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THE CLASSIFICATION OF V -TRANSVERSE KNOTS AND LOOSE LEGENDRIANS

PATRICIA CAHN AND VLADIMIR CHERNOV

ABSTRACT. We classify knots in a 3-manifold M that are transverse to a nowhere zero vector field V up to the corresponding isotopy relation. When V is the coorienting vector field of a contact structure, these knots are the same as pseudo-Legendrian knots, which were introduced by Benedetti and Petronio. We show that two loose Legendrian knots with the same overtwisted disk in their complement are Legendrian isotopic if and only if they are pseudo-Legendrian isotopic.

V -transverse knots are naturally framed. We show that each framed isotopy class corresponds to infinitely many V -transverse isotopy classes whose elements are pairwise distinct up to V -transverse homotopy, provided that one of the following conditions holds: V is a coorienting vector field of a tight contact structure; the manifold M is irreducible and atoroidal; or, the Euler class of a 2-dimensional bundle orthogonal to V is a torsion class.

We also give examples of infinite sets of distinct V -transverse isotopy classes whose representatives are all V -transverse homotopic and framed isotopic.

1. INTRODUCTION

We work in the smooth category. All manifolds and mappings are C^∞ . Throughout the paper M is an oriented manifold that is not necessarily compact or closed unless explicitly stated otherwise.

A *knot* in M is an embedding $K : S^1 \rightarrow M$ and a *curve* is an immersion $c : S^1 \rightarrow M$. Suppose M is equipped with an oriented d -dimensional distribution C . We denote the corresponding d -dimensional subspace of $T_p M$ by C_p . We say that K is C -*transverse* if for all t the velocity vector $K'(t)$ is transverse to $C_{K(t)}$. An *isotopy of knots* (resp. C -*transverse knots*) is a path in the space of knots (resp. C -transverse knots). A *homotopy of knots* (resp. C -*transverse knots*) is a path in the space of curves (resp. C -transverse curves).

When C is a contact structure on a 3-manifold, the C -transverse are just ordinary transverse knots. Given a nowhere vanishing vector field V on M we say that a knot is V -*transverse* if it is L -transverse, where L is an oriented line field spanned by V . In particular, when V is a coorienting vector field of a contact structure, V -transverse knots are the same as *Pseudo-Legendrian knots*, which were introduced by Benedetti and Petronio [2, 3]. (Benedetti and Petronio distinguish strong pseudo-Legendrian isotopy, which is the same as our V -transverse isotopy, and weak pseudo-Legendrian isotopy, which is a path (K_t, V_t) , where each K_t is V_t -transverse.) We show in Theorem 11.2 that two loose Legendrian knots with the same overtwisted disk in the complement are Legendrian isotopic exactly when they are pseudo-Legendrian

isotopic. Another nice fact about pseudo-Legendrian knots is that the groups of Vassiliev invariants of Legendrian and of Pseudo-Legendrian knots are canonically isomorphic [16].

Our goal is to classify V -transverse knots for a given nowhere vanishing vector field V on M up to V -transverse isotopy.

A *framed knot* is a knot K equipped with a section of the normal bundle of K . Each V -transverse knot in M has a natural framing, given by projecting V onto the 2-plane orthogonal to the velocity vector of K . Therefore an isotopy in the category of V -transverse knots is also an isotopy of framed knots. However we will see examples of V -transverse knots that are isotopic as framed knots but not isotopic as V -transverse knots. So, one natural question to ask is how many isotopy classes of V -transverse knots are there in a given framed isotopy class?

Like knots, any V -transverse curve has a natural framing.

Consider V -transverse knots in \mathbb{R}^3 with respect to the vector field $\frac{\partial}{\partial z}$. The projection of a V -transverse knot to the xy -plane is an immersed curve. Homotopy classes of immersed curves $c(t)$ in the plane are completely determined by their rotation number, i.e., the degree of the map $S^1 \rightarrow S^1$ determined by mapping t to the direction of $c'(t)$. During a framed homotopy or isotopy, the rotation number of the projection of a blackboard-framed knot to the xy -plane can change by a multiple of 2. Therefore, in a given framed isotopy class, there are infinitely many V -transverse homotopy classes, and hence also infinitely many V -transverse isotopy classes. Two V -transverse knots in \mathbb{R}^3 are isotopic as V -transverse knots if and only if they are isotopic as framed knots and homotopic as V -transverse curves. This theorem was proven by Trace [18].

We would like to know whether the intersection of a framed isotopy class and a connected component of the space of V -transverse curves in an arbitrary 3-manifold can contain more than one isotopy class of V -transverse knots, and if so, how many. To do this we study actions on following sets:

- The set of homotopy classes of V -transverse curves in a given connected component of the space of framed curves;
- The set of isotopy classes of V -transverse knots in a given connected component of the space of framed knots;
- The set of isotopy classes of V -transverse knots in the intersection of a given connected component of the space of V -transverse curves and a given connected component of the space of framed knots.

In particular, we show that for a fixed framed isotopy class \mathcal{K}_f and any V -transverse $K \in \mathcal{K}_f$, every V -transverse isotopy class in \mathcal{K}_f contains a representative of the form K^i for some $i \in \mathbb{Z}$, where K^i is obtained from K by adding i pairs of kinks as pictured in Figure 1 (more detail is given in Section 3). We first ask which of the K^i are in distinct V -transverse homotopy classes, and show that either all the K^i are distinct as V -transverse curves, or there exists some (smallest) nonzero $k \in \mathbb{Z}$ such that the knots K^{nk} are V -transverse homotopic for all $n \in \mathbb{Z}$. Then we ask which of the knots K^{nk} are V -transverse isotopic. Again, either these knots are all distinct as V -transverse knots or there exists some (smallest) nonzero $l \in \mathbb{Z}$, with

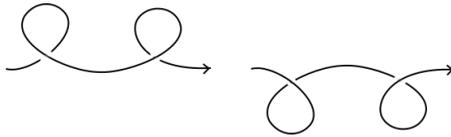


FIGURE 1. In this figure V is pointing out of the page.

$k|l$, such that the knots K^{nl} are V -transverse isotopic for all $n \in \mathbb{Z}$. The integers k and l depend on the Euler class of the 2-plane bundle V^\perp on M , and the precise formulation of the results described in this paragraph is given in Theorem 3.1.

If the knots K^i are all distinct as V -transverse curves (i.e., the integer k above does not exist), or if the integers k and l described above exist and are equal, then two V -transverse knots in \mathcal{K}_f are V -transverse isotopic if and only if they are V -transverse homotopic. Examples where the K^i are all distinct are given in Theorem 5.2. In particular this occurs if the Euler class of V^\perp is a torsion element of $H^2(M)$, or if M is closed, irreducible, and atoroidal, or if V is the coorienting vector field of a tight contact structure on M . Examples where k and l exist and are equal are given in Theorem 8.1.

On the other hand, if the integer k exists and l does not, or $l \neq k$, then there exist V -transverse knots in \mathcal{K}_f which are V -transverse homotopic but not V -transverse isotopic. Such examples are given in Theorems 6.1 and 8.2. These examples illustrate that the number of V -transverse isotopy classes corresponding to a given V -transverse homotopy class can be infinite. In fact, our example in Theorems 8.1 and 8.2 show that the number of distinct V -transverse isotopy classes corresponding to a single V -transverse homotopy class depends on the isotopy class of the knot, and not just on its homotopy class.

2. THE FUNDAMENTAL GROUPS OF THE SPACES OF FRAMED KNOTS AND CURVES

It follows from the existence of the self-linking number that the number $|K|$ of framed knots in S^3 with given underlying knot K is infinite. The second author previously defined affine self-linking invariants and used them [5, Theorem 2.4] to show that $|K|$ is infinite for every knot in an orientable manifold unless the manifold contains a connected sum factor of $S^1 \times S^2$. The knot K need not be zero-homologous and the manifold is not required to be compact. In our work with Sadykov [4] we used the results of McCullough [14] and strengthened the above result. We showed that

2.1. Lemma. *Let M be a not necessarily compact orientable 3-manifold. Given a knot K in M we have $|K| = \infty$ unless K intersects a nonseparating 2-sphere at exactly one point in which case $|K| = 2$.*

Note that if K intersects a nonseparating sphere at exactly one point then M contains $S^1 \times S^2$ as a connected sum factor.

2.1. Some important coverings. Let \mathcal{C} be a connected component of the space of curves in M . Let \mathcal{C}_f be a connected component of the space of framed curves in M which realize elements of \mathcal{C} after one forgets their framing. Let \mathcal{K} be a connected component of the space of knots in M such that $\mathcal{K} \subset \mathcal{C}$ and let \mathcal{K}_f be a connected component of the space of framed knots in M whose elements realize elements of \mathcal{K} as unframed knots, and such that $\mathcal{K}_f \subset \mathcal{C}_f$.

Let $\tilde{\mathcal{C}}_f$ be the quotient space obtained from \mathcal{C}_f by identifying framed curves which agree as unframed curves and whose framings are homotopic as nowhere zero sections of the normal bundle of the curve. Then the map $p : \tilde{\mathcal{C}}_f \rightarrow \mathcal{C}$ which forgets the framing of an equivalence class of framed curves is a regular covering map [6, Lemma 3.2]. We can restrict p to get another covering map $p : \tilde{\mathcal{K}}_f \rightarrow \mathcal{K}$. The proof of the following Lemma is straightforward.

