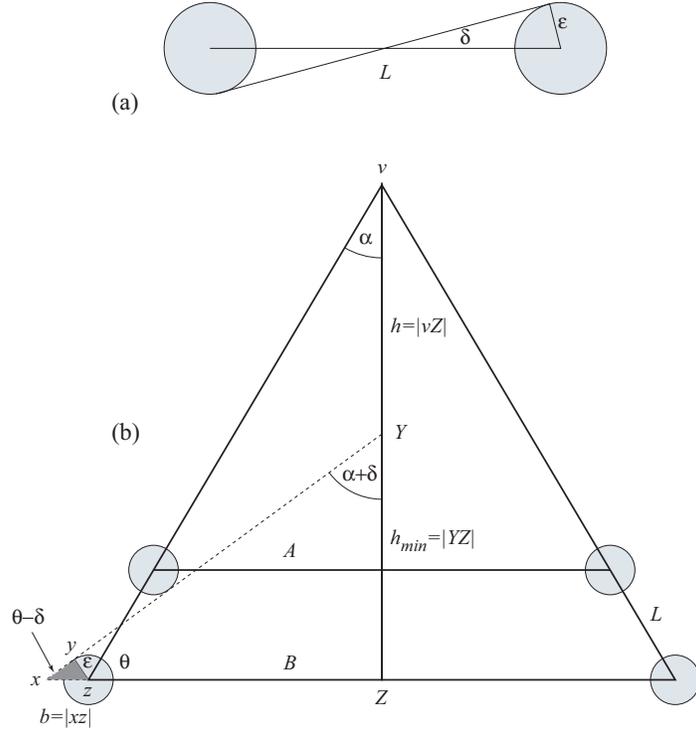
FIGURE 6. Trapezoid Lemma: (a) Line through two disks deviates at most δ; (b) Trapezoid structure, with h_{min} computation illustrated.

3.2.4. *Main Theorem.* We connect 3D to 2D via the plane determined by the 2-link in the proof of the main theorem below.

Theorem 1. *The 2-link chain is interlocked with the 16-link trapezoid chain.*

Proof. Let H be the plane containing the 2-link chain. We know that the links of the 2-chain must pass through ϵ-balls around the four vertices of the trapezoid. H meets these balls in disks each of radius $\leq \epsilon$. The Trapezoid Lemma shows that the height of the triangle approaches h as ϵ approaches 0. Thus, by choosing ϵ small enough, we limit the amount that the apex v of the 2-link chain can be separated from or pushed toward the trapezoid to any desired amount.

We previously established (in Corollary 2 and Lemma 3) that the 2-chain links are interlocked with the 3/4-tangles and jag loops through which they pass, under the assumption that the trapezoid is nearly rigid. The near-rigidity of the trapezoid could only be destroyed by a 2-chain link escaping from one of the jag loops through which it is threaded. But up until the time of this first escape, the trapezoid is nearly rigid; and so there can be no first escape.

Thus, choosing ϵ small enough to prevent any of the vertices of the 2-link chain from entering the ϵ-balls ensures that the 2-link chain is interlocked with the trapezoid chain. □

4. Discussion

We do not believe that $k = 16$ is minimal. We have designed two different 11-chains both of which appear to interlock with a 2-chain. However, both are based on a triangular skeleton rather than on a trapezoidal skeleton, and place the apex v of the 2-chain close to the 11-chain. It seems it will require a different proof technique to establish interlocking, for the simplicity of the proof presented here relies on the vertices of the 2-chain remaining far from the entangling chain.

Another direction to explore is closed chains, for which it is reasonable to expect fewer links. Replacing the 3/4-tangles with "knitting needles" configurations [2][1] produces a closed chain that appears interlocked, but we have not determined the minimum number of links that can achieve this.

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