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Unfolding Smooth Prismatoids

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Abstract

We define a notion for unfolding smooth, ruled surfaces, and prove that every smooth prismatoid (the convex hull of two smooth curves lying in parallel planes), has a nonoverlapping "volcano unfolding." These unfoldings keep the base intact, unfold the sides outward, splayed around the base, and attach the top to the tip of some side rib. Our result answers a question for smooth prismatoids whose analog for polyhedral prismatoids remains unsolved.

1 Introduction

It is a long-unsolved problem to determine whether or not every convex polyhedron can be cut along its edges and unfolded flat into the plane to a single nonoverlapping simple polygon (see, e.g., [O'R00]). These unfoldings are known as *edge unfoldings* because the surface cuts are along edges; the resulting polygon is called a *net* for the polyhedron. Only a few classes of polyhedra are known to have such unfoldings: pyramids, prismoids, and domes [DO04]. Even for the relatively simple class of prismatoids, nonoverlapping edge unfoldings are not established. (All these classes of polyhedra, except domes, will be defined below.) In this paper, we generalize edge unfoldings to certain piecewise-smooth ruled surfaces,[1] and show that smooth prismatoids can always be unfolded without overlap. Our hope is that the smooth case will inform the polyhedral case.

Pyramids. We start with pyramids and their smooth analogues, cones. A *pryamid* is a polyhedron that is the convex hull of a convex *base* polygon B and a point v, the *apex*, above the plane containing the base. The *side faces* are all triangles. It is trivial to unfold a pyramid without overlap: cut all side edges and no base edge. This produces what might be called a *volcano* unfolding (it blows out the side faces around the base). Examples are shown in Fig. 1(a,b) for regular polygon bases.

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[1] A *ruled surface* is one that can be swept out by a line moving in space. A patch of a ruled surface may therefore be viewed as composed of line segments.

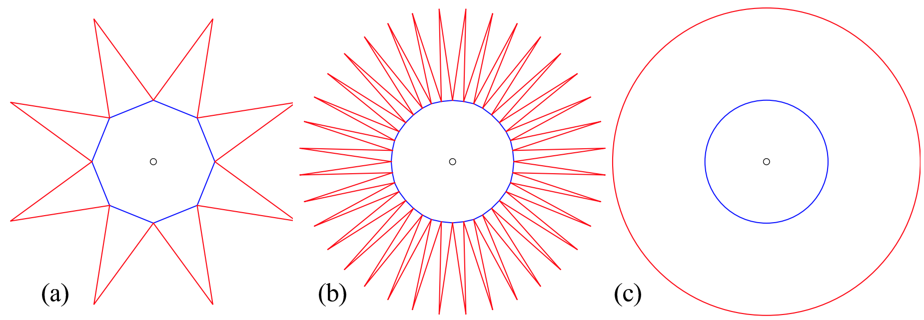


Figure 1: Unfoldings of regular pyramids (a-b) approaching the unfolding of a cone (c).

Cones. We generalize pyramids to *cones*: shapes that are the convex hull of a smooth convex curve base B lying in the xy-plane, and a point apex v above the plane. We define the volcano unfolding of a cone to be the natural limiting shape as the number of vertices of base polygonal approximations goes to infinity, and each side triangle approaches a segment *rib*. This limiting process is illustrated in Fig. 1(c). For any point $b \in \partial B$, the segment vb is unfolded across the tangent to B at b. Note that this net for a cone is no longer an unfolding that could be produced by paper, because the area increases. (For a right circular cone of unit-radius base and unit height, the surface area of the side of the cone is 2π, but the area of the unfolding annulus is $\pi(2^2 - 1^2) = 3\pi$.) In a sense that can be made precise, the density of the paper is thinned toward the tips of the spikes, so that the integral of this density is the paper area unfolded.

Truncated Pyramids. The first extension of pyramids is to *truncated pyramids* those whose apexes are sliced off by a plane parallel to the base. A cone truncation parallel to the base produces the smooth analog; see Fig. 2. The

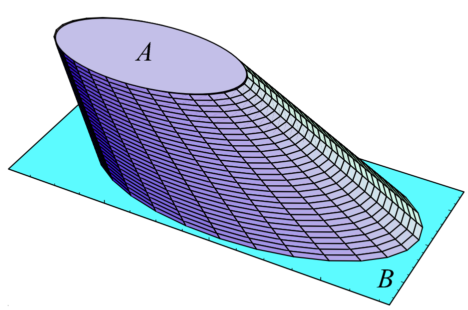


Figure 2: Cone truncated by plane parallel to base.

goal now is to perform a volcano unfolding, with the addition of attaching the top A to the top of some side face for a truncated polyhedron, or to the end of some side segment for a truncated cone. Fig. 3 shows the volcano unfolding of a truncated cone with an irregular base, and two possible locations for placement of the top, one that overlaps and one that does not. Note that the boundary of the unfolding U is not convex, a point to which we will return.