2.2. Lemma. $p_* : \pi_1(\tilde{\mathcal{K}}_f) \rightarrow \pi_1(\mathcal{K})$ is an isomorphism if and only if \mathcal{K} does not intersect some nonseparating 2-sphere at exactly one point.

3. ACTIONS ON THE SPACES OF V -TRANSVERSE KNOTS AND CURVES

Given a V -transverse knot or curve K , let K^i denote the V -transverse knot or curve obtained by adding i pairs of the kinks on the left in Figure 1 if $i > 0$ and $|i|$ pairs of the kinks on the right in Figure 1 if $i < 0$. Let \mathcal{K}_f be a framed isotopy class of knots in M and let S be the set of isotopy classes of V -transverse knots contained in \mathcal{K}_f . There is a \mathbb{Z} -action $\text{Act}_{\mathcal{K}_f, V} : \mathbb{Z} \times S \rightarrow S$ defined by $\text{Act}_{\mathcal{K}_f, V}(i)([K]) = [K^i]$. Here square brackets denote the V -transverse isotopy class of a knot. Note that it takes a bit of effort to verify that $\text{Act}_{\mathcal{K}_f, V}$ is a well-defined action and that it is indeed a \mathbb{Z} -action rather than separate actions of the semigroups of nonnegative and nonpositive integers.

Similarly, we can define an action $\text{Act}_{\mathcal{C}_f, V} : \mathbb{Z} \times \bar{S} \rightarrow \bar{S}$ where \bar{S} is the set of homotopy classes of V -transverse curves in \mathcal{C}_f . The action is defined by $\text{Act}_{\mathcal{C}_f, V}(i)(\langle C \rangle) = \langle C^i \rangle$ where $\langle C \rangle$ denotes the V -transverse homotopy class of C .

We will compute the stabilizers of the above \mathbb{Z} -actions using the homomorphisms below.

Let V be a nonvanishing vector field on M . Let \mathcal{K} be a connected component of the space of knots in M , and let \mathcal{K}_f be a component of the space of framed knots in M whose elements are elements of \mathcal{K} when one forgets their framing. We will define a homomorphism $\theta_{\mathcal{K}_f, V} : H_1(\mathcal{K}_f) \rightarrow \mathbb{Z}$. Let $\alpha \in \pi_1(\mathcal{K}_f)$. This gives rise to a map $\bar{\alpha} : S^1 \times S^1 \rightarrow M$ where $\bar{\alpha}|_{S^1 \times t} = \alpha(t)$. Note also that we abuse notation and use α to denote both a loop and its homotopy class. If α and α' are homotopic loops in $\pi_1(\mathcal{K}_f)$ then $\bar{\alpha}$ and $\bar{\alpha}'$ are homotopic. Now define $\theta_{\mathcal{K}_f, V} = \frac{1}{2}e_{V^\perp}(\bar{\alpha}_* [S^1 \times S^1])$ where $e_{V^\perp} \in H^2(M)$ is the Euler class of the distribution V^\perp of 2-planes orthogonal to V and $[S^1 \times S^1] \in H_2(S^1 \times S^1)$ is the fundamental class of the torus. Note that $e_{V^\perp} \in H^2(M)$ is even. One can verify that $\theta_{\mathcal{K}_f, V} : \pi_1(\mathcal{K}_f) \rightarrow \mathbb{Z}$ is a homomorphism. Since \mathbb{Z} is abelian we can regard $\theta_{\mathcal{K}_f, V}$ as a homomorphism from $H_1(\mathcal{K}_f)$ to \mathbb{Z} .

Now let \mathcal{C} be a connected component of the space of curves in M and let \mathcal{C}_f be a component of the space of framed curves whose elements are elements of \mathcal{C} when

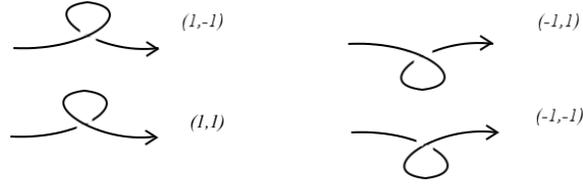


FIGURE 2. Four different types of kinks.

one forgets their framing. We define $\theta_{\mathcal{C}_f, V} : \pi_1(\mathcal{C}_f) \rightarrow \mathbb{Z}$ as above and regard it as a map out of $H_1(\mathcal{C}_f)$.

Recall that a G -torsor is a set X equipped with an action of G such that for any $x, y \in X$ there exists a unique $g \in G$ such that $y = gx$. Hence $|X| = |G|$.

The theorem below describes the number of homotopy (resp. isotopy) classes of V -transverse curves (resp. knots) in a given homotopy (resp. isotopy) class of framed curves (resp. knots). It also describes how many isotopy classes of V -transverse curves there are in the intersection of a given framed isotopy class and a given V -transverse homotopy class.

Let $\Gamma_{\mathcal{C}_f, V} = \mathbb{Z}/\text{Im } \theta_{\mathcal{C}_f, V}$ and $\Gamma_{\mathcal{K}_f, V} = \mathbb{Z}/\text{Im } \theta_{\mathcal{K}_f, V}$. When \mathcal{C} is a path connected component of the space of curves such that $\mathcal{K}_f \subset \mathcal{C}$ after forgetting the framing, let $\bar{\Gamma}_{\mathcal{K}_f, V} = \text{Im } \theta_{\mathcal{C}_f, V} / \text{Im } \theta_{\mathcal{K}_f, V}$.

3.1. Theorem. *Let \mathcal{C}_f be a connected component of the space of framed curves in M and let \mathcal{K}_f be a connected component of the space of framed knots in M such that $\mathcal{K}_f \subset \mathcal{C}_f$. Let \mathcal{T} be a connected component of the space of V -transverse curves in M such that $\mathcal{T} \subset \mathcal{C}_f$.*

- (1) *The set of homotopy classes of V -transverse curves in M that lie in \mathcal{C}_f is a $\Gamma_{\mathcal{C}_f, V}$ -torsor, and the number of such classes is $|\Gamma_{\mathcal{C}_f, V}|$.*
- (2) *The set of isotopy classes of V -transverse knots in M that lie in \mathcal{K}_f is a $\Gamma_{\mathcal{K}_f, V}$ -torsor, and the number of such classes is $|\Gamma_{\mathcal{K}_f, V}|$.*
- (3) *The set of isotopy classes of V -transverse knots in M that lie in $\mathcal{K}_f \cap \mathcal{T}$ is a $\bar{\Gamma}_{\mathcal{K}_f, V}$ -torsor, and the number of such classes is $|\bar{\Gamma}_{\mathcal{K}_f, V}|$.*

3.2. Lemma. *If K and L are V -transverse knots which are isotopic as framed knots, then K is V -transversely isotopic to L^i for some $i \in \mathbb{Z}$.*

Proof. We can choose a set of coordinate charts $\{(U_i, \phi_i)\}_{i=1}^n$ for M such that $V = \phi_{i*}^{-1}(\partial/\partial z)$ in each chart. We will imitate the framed isotopy K_t from K to L by a V -transverse isotopy \bar{K}_t in such a way that the knot \bar{K}_1 agrees with L outside some coordinate chart (U_i, ϕ_i) , and inside that chart L and \bar{K}_1 differ by a collection of small kinks (see Figure 2). We will then argue that these kinks cancel via an isotopy in such a way that $\bar{K}_1 = L^i$.

(Remark: The above framed and V -transverse isotopies can be made C^0 -close.)

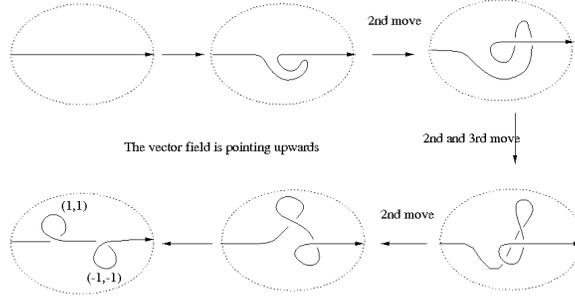


FIGURE 3. Creation and cancelation of a pair of opposite kinks

In each chart (U_i, ϕ_i) , the projection of the framed isotopy \overline{K}_t to the xy -plane (after forgetting the framing) can be viewed as a sequence of type 1, 2 and 3 Reidemeister moves, in addition to ambient isotopy.

The type 2 and 3 Reidemeister may appear in the projection of a V -transverse isotopy to the xy -plane, but type 1 does not appear, because the projection of a V -transverse isotopy to the xy -plane is always an immersed curve.

There are four different kinds of kinks that may appear in a type 1 Reidemeister move, and these kinks are pictured in Figure 2. Each kink is labeled by an ordered pair, where the first number is the contribution of the kink to the rotation number of the projection to the xy -plane, and the second is the local writhe number.

Pairs of kinks with opposite rotation number *and* opposite local writhe number can be created or cancelled by a V -transverse isotopy, see Figure 3.

Therefore if a type 1 move creates a kink of type (ϵ_1, ϵ_2) during K_t , we instead create a pair of kinks (ϵ_1, ϵ_2) and $(-\epsilon_1, -\epsilon_2)$ in \overline{K}_t . Then we make the extra kink of type $(-\epsilon_1, -\epsilon_2)$ very small and carry it along during the V -transverse isotopy.

If there is a type 1 move in K_t which deletes a kink, we do not delete that kink in \overline{K}_t and instead make it small and carry it along during the V -transverse isotopy.

At the end of the isotopy \overline{K}_t we see L with many extra kinks. We may slide these kinks along L using a V -transverse isotopy so that they all appear in the same chart, and in an unknotted portion of L in that chart.

Let a be the number of $(1, 1)$ kinks, b the number of $(-1, -1)$ kinks, c the number of $(-1, 1)$ kinks, and d the number of $(1, -1)$ kinks. Possibly by sliding kinks past one another, we cancel all pairs of kinks that have both opposite rotation number and opposite writhe.

Now we have a or $b = 0$, and c or $d = 0$. For all t the knots K_t and \overline{K}_t are contained in a thin solid torus T_t , which we can identify with the standard solid torus in \mathbb{R}^3 . Since both K_t and \overline{K}_t are framed isotopies, we can compare their self-linking numbers at each time t *after* identifying T_t with the standard solid torus in \mathbb{R}^3 . The difference between their self-linking numbers does not depend on the choice of identification of T_t with the standard solid torus. We call this number

$s(t)$. Since $s(0) = 0$, we must have $s(1) = 0$. On the other hand each kink of type (ϵ_1, ϵ_2) in \overline{K}_1 contributes ϵ_2 to the value of $s(1)$. Hence $a + c = b + d$.

In the case where a and c are equal to 0, we have $b + d = 0$. But both b and d are nonnegative, so $a = b = c = d = 0$. In this case $\overline{K}_1 = L$. This also occurs in the case where b and d are equal to 0.

In the case where a and d equal 0, we have $b = c$. In this case $\overline{K}_1 = L^{-b}$. In the case where b and c equal 0, we have $a = d$ and $\overline{K}_1 = L^a$. \square

We get a similar theorem by approximating a framed homotopy by a V -transverse homotopy. First we must classify the connected components of the space of framed curves in M . Let K be an unframed curve in M . Given two framed curves K_1 and K_2 in M that coincide pointwise with K as unframed knots, one can regard their framings as simple closed curves c_1 and c_2 on a thin torus neighborhood of K . Put $m(K_1, K_2)$ to be the intersection number of c_1 and c_2 . The second author [17, Proposition 5.1.10] proved that the K_i are homotopic as framed curves if and only if $m(K_1, K_2)$ is even. One can check that if K_1 and K_2 differ by one of the four kinks in Figure 2 then after performing a small V -transverse isotopy to K_2 so that it coincides with K_1 as an unframed knot, we have $m(K_1, K_2) = \pm 1$.

3.3. Lemma. *Suppose that K and L are V -transverse curves which are in the same connected component \mathcal{C}_f of the space of framed curves in M . Then K is V -transversely isotopic to L^i for some $i \in \mathbb{Z}$.*

Proof. Again, we cover M with charts (U_i, ϕ_i) such that in each chart $V = \phi_{i*}^{-1} \partial / \partial z$. Second and third Reidemeister moves, and crossing changes are V -transverse. We adjust the first Reidemeister move as in the proof of Lemma 3.2. At the end of our V -transverse homotopy, we are left with a copy of L with extra kinks. One can pass through a double point of a kink using a V -transverse homotopy, so we may cancel all pairs of kinks with opposite contributions to the rotation number, i.e., pairs of types (ϵ_1, ϵ_2) and $(-\epsilon_1, \pm \epsilon_2)$. We are left with kinks which all have the same rotation number. Because K and L are in the same component of the space of framed curves the number of kinks remaining must be even. Now, we can pass through double points to obtain L^i for some $i \in \mathbb{Z}$. \square

We will use the theorem below to prove Theorem 3.1.

3.4. Theorem. *Let K be a V -transverse knot, let \mathcal{K}_f be the component of the space of framed knots in M containing K , and let \mathcal{C}_f be the component of the space of framed curves in M containing K . Then $[K^i] = [K^j]$ if and only if $i - j \in \text{Im}(\theta_{\mathcal{K}_f, V})$ and $\langle K^i \rangle = \langle K^j \rangle$ if and only if $i - j \in \text{Im}(\theta_{\mathcal{C}_f, V})$.*

Proof. First we show that if $[K^i] = [K^j]$ then $i - j \in \text{Im}(\theta_{\mathcal{K}_f, V})$. We write $K : ([0, 1], \partial) \rightarrow M$. We assume that $K^i(s) = K^j(s)$ for all $s \in [\epsilon, 1]$ for some $0 < \epsilon < 1$, and the image $K^i([0, \epsilon])$ consists of i pairs of kinks in Figure 1, where the type of kink depends on the sign of i . Similarly $K^j([0, \epsilon])$ consists of j pairs of kinks. A chart containing $K^i([0, \epsilon])$ and $K^j([0, \epsilon])$ is shown in Figure 4. Let $\alpha = K^i([0, \epsilon])$ and let $\beta = K^j([0, \epsilon])$. In this chart V points out of the page.

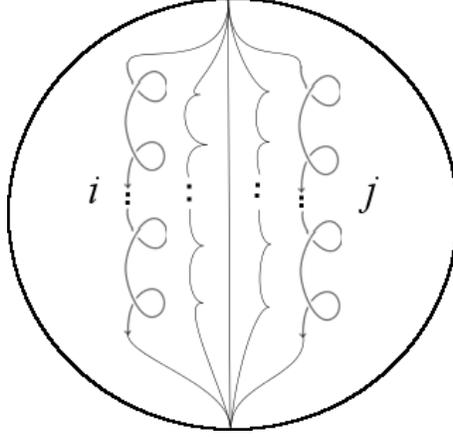


FIGURE 4. A chart containing the paths α (left) and β (right), with an isotopy between them. In this chart V points out of the page. Of course if i or j are negative we use the kinks on the right hand side of Figure 1.

Suppose we have a V -transverse isotopy K_t taking K^i to K^j . Since K^i and K^j agree pointwise with K on the interval $[\epsilon, 1]$, this isotopy yields a map $\psi : T^* \rightarrow M$, where T^* is a torus with one hole and $\psi(\partial T^*) = \alpha\beta^{-1}$. More precisely, we first define ψ on $[0, 1] \times [0, 1]$ by $\psi(s, t) = K_t(s)$. Note $\psi(s, 0) = K^i(s)$ and $\psi(s, 1) = K^j(s)$. Let

$$T^* = \frac{[0, 1] \times [0, 1]}{s \times 0 \sim s \times 1 \text{ for } s \in [\epsilon, 1], \text{ and } 0 \times t \sim 1 \times t \text{ for all } t \in [0, 1]}.$$

Since $K^i(t) = K^j(t)$ for all $t \in [\epsilon, 1]$ we may view ψ as a map out of T^* .

Since the isotopy is V -transverse, we get a section of V^\perp on $\psi(T^*)$ by projecting the tangent vector of the knot to V^\perp . We can extend ψ to a map of a torus T using a framed isotopy from K^i to K^j . We can assume this isotopy takes place within a chart (U, ϕ) on M such that $V = \phi_*^{-1} \partial / \partial z$ on U . We choose a trivialization of V^\perp on U . We have a section of V^\perp on the loop $\alpha\beta^{-1}$ given by the tangent vectors to K^i and K^j . The Euler class of $\psi(T)$, which is the obstruction to extending our section on $\psi(T^*)$ to a section on $\psi(T)$, is the degree of the map $\alpha\beta^{-1} \rightarrow S^1$ that sends a point on α or β^{-1} to the corresponding velocity vector of K^i or $(K^j)^{-1}$, is $2i - 2j$ (because of our choice of trivialization the Euler class is just the rotation number of the projection of $\alpha\beta^{-1}$ to D^2). Hence $i - j$ is in the image of $\theta_{\mathcal{K}_f, V}$.

Conversely suppose that $i - j \in \text{Im}(\theta_{\mathcal{K}_f, V})$. We will show $[K^i] = [K^j]$. Let $\gamma \in \pi_1(\mathcal{K}_f, K^i)$ such that $\theta_{\mathcal{K}_f, V}(\gamma) = i - j$. Suppose that K^i and K^j agree on the interval $[\epsilon, 1]$, so the additional kinks are added in the interval $[0, \epsilon]$. We can choose a representative $g \in \gamma$ such that

- $g_0(s) = K^i(s)$ and $g_\delta(s) = K^j(s)$ for some small $\delta \in (0, 1)$
- $g_t(s) = K^i(s)$ for all $s \in [\epsilon, 1]$ and all $t \in [0, \delta]$

Then for $t \in [\delta, 1]$, $g_t(s)$ is a framed isotopy from $K^j(s)$ to $K^i(s)$.

For $t \in [\delta, 1]$, we imitate the framed isotopy $g_t(s)$ by a V -transverse isotopy $g_{V,t}(s)$ as in the proof of Lemma 3.2, so that $g_{V,\delta}(s) = K^j(s)$ and $g_{V,1}(s) = K^{i+k}(s)$ for some $k \in \mathbb{Z}$.

As before, we have a map of a punctured torus $\psi : T^* \rightarrow M$, and extend this to a map of a torus T using a framed isotopy from $K^{i+k}(s)$ to $K^j(s)$. We have a section of V^\perp on $\psi(T^*)$. The obstruction to extending this section to $\psi(T)$, i.e. the value of $e_{V^\perp}(\psi(T))$, is now $2(i+k-j)$. Therefore $\theta_{\mathcal{K}_f, V}(\gamma) = i+k-j$. But we assumed that $\theta_{\mathcal{K}_f, V}(\gamma) = i-j$ so $k=0$. Hence K^i and K^j are V -transverse isotopic.

The next step is to show $\langle K^i \rangle = \langle K^j \rangle$ if and only if $i-j \in \text{Im } \theta_{\mathcal{C}, V}$.

The proof that $\langle K^i \rangle = \langle K^j \rangle$ implies $i-j \in \text{Im } \theta_{\mathcal{C}, V}$ is similar to the proof that $[K^i] = [K^j]$ then $i-j \in \text{Im}(\theta_{\mathcal{K}_f, V})$.

Now we assume $i-j \in \text{Im } \theta_{\mathcal{C}_f, V}$ and show $\langle K^i \rangle = \langle K^j \rangle$. Let $\gamma \in \pi_1(\mathcal{C}_f, K^i)$ such that $\theta_{\mathcal{C}_f, V}(\gamma) = i-j$. We can choose a representative $g \in \gamma$ such that $g_0(s) = K^i(s)$ and $g_x(s) = K^j(s)$ for some $x \in (0, 1)$. Then for $t \in [x, 1]$, $g_t(s)$ is a homotopy from $K^j(s)$ to $K^i(s)$ through framed curves in M .

For $t \in [x, 1]$, we imitate the framed homotopy $g_t(s)$ by a V -transverse homotopy $g_{V,t}(s)$ as in the proof of Lemma 3.3, so that $g_{V,x}(s) = K^j(s)$ and $g_{V,1}(s) = K^{i+k}(s)$ for some $k \in \mathbb{Z}$. As before, we see $\theta_{\mathcal{C}_f, V}(\gamma) = i+k-j$, but we assumed $\theta_{\mathcal{C}_f, V}(\gamma) = i-j$ so $k=0$. Hence $\langle K^i \rangle = \langle K^j \rangle$. \square

4. PROOF OF THEOREM 3.1

Now we prove:

The set of homotopy classes of V -transverse curves in M that lie in \mathcal{C}_f is a $\Gamma_{\mathcal{C}_f, V}$ -torsor, and the number of such classes is $|\Gamma_{\mathcal{C}_f, V}|$.

Proof. Recall $\Gamma_{\mathcal{C}_f, V} = \mathbb{Z}/\text{Im } \theta_{\mathcal{C}_f, V}$. We have an action of \mathbb{Z} on the set \bar{S} of V -transverse homotopy classes of V -transverse curves in \mathcal{C}_f , which is given by $i \cdot \langle K \rangle = \langle K^i \rangle$. Now we show that we have an action of $\mathbb{Z}/\text{Im } \theta_{\mathcal{C}_f, V}$ on \bar{S} given by $\bar{i} \cdot \langle K \rangle = \langle K^i \rangle$. Let $\bar{i} = \bar{j} \in \mathbb{Z}/\text{Im } \theta_{\mathcal{C}_f, V}$. Then $i-j \in \text{Im } \theta_{\mathcal{C}_f, V}$. By Theorem 3.4, we have $i \cdot \langle K \rangle = j \cdot \langle K \rangle$, so the action of $\mathbb{Z}/\text{Im } \theta_{\mathcal{C}_f, V}$ on \bar{S} is well-defined. Theorem 3.4 also implies the action is free. This action is transitive by Lemma 3.3. \square

The proof of the statement below is similar.

The set of isotopy classes of V -transverse knots in M that lie in \mathcal{K}_f is a $\Gamma_{\mathcal{K}_f, V}$ -torsor, and the number of such classes is $|\Gamma_{\mathcal{K}_f, V}|$.

Proof. The action of $\mathbb{Z}/\text{Im } \theta_{\mathcal{K}_f, V}$ on the set S of V -transverse isotopy classes of V -transverse knots in \mathcal{K}_f defined by $\bar{i} \cdot [K] = [K^i]$ is well-defined and free by Theorem 3.4 and transitive by Lemma 3.2. \square

Let \mathcal{C}_f be a component of the space of framed curves such that the elements of the framed isotopy class \mathcal{K}_f are contained in \mathcal{C}_f . Let \mathcal{T} be a connected component of the space of V -transverse curves contained in \mathcal{C}_f . Recall that $\text{Im } \theta_{\mathcal{K}_f, V}$ is a subgroup of $\text{Im } \theta_{\mathcal{C}_f, V}$ and $\bar{\Gamma}_{\mathcal{K}_f, V} = \text{Im } \theta_{\mathcal{C}_f, V} / \text{Im } \theta_{\mathcal{K}_f, V}$.

The set of isotopy classes of V -transverse knots in M that lie in $\mathcal{K}_f \cap \mathcal{T}$ is a $\overline{\Gamma}_{\mathcal{K}_f, V}$ -torsor, and the number of such classes is $|\overline{\Gamma}_{\mathcal{K}_f, V}|$.

Proof. Let T be the set of isotopy classes of V -transverse knots that lie in $\mathcal{T} \cap \mathcal{K}_f$. There is an action of $\text{Im } \theta_{\mathcal{C}_f, V}$ on T given by $i \cdot [K] = [K^i]$. This gives us a well-defined free action of $\text{Im } \theta_{\mathcal{C}_f, V} / \text{Im } \theta_{\mathcal{K}_f, V}$ on T by Theorem 3.4. The action is transitive because $[L] \in \mathcal{K}_f$ implies $[L] = [K^i]$ for $i \in \mathbb{Z}$ by Lemma 3.2, and since $[L] \in \mathcal{T}$, we have $i \in \text{Im } \theta_{\mathcal{C}_f, V}$. \square

5. CASES WHEN V -TRANSVERSE KNOT THEORY IS SIMPLE

Let us recall a few classical definitions.

5.1. Definition. A 3-manifold M is *irreducible* if every 2-sphere embedded into M bounds a ball.

A closed irreducible 3-manifold is *atoroidal* if it does not admit *essential mappings* $\mu : S^1 \times S^1 \rightarrow M$, i.e. mappings such that $\mu_* : \pi_1(S^1 \times S^1) \rightarrow \pi_1(M)$ is injective.

The 2-dimensional subbundle $C \subset TM$ is called a *contact structure* if it can be locally presented as $\ker \alpha$, for some 1-form α with $\alpha \wedge d\alpha \neq 0$. All contact manifolds (M, C) have a natural orientation given by $\alpha \wedge d\alpha$. If C can be globally presented as $\ker \alpha$ with $\alpha \wedge d\alpha \neq 0$, then it is called a *coorientable (transversely orientable) contact structure*. Clearly, if α_1, α_2 are two forms such that $C = \ker \alpha_1 = \ker \alpha_2$, then $\alpha_1 = g\alpha_2$, for some nowhere zero smooth function g .

The choice of the class of such α up to multiplication by a smooth positive function g is called a *coorientation of C* . A *coorienting vector field* of a cooriented contact structure C is a vector field V transverse to C such that for all $x \in M$ the vector $V_x \in T_x M$ points into the half-space of $T_x M \setminus C_x$ where α_x is positive. *In this paper all contact structures are assumed to be cooriented.*

A contact structure is said to be *overtwisted* if there exists a 2-disk D embedded into M such that the boundary ∂D is tangent to C while the disk D is transverse to C along ∂D . Non-overtwisted contact structures are called *tight*.

A knot f in a contact (M, C) is *Legendrian* if df maps $T_p S^1$ to $C_{f(p)}$ for all p . A Legendrian knot is *loose* if its complement contains an overtwisted disk.

5.2. Theorem. *Assume that V is a nowhere zero vector field on M satisfying one of the following three conditions.*

- (1) *The Euler class $e_{V^\perp} \in H^2(M)$ is a torsion element (in particular if $e_{V^\perp} = 0$.)*
- (2) *The manifold M is closed, irreducible and atoroidal.*
- (3) *V is a coorienting vector field of a contact structure C such that (M, C) is a total space of a (not necessarily finite) covering of a tight contact 3-manifold (in particular if (M, C) is itself a tight contact 3-manifold).*

Let \mathcal{C}_f be a connected component of the space of framed curves and let \mathcal{K}_f be a connected component of the space of framed knots such that $\mathcal{K}_f \subset \mathcal{C}_f$. Then $\theta_{\mathcal{K}_f, V}$ and $\theta_{\mathcal{C}_f, V}$ are zero homomorphisms, so $\Gamma_{\mathcal{C}_f, V} = \mathbb{Z}, \Gamma_{\mathcal{K}_f, V} = \mathbb{Z}, \overline{\Gamma}_{\mathcal{K}_f, V} = 0$. In

particular each framed knot corresponds to infinitely many V -transverse knots and they are pairwise not V -transverse homotopic.

Proof. It suffices to show that if any one of these three conditions holds, then for every $\bar{\alpha} : S^1 \times S^1 \rightarrow M$ we have $e_{V^\perp}(\bar{\alpha}_*[S^1 \times S^1]) = 0$. If condition 1 holds then this is certainly true. If condition 2 holds then this true because $\bar{\alpha}_*([S^1 \times S^1]) = 0$, see for example [6, pages 2784-2785]. If condition 3 holds then $e_{V^\perp} = e_C$ and the desired statement was proved in [6, Corollary 3.10]. (Note that if $\bar{\alpha} : S^1 \times S^1 \rightarrow M$ is an embedding, then $e_C(\bar{\alpha}_*[S^1 \times S^1]) = 0$ by the Bennequin type inequality proved by Eliashberg [10, Theorem 2.2.1].) \square

6. EXAMPLES WHERE $\theta_{\mathcal{K}_f, V} = 0$ AND $\theta_{\mathcal{C}_f, V} \neq 0$.

Let F be a nonorientable surface of genus bigger than one. Let $\text{pr} : M \rightarrow F$ to be a locally trivial S^1 -bundle with an oriented total space M . Let μ be a curve in M that projects to the dashed curve in Figure 5. Take $k \in \mathbb{N}$ and let $C = C_{2k}$ be a cooriented contact structure on M whose Euler class is the Poincaré dual of $2k[\mu] \in H_1(M)$. The fact that such C exists is proved, for example, in [17, Propoposition 4.0.1] and follows from the results of Lutz [13]. Take V to be the vector field that coorients C . Let K_f be any framed knot that projects to a loop homotopic to the solid loop ν in Figure 5, let \mathcal{K}_f be the connected component of the space of framed knots that contains K_f , and let \mathcal{C}_f be the connected component of the space of framed curves that contains \mathcal{K}_f .

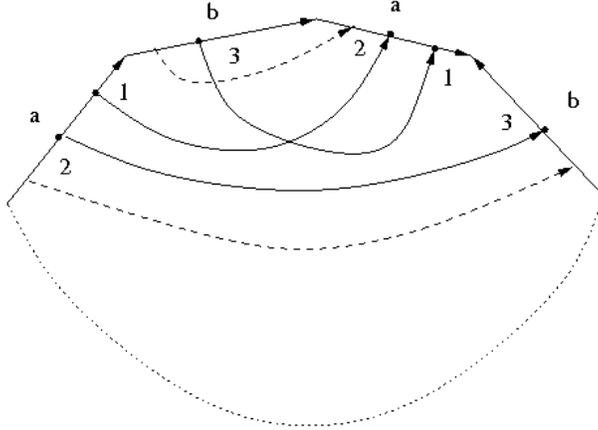


FIGURE 5.

6.1. Theorem. *Let $M, V, \mathcal{K}_f, \mathcal{C}_f$ be as described above. Then $\text{Im}(\theta_{\mathcal{C}_f, V}) = k\mathbb{Z} < \mathbb{Z}$ and $\text{Im}(\theta_{\mathcal{K}_f, V}) = \{0\} < \mathbb{Z}$. So $\Gamma_{\mathcal{C}_f, V} = \mathbb{Z}_k, \Gamma_{\mathcal{K}_f, V} = \mathbb{Z}$, and $\bar{\Gamma}_{\mathcal{K}_f, V} = k\mathbb{Z}$.*

For the proof of Theorem 6.1 see Section 7. The properties of this example used in the proof of Theorem 6.1 are that the curve μ intersects ν transversely in one point, and the two curves obtained by smoothing ν at its one self-intersection point are orientation reversing.

6.2. Remark. Let K be a V -transverse knot that realizes the class \mathcal{K}_f described in Theorem 6.1 and let $i, j \in \mathbb{Z}$ be such that $i - j$ is divisible by k . Then Theorem 3.1 implies that $[K^i] \neq [K^j]$ even though $\langle K^i \rangle = \langle K^j \rangle$. Thus we have constructed examples where regular knot theory does not reduce to the framed knot theory and the classification of the connected components of the space of V -transverse curves.

7. PROOF OF THEOREM 6.1

We first show that $\text{Im } \theta_{\mathcal{C}_f, V} = k\mathbb{Z}$. This requires understanding the groups $\pi_1(\mathcal{C}_f)$ and $\pi_1(\mathcal{C})$.

7.1. The loops $\gamma_1, \gamma_2, \gamma_3$ in \mathcal{C} . Let \mathcal{C} be a connected component of the space of unframed curves in M obtained by forgetting the framing on curves from \mathcal{C}_f and let $K \in \mathcal{C}$ be the unframed knot obtained by forgetting the framing on K_f .

We introduce three loops $\gamma_i, i = 1, 2, 3$ following [5].

Let γ_1 be the isotopy of K to itself induced by a full rotation of its parameterizing circle.

Let γ_2 be the deformation of K described in Figure 6.

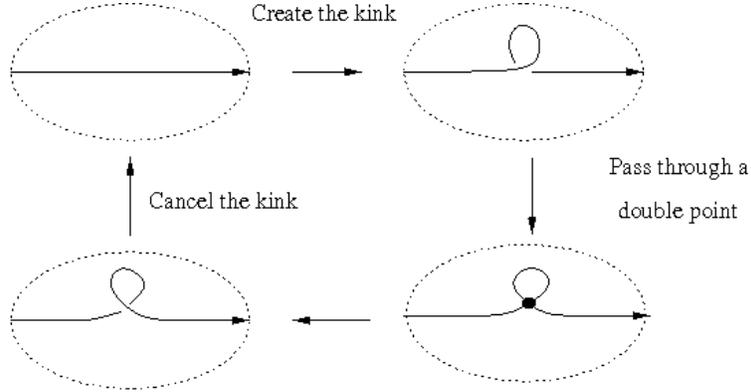


FIGURE 6. The loop γ_2 .

Since M is orientable, we get that the S^1 -fibration over S^1 (parameterizing the knot K) induced from pr by $\text{pr} \circ K : S^1 \rightarrow F$ is trivializable. Hence we can coherently orient the fibers of the induced fibration $\overline{\text{pr}} : S^1 \times S^1 \rightarrow S^1$. The orientation of the S^1 -fiber of $\overline{\text{pr}}$ over $t \in S^1$ induces the orientation of the S^1 -fiber of pr that contains $K(t)$. Let γ_3 be the homotopy of K that slides every point $K(t)$ of K around the fiber that contains $K(t)$ with unit velocity in the direction specified by the orientation of the fiber of $\overline{\text{pr}}$ over $t \in S^1$.

7.1. Lemma. Every element $\alpha \in \pi_1(\mathcal{C}, K)$ can be written as $\gamma_1^r \gamma_2^s \gamma_3^t$ for some $r, s, t \in \mathbb{Z}$.

Proof. In [5, Proof of Lemma 6.11 page 808] we proved that for each α there exists i such that $\alpha^i = \gamma_1^r \gamma_2^s \gamma_3^t$. Analyzing the proof we see that if $\text{pr}(K) \in \pi_1(F, \text{pr}(K(1)))$ is not a nontrivial power of another element of $\pi_1(F, \text{pr}(K(1)))$, then i can be taken to be 1. \square

7.2. Remark. The statement of Lemma 7.1 remains true for every K in M such that $\text{pr} K$ is an orientation preserving loop on the surface F and $\text{pr}(K) \neq 1 \in \pi_1(F, \text{pr}(K(1)))$ is not a nontrivial power of another element of $\pi_1(F, \text{pr}(K(1)))$.

Let $\alpha \in \pi_1(\mathcal{C}_f, K_f)$, and regard α as an element of $\pi_1(\mathcal{C}, K)$ by forgetting the framing. By Lemma 7.1 we have $\alpha = \gamma_1^r \gamma_2^s \gamma_3^t$ for some $r, s, t \in \mathbb{Z}$. Define a homomorphism $\delta_1 : \pi_1(\mathcal{C}, K) \rightarrow \mathbb{Z}$ given by the lifting correspondence for the covering $p : \tilde{\mathcal{C}}_f \rightarrow \mathcal{C}$ where we identify $0 \in \mathbb{Z}$ with the equivalence class of K_f in $\tilde{\mathcal{C}}_f$. Note that $\delta_1(\gamma_i)$ is 0 for $i = 1, 3$ and is 2 for $i = 2$. Since α is a loop in the space of framed curves, we must have $\delta_1(\alpha) = 0$. Hence $s = 0$ and $\alpha = \gamma_1^r \gamma_3^t$.

Now to compute the value of $\theta_{\mathcal{C}_f, V}$ on α , we only need to know the value of $e_{\mathcal{C}}$ on γ_1 and γ_3 . Generically the torus corresponding to the loop γ_1 does not intersect μ so $e_{\mathcal{C}}(\gamma_1 * [S^1 \times S^1]) = 0$. On the other hand, the torus corresponding to the loop γ_3 intersects μ transversely at one point, so $e_{\mathcal{C}}(\gamma_3 * [S^1 \times S^1]) = 2k$. Hence $\text{Im } \theta_{\mathcal{C}_f, V} = k\mathbb{Z}$.

Next we show $\text{Im } \theta_{\mathcal{K}_f, V} = \{0\}$, so suppose $\alpha \in \pi_1(\mathcal{K}_f, K_f)$. Again $\alpha = \gamma_1^r \gamma_3^t$, but now $t = 0$ as the following lemma shows.

7.3. Lemma. *The loop $\alpha = \gamma_1^r \gamma_3^t$ is homotopic to a loop in \mathcal{K} if and only if $t = 0$.*

Proof. Let K_s be a singular knot with one double point at $s \in M$ and view K_s as a pair of maps $(K_{s,1}, K_{s,2}) \in \pi_1(M, s)$. Let $\sigma(K_s) = 1$ if both $K_{s,i}$ are noncontractible and let $\sigma(K_s) = 0$ otherwise. We say a loop or path $\gamma : [0, 1] \rightarrow \mathcal{C}$ is *generic* if, whenever $\gamma(t)$ is a singular knot, $\gamma(t)$ has exactly one transverse double point and no other multiple points, and the set of times such that $\gamma(t)$ is singular is a discrete set $\{t_1, \dots, t_n\}$. In particular any loop in $\pi_1(\mathcal{C}, K)$ has a generic representative.

A transverse double point s of a singular knot can be resolved in two essentially different ways. We say that a resolution of a double point is positive (resp. negative) if the tangent vector to the first strand, the tangent vector to the second strand, and the vector from the second strand to the first form the positive 3-frame. (This does not depend on the order of the strands).

We assign a sign to each singular knot $\gamma(t_i)$ as follows: if, for $t_i < t^+ < t_{i+1}$, $\gamma(t^+)$ is obtained from the singular knot $\gamma(t_i)$ by a positive resolution of its double point, put $\epsilon(t_i) = 1$. Otherwise $\epsilon(t_i) = -1$. For any generic $\gamma : [0, 1] \rightarrow \mathcal{C}$ define $\delta_2(\gamma) = \sum_{i=1}^n \epsilon(t_i) \sigma(\gamma(t_i))$.

The set of singular knots forms the *discriminant* D in \mathcal{C} . The codimension two (with respect to D) stratum of the discriminant consists of singular knots with two distinct transverse double points. It is easy to see that $\delta_2(\alpha') = 0$, for every small generic loop α' going around the codimension two stratum. This implies (cf. Arnold [1]) that if γ is a generic loop in \mathcal{C} that starts at a nonsingular knot K , then $\delta_2(\alpha)$ depends only on the element of $\pi_1(\mathcal{C}, K)$ realized by a generic loop α .

Clearly $\delta_2(\gamma) = 0$ for any γ which is homotopic to a loop in \mathcal{K} . However $\delta_2(\gamma_3) = 2$. The knot $K = \gamma_3(0)$ crosses the fiber over the self-intersection point p of ν twice. During the homotopy $\gamma_3(t)$, these two points move along the fiber at unit speed in opposite directions because the two loops in F one gets by smoothing ν at p are orientation reversing. Therefore $\gamma_3(t)$ is singular at two times t_1 and t_2 , and $\delta_2(\gamma_3) = 2$ because $\epsilon(t_1)$ and $\epsilon(t_2)$ are equal and $\sigma(\gamma_3(t_1)) = \sigma(\gamma_3(t_2)) = 1$. The last identity holds because the two loops adjacent to a double point of singular knots $\gamma_3(t_i), i = 1, 2$ project to orientation reversing loops on F and hence are not contractible in M .

Hence γ_3 is not homotopic to a loop in \mathcal{K} .

Now we have $\delta_2(\alpha) = r\delta_2(\gamma_1) + t\delta_2(\gamma_3)$. But $\delta_2(\gamma_1) = 0$, so $\delta_2(\alpha) = 2t$. Thus α is not homotopic to a loop in \mathcal{K} unless $t = 0$. \square

Now $\theta_{\mathcal{K}_f, V}(\alpha) = r\theta_{\mathcal{K}_f, V}(\gamma_1) = r \cdot \frac{1}{2}e_C(\gamma_{1*}[S^1 \times S^1]) = 0$. This finishes the proof of Theorem 6.1.

8. EXAMPLES WHERE $\theta_{\mathcal{K}_f, V}$ DEPENDS ON \mathcal{K}_f AND NOT JUST ON \mathcal{C}_f .

Let M be a 3-manifold which is a locally-trivial S^1 -fibration $\text{pr} : M \rightarrow F$ over a closed orientable surface F of at least two. Let μ be an oriented curve that projects to the dashed curve in Figure 7.

Take C to be the contact structure such that the Poincaré dual of its Euler class is $2k[\mu] \in H_1(M)$. The fact that such C exists is proved, for example, in [17, Proposition 4.0.1] and follows from the results of Lutz [13].

Let V be the coorienting vector field of C . Let $\mathcal{K}_{1,f}$ be a component of the space of framed knots in M that contains the framed knot which is the S^1 -fiber of pr with some framing, and call this knot $K_{1,f}$.

Let $\lambda : S^1 \rightarrow F$ be the solid curve in Figure 7. Let $K_{2,f}$ be a framed knot whose projection $\text{pr} \circ K$ is λ , and assume that $K_{2,f}$ is obtained from $K_{1,f}$ by an isotopy and one passage through a transverse double point. Let $\mathcal{K}_{2,f}$ be the component of the space of framed knots that contains $K_{2,f}$. Let \mathcal{C}_f be the component of the space of framed curves that contains $K_{1,f}$ and $K_{2,f}$.

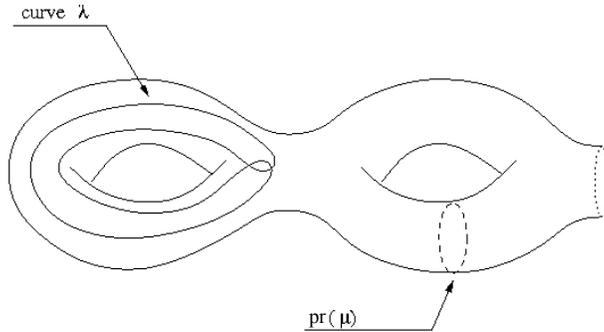


FIGURE 7.

8.1. Theorem. *Let $M, k, V, K_{1,f}, \mathcal{K}_{1,f}$, and \mathcal{C}_f be as defined above. Then $\text{Im}(\theta_{\mathcal{C}_f, V}) = k\mathbb{Z} < \mathbb{Z}$ and $\text{Im}(\theta_{\mathcal{K}_{1,f}, V}) = k\mathbb{Z} < \mathbb{Z}$. Therefore $\Gamma_{\mathcal{C}_f, V} = \mathbb{Z}_k, \Gamma_{\mathcal{K}_{1,f}, V} = \mathbb{Z}_k$, and $\bar{\Gamma}_{\mathcal{K}_{1,f}, V} = \{0\}$.*

8.2. Theorem. *Let $M, k, V, K_{2,f}$, and $\mathcal{K}_{2,f}$, be as defined above. Then $\text{Im}(\theta_{\mathcal{K}_{2,f}, V}) = \{0\} < \mathbb{Z}$ and hence $\Gamma_{\mathcal{K}_{2,f}, V} = \{0\}$, and $\bar{\Gamma}_{\mathcal{K}_{2,f}, V} = k\mathbb{Z}$.*

For the proof of Theorem 8.1 see Section 9.

For the proof of Theorem 8.2 see Section 10.

9. PROOF OF THEOREM 8.1

Assume that K is the knot homotopic to the i -th power of the S^1 -fiber of $\text{pr} : M \rightarrow F$. Let \mathcal{C} is the connected component of the space of curves containing K . Take $\rho : S^1 \rightarrow F$ to be a loop based at $\text{pr}(K(1))$.

We introduce the loop γ_ρ following [5]. Put γ_ρ to be the loop in \mathcal{C} based at K constructed as follows. First apply a homotopy ϕ (that fixes $K(1)$) so that K becomes the $(1, i)$ knot on the surface of the torus boundary of $\text{pr}^{-1}(D^2)$. Here D^2 is the small disk centered at $\text{pr}(K(1))$. Transport the disk D along the curve ρ via a path $D_{\rho(t)}$. Consider the isotopy I_ρ of the knot such that at each moment it is the $(1, i)$ knot on the boundary torus of $\text{pr}^{-1}(D_{\rho(t)})$. Now apply the homotopy ϕ^{-1} . So the loop γ_ρ is $\phi^{-1}I_\rho\phi$.

9.1. Lemma. *Every element $\alpha \in \pi_1(\mathcal{C}, K)$ can be written as $\gamma_2^s \gamma_3^t \gamma_\rho$ for some $s, t \in \mathbb{Z}$ and ρ a loop on F based at $\text{pr}(K(1))$.*

9.2. Remark. In the above Lemma, γ_ρ can in fact be any loop in $\pi_1(\mathcal{C}, K)$ such that the projection of trace of the basepoint of the knot is ρ . Similarly γ_3 can be any loop in $\pi_1(\mathcal{C}, K)$ such that the trace of the basepoint is homotopic to the fiber.

Proof. In [5, Proof of Lemma 6.11 page 809] we proved that when the surface F is not necessarily orientable, then there exist ρ, s, t such that $\alpha^2 = \gamma_2^s \gamma_3^t \gamma_\rho^2$. (Note that there is a typo in the formula proved in [5] and the term γ_3^t is missing.) Analyzing the proof we see that if the surface F is orientable then the desired statement holds true. \square

Let $\alpha \in \pi_1(\mathcal{C}_f, K_f)$, and regard α as an element of $\pi_1(\mathcal{C}, K)$ by forgetting the framing. By Lemma 9.1 we have $\alpha = \gamma_2^s \gamma_3^t \gamma_\rho$ for some $s, t \in \mathbb{Z}$ and loop ρ on F . Define a homomorphism $\delta_1 : \pi_1(\mathcal{C}, K_1) \rightarrow \mathbb{Z}$ given by the lifting correspondence for the covering $p : \tilde{\mathcal{C}}_f \rightarrow \mathcal{C}$ where we identify $0 \in \mathbb{Z}$ with the equivalence class of $K_{1,f}$ in $\tilde{\mathcal{C}}_f$. Note that $\delta_1(\gamma_3) = \delta_1(\gamma_\rho) = 0$ and $\delta_1(\gamma_2) = 2$. Since α is a loop in the space of framed curves, we must have $\delta_1(\alpha) = 0$. Hence $s = 0$ and $\alpha = \gamma_3^t \gamma_\rho$.

Now to compute the value of $\theta_{\mathcal{C}_f, V}$ on α , we only need to know the value of e_C on γ_3 and γ_ρ . The torus corresponding to the loop γ_3 does not intersect μ so $e_C(\gamma_3 * [S^1 \times S^1]) = 0$. On the other hand, the intersection index of the torus corresponding to the loop γ_ρ with μ equals to the intersection index of ρ and $\text{pr}(\mu)$. It is easy to construct ρ which has one intersection point with $\text{pr}(\mu)$. So $\text{Im} \theta_{\mathcal{C}_f, V} = k\mathbb{Z}$.

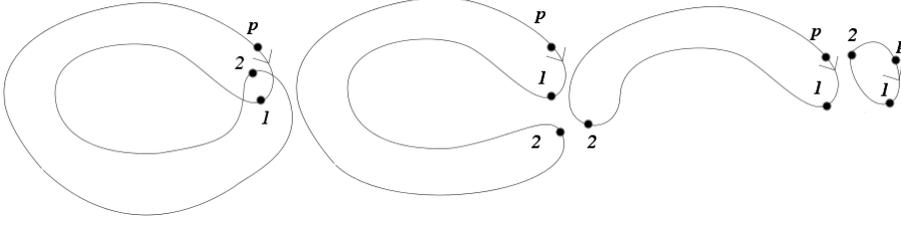


FIGURE 8. The projection of the homotopy ϕ^{-1} . In this picture p is $K_{2,f}(1)$, the point 1 is the image of $x_1 \in S^1$, and the point 2 is the image of $x_2 \in S^1$.

Now γ_3 is an isotopy and γ_ρ is an isotopy when it is based at K_1 . So $\text{Im } \theta_{\mathcal{K}_f, V} = k\mathbb{Z}$. \square

10. PROOF OF THEOREM 8.2

Let $\beta \in \pi_1(F, \text{pr}(K_{2,f}(1)))$ be the embedded loop in F that wraps one time clockwise around the left hole in Figure 8.1 and is equal to the boundary component of a regular neighborhood of the image of the projection of $K_{2,f}$ to F which contains the point $K_{2,f}(1)$.

By Lemma 9.1 every element $\alpha \in \pi_1(\mathcal{C}, K_{2,f})$ can be written as $\gamma_2^s \gamma_3^t \gamma_\rho$ for some $s, t \in \mathbb{Z}$ and ρ a loop on F based at $\text{pr}(K(1))$, and we have shown that $s = 0$ for a loop in the space of framed curves. We would like to know which such $\alpha = \gamma_3^t \gamma_\rho$ contain representatives in $\pi_1(\mathcal{K}_{2,f}, K_{2,f})$. We will prove that if ρ does not commute with β , then α does not contain such a representative. Note that if ρ and β do commute, and since β is not a power of another class, then $\rho = \beta^k$ and thus has algebraic intersection 0 with $\text{pr } \mu$. Therefore $\theta_{\mathcal{K}_{2,f}, V}(\alpha) = 0$ for any α in $\pi_1(\mathcal{K}_{2,f}, K_{2,f})$.

Our definition of γ_ρ requires a choice of homotopy ϕ (that fixes $K_{2,f}(1)$) so that $K_{2,f}$ becomes the $(1, i)$ knot on the surface of the torus boundary of $\text{pr}^{-1}(D^2)$. In the case where our basepoint is $K_{2,f}$ we can describe ϕ as in Figure 8: ϕ is the homotopy which pulls a small arc of the knot $K_{1,f}$ around a loop whose projection to F is β^{-1} , and passes this arc through one transverse double point of the knot near p .

10.1. Lemma. *Let ρ be any loop in $\pi_1(F, \text{pr } K_{2,f}(1))$ that does not commute with β . There is no representative of the homotopy class $\gamma_\rho \in \pi_1(\mathcal{C}_f, K_{2,f})$ which is contained in $\mathcal{K}_{2,f}$.*

Proof. The loop $\gamma_\rho = \phi^{-1} I_\rho \phi$ gives rise to a map of a cylinder $C : S^1 \times I \rightarrow M \times I$ given by $C(x, t) = \gamma_\rho(t)(x) \times t$. Because $\gamma_\rho(t)$ is a singular knot with one double point at two distinct values of t and the images of these two singular knots coincide, the image of $C(x, t)$ intersects itself transversely at two points $p \times t_1$ and $p \times t_2$. We assume $p = \text{pr}(K_{2,f}(1))$.

Let x_1 and x_2 be the elements of S^1 which are the preimages of the double points of the two singular knots $\gamma_\rho(t_1)$ and $\gamma_\rho(t_2)$ (we can assume the preimages of the double points at both times t_1 and t_2 are equal).

Now $x_1 \times t_1$ and $x_2 \times t_1$ are the preimages under C of $p \times t_1$, and similarly $x_1 \times t_2$ and $x_2 \times t_2$ are the preimages of $p \times t_2$.

If there is a homotopy from the loop $\gamma_\rho = \phi^{-1}I_\rho\phi$ to a loop η in $\mathcal{K}_{2,f}$ based at $K_{2,f}$ then $C : S^1 \times I \rightarrow M \times I$ is regularly and properly homotopic to an embedding. Our strategy is to show no such embedding exists.

To prove that C is not regular homotopic to an embedding, we will use an adaptation of Wall's self-intersection invariant μ for properly embedded annuli, defined by Schneiderman [15, Section 4.1]. First we recall the definition of μ , and use the notation of [15].

Let X be a 4-manifold and let $A : S^1 \times [0, 1], S^1 \times \{0, 1\} \rightarrow (X, \partial X)$ be a properly immersed annulus. Let x be a basepoint of X and let a be a basepoint of $\text{Im } A$. A *whisker* for A is a choice of path from x to a ; fix some whisker ω . For each self-intersection point p of A , the *sheets* at p are the two transversely intersecting immersed 2-disks in a small neighborhood of p in A . For each self-intersection p define a loop $g_p \in \pi_1(X, x)$ as follows: go along ω to a , go along a path in A to p , switch sheets, return to a without passing through any other double points of A , and then return to x along ω^{-1} . The loop g_p is well defined up to powers the loop $\kappa = \omega A_*(S^1 \times t_a)\omega^{-1}$ where $a \in A(S^1 \times t_a)$. Define a sign $\epsilon(p)$ by comparing the orientation of X at p with the orientation given by the two sheets of A at p . Now

$$\mu(A) = \sum_{p \in A \cap A} \epsilon(p)[g_p].$$

Let $\Lambda_\kappa = \mathbb{Z}[\pi_1(X, x)]/\{g - \kappa^n g^{\pm 1} \kappa^m\}$ where $\mathbb{Z}[\pi_1(X, x)]$ denotes the free abelian group generated by the elements of $\pi_1(X, x)$. Note that if one wants an invariant of homotopy rather than just regular homotopy one should add $\mathbb{Z}[1]$ to the denominator of the quotient; for our purposes a regular homotopy invariant is enough.

Following Wall [19], Schneiderman [15, Proposition 4.1.2] proves that $\mu(A)$, when viewed as an element of the quotient Λ_κ , is an invariant of regular homotopy, and whenever μ vanishes on A , the double points of A can be paired off with Whitney disks. In higher dimensions, because of the Whitney trick, μ vanishes if and only if A is regularly homotopic to an embedding; in dimension 4, μ vanishing is just a necessary condition for A to be regularly homotopic to an embedding.

Now we compute $\mu(C)$. To do this, we let $K_{2,f}(1) \times 0$ be both our basepoint of $M \times I = X$ our basepoint of C , so that our whisker is the trivial path. Now we define several paths in $S^1 \times [0, 1]$ which are pictured in Figure 9. We assume that if one begins at $1 \in S^1$ and moves along S^1 according to its orientation, one encounters x_1 before x_2 . Let $\sigma(x)$ denote the corresponding path in $S^1 \times 0$ from 1×0 to $x_1 \times 0$ (which does not cross $x_2 \times 0$). Let $\tau(x)$ denote the path in $S^1 \times 0$ that begins at x_1 , continues in the direction of the orientation of S^1 , and ends at x_2 . Let $y_1(t)$ denote the path in $S^1 \times I$ corresponding to fixing $x_1 \in S^1$ and letting t vary from 0 to t_1 . Let $y_2(t)$ denote the path in $S^1 \times I$ corresponding to fixing $x_1 \in S^1$ and letting t vary from t_1 to t_2 . Let $y_3(t)$ denote the path in $S^1 \times I$ corresponding

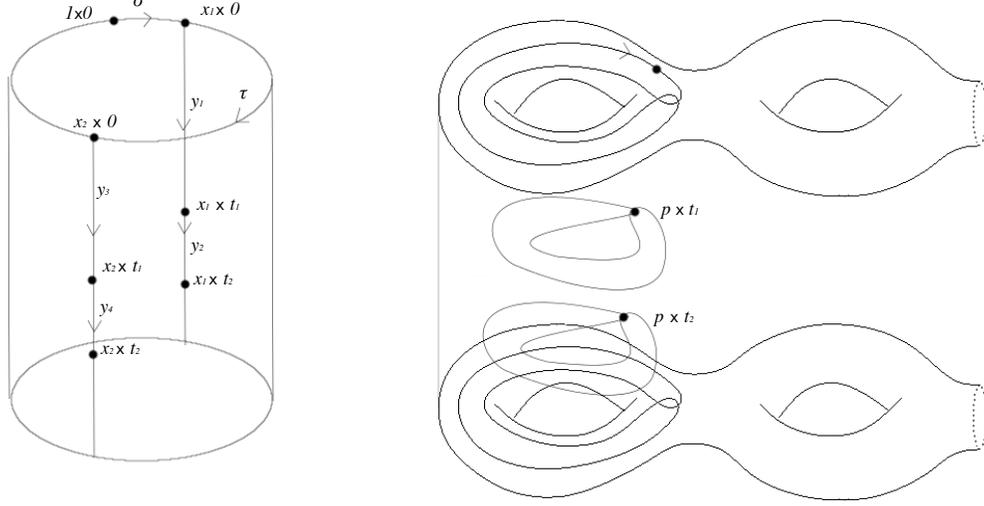


FIGURE 9. The track of the homotopy γ_ρ in $M \times [0, 1]$. Note that only the singular knots $\gamma_\rho(t_1) \times t_1$ and $\gamma_\rho(t_2) \times t_2$ are pictured.

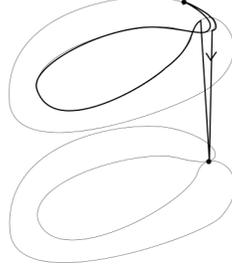


FIGURE 10. The term $g_{p \times t_1}$ projects to β^{-1} on F .

to fixing $x_2 \in S^1$ and letting t vary from 0 to t_1 . Let $y_4(t)$ denote the path in $S^1 \times I$ corresponding to fixing $x_2 \in S^1$ and letting t vary from t_1 to t_2 .

Now the term $g_{p \times t_1}$ of $\mu(C)$ corresponding to $p \times t_1$ is

$$C_*(\sigma y_1)C_*(y_3^{-1})C_*(\tau^{-1}\sigma^{-1}).$$

This loop is illustrated in Figure 10. Similarly $g_{p \times t_2}$ is

$$C_*(\sigma y_1 y_2)C_*(y_4^{-1}y_3^{-1})C_*(\tau^{-1}\sigma^{-1}).$$

Finally we show $\mu(C) \neq 0$. Suppose $g_{p \times t_1} = g_{p \times t_2} \in \Lambda_{K_{2,f}}$. Then substituting into one of the relations for $\Lambda_{K_{2,f}}$ we have $g_{p \times t_1} = K_{2,f}^n g_{p \times t_2} K_{2,f}^m$.

Let $\text{pr} : M \times I \rightarrow F$ denote the composition of the projections $M \times I \rightarrow M$ and $\text{pr} : M \rightarrow F$. Since $\text{pr}_*(K_{2,f}) = 1$, we have $\text{pr}_*(g_{p \times t_1}) = \text{pr}_*(g_{p \times t_2})$. One can check that $\text{pr}_*(g_{p \times t_1}) = \beta^{-1}$, and this is illustrated in Figure 10.

We will now compute $\text{pr}_*(g_{p \times t_2})$ and show it not equal to β^{-1} .

Now observe that $\text{pr} C_*(y_2) = \rho$ and $\text{pr} C_*(y_4) = \beta\rho\beta^{-1}$. This is because during the homotopy ϕ^{-1} , the track of the projection $\text{pr}(K_{2,f}(x_1))$ is the trivial loop, the track of the projection $\text{pr}(K_{2,f}(x_2))$ is β , and during the homotopy I_ρ , the track of both $\text{pr}(K_{2,f}(x_i))$ for $i = 1, 2$ is ρ . See Figure 8.

Therefore $\text{pr} g_{p \times t_2} = \text{pr}(C_*(\sigma y_1 y_2) C_*(y_4^{-1} y_3^{-1}) C_*(\tau^{-1} \sigma^{-1})) = \rho\beta\rho^{-1}\beta^{-1}\beta^{-1}$. This cannot be equal to β^{-1} , since β and ρ do not commute in $\pi_1(F, K_{2,f}(1))$. \square

11. V -TRANSVERSE KNOTS AND LOOSE LEGENDRIAN KNOTS

First we recall some definitions and results from Cieliebak and Eliashberg's book [7].

Consider a contact manifold (M^{2n+1}, C) and a manifold Λ^n of dimension n . A *formal Legendrian embedding* of Λ into (M, C) is a pair (f, F^s) where $f : \Lambda \rightarrow M$ is a smooth embedding and $F^s : T\Lambda \rightarrow TM$ is a homotopy of monomorphisms over f starting at $F^0 = df$ and ending at an isotropic monomorphism $F^1 : T\Lambda \rightarrow C$ covering f . A genuine Legendrian embedding f can be viewed as a formal Legendrian embedding $(f, F^s = df)$.

Two formal Legendrian embeddings are called *formally isotopic* if they are isotopic as formal Legendrian embeddings.

The following Theorem [7, Theorem 7.19 part b] is attributed to Dymara [8] and Eliashberg-Fraser [11].

11.1. Theorem. *Let (M, C) be a closed connected overtwisted 3-manifold and $D \subset M$ be an overtwisted disk.*

Let (f_t, F_t^s) , $s, t \in [0, 1]$ be a formal Legendrian isotopy in M connecting two genuine Legendrian embeddings $f_0, f_1 : S^1 \rightarrow M \setminus D$. Then there exists a Legendrian isotopy $\tilde{f}_t : S^1 \rightarrow M \setminus D$ connecting $\tilde{f}_0 = f_0$ and $\tilde{f}_1 = f_1$ which is homotopic to (f_t, F_t^s) through formal Legendrian isotopies with fixed endpoints.

Let (M, C) be a not necessarily overtwisted contact structure and let V be a coorienting vector field of the contact structure. Clearly every Legendrian knot is V -transverse and a Legendrian isotopy is a V -transverse isotopy.

We use Theorem 11.1 to obtain the following result.

11.2. Theorem. *Let (M, C) be a closed overtwisted contact manifold with an overtwisted disk D , let V be the coorienting vector field of C . Let f_1, f_2 be Legendrian knots in $M \setminus D$ that are V -transverse isotopic. Then they are Legendrian isotopic in $M \setminus D$.*

Proof. Take an auxiliary Riemannian metric so that V is orthogonal to C .

We use the V -transverse isotopy g_t from f_1 to f_2 to produce a formal Legendrian isotopy from f_1 to f_2 . Let $\bar{g}'_t(u)$ be the normalized orthogonal projection of $g'_t(u)$ to the contact plane $C_{g_t(u)}$. Let $V^t(u)$ denote the element of the vector field V at

$g_t(u)$, which again is orthogonal to $C_{g_t(u)}$. Let α_u be the angle between $g'_t(u)$ and $\tilde{g}'_t(u)$. Let

$$V^{s,t}(u) = \cos((1-s)\alpha_u)\tilde{g}'_t(u) + \sin((1-s)\alpha_u)V^t(u),$$

which is a unit vector in $T_{g_t(u)}M$. Then $V^{0,t}(u)$ is equal to $g'_t(s)$ and $V^{1,t}(u)$ is tangent to $C_{g_t(s)}$. Hence for each t , $(g_t(u), V^{s,t}(u))$ is a formal Legendrian embedding. Note that we abuse notation and let $V^{s,t}(u)$ denote the map from $TS^1 \rightarrow TM$ that sends the unit vector (giving the desired orientation of the circle) at $u \in S^1$ to $V^{s,t}(u)$. \square

Dymara [9, Theorem 4.1] proved the following result.

11.3. Theorem. *Let (M, C) be a contact manifold with an overtwisted disk D and trivializable contact bundle. Let \mathcal{K} be a connected component of the space of unframed knots, and let $f_0, f_1 \in \mathcal{K}$ be two Legendrian knots in $M \setminus D$. Assume that the following three conditions hold.*

- a:** $|K| = \infty$ for $K \in \mathcal{K}$ (see Lemma 2.1)
- b:** f_1 and f_2 are isotopic as framed knots
- c:** the rotation numbers of f_1 and f_2 with respect to some trivialization of C are equal.

Then f_1 and f_2 are isotopic as Legendrian knots.

This Theorem of Dymara in the case of closed (M, C) can be viewed as a straightforward Corollary of our Theorem 11.2 and Theorem 5.2 part 1. Note that for trivializable C the V -transverse homotopy classes of knots in \mathcal{C}_f (the connected component of the space of framed immersions containing f_0, f_1) are enumerated by the rotation number of the V -transverse knot obtained by projecting the velocity vectors of the knots to the planes of C .

The following classification of loose knots up to contactomorphism was given in the work of Etnyre [12, Theorem 1.4]. (According to [12] different proofs of this result were independently obtained by Geiges and Klukas.) Zero homologous framed knots corresponding to a given unframed knot K are enumerated by the self-linking number, which is the Thurston-Bennequin invariant tb of a Legendrian knot with the natural framing; and the sum of the tb and rot of a Legendrian knot is always odd.

11.4. Theorem. *Let (M, C) be an overtwisted contact manifold. For each null homologous knot type \mathcal{K} and a pair of integers (t, s) satisfying $(t+s)$ is odd, there is a unique, up to contactomorphism, loose Legendrian knot in \mathcal{K} satisfying $\text{tb}(K) = t$ and $\text{rot}(K) = r$.*

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