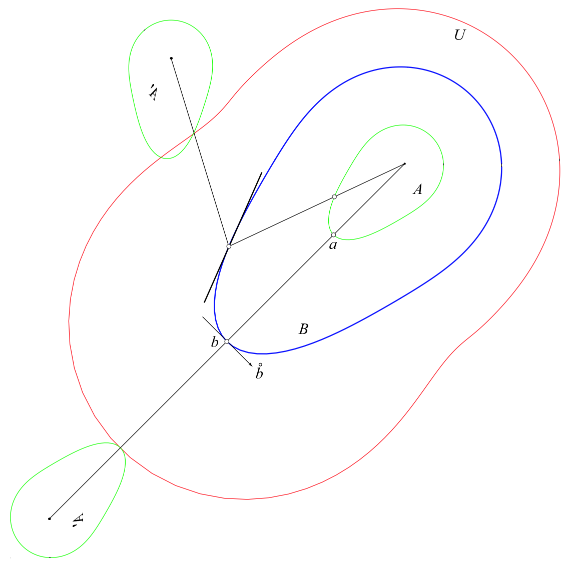


Figure 3: A volcano unfolding of a truncated cone with two possible placements of top A "flipped-out" to A'. U is the boundary of the unfolding.

The next step in generalization is to polyhedra known as *prismoids*, of which a truncated pyramid is a special case. A prismoid can be defined as the convex hull of two convex polygons A and B lying in parallel planes, with A angularly similar to B. This last condition ensures that the side faces are trapezoids, each with an edge of A parallel to an edge of B. An algorithm for edge-unfolding prismoids is available [DO04]. It is a volcano unfolding, with the top A attached to one carefully chosen side face. See Fig. 4.

Prismatoids The natural generalization of a prismoid is a *prismatoid*, the convex hull of two convex polygons A and B lying in parallel planes, with no particular relationship between A and B. As mentioned above, there is no algorithm for edge-unfolding prismatoids. One complication for a volcano unfolding is that the side faces are generally triangles, with base edges either on B or on A. Our concentration in this paper is on *smooth prismatoids*, which we define as the convex hull of two smooth convex curves A above and B below, lying in parallel planes. By *smooth* we mean C^2: possessing continuous first and second derivatives. A volcano unfolding of a smooth prismatoid unfolds every rib segment ab of the convex hull, $a \in \partial A$ and $b \in \partial B$, across the tangent to B at b, into the xy-plane, surrounding the base B, with the top A attached to one

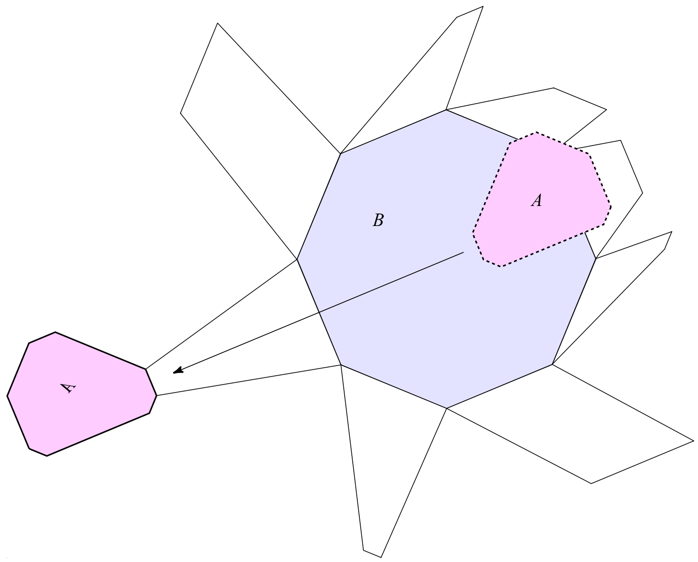


Figure 4: Overhead view of a prismoid unfolding. [From [DO04].]

appropriately chosen rib. The main result of this paper is that every smooth prismatoid has a nonoverlapping volcano unfolding (Theorem 4.7).

If a particular unfolding of a smooth prismatoid does overlap, then it can be converted into an overlapping unfolding of a polyhedral prismadoid, by sufficiently fine polygonal approximations to the base and top curves of the smooth prismatoid. (Cf. Fig. 3.) Thus, if there were a smooth prismatoid with no nonoverlapping unfolding, this would imply the same result for polyhedral prismatoids. However, the reverse implication does not hold: Our theorem does not imply that every polyhedral prismatoid can be unfolded without overlap. We do hope, however, that the smooth case will inform design of an algorithm to handle the polyhedral case.

2 Basic Properties

We use $\mathcal{P}$ to denote a smooth prismatoid. Its smooth convex base in the xy-plane is B, its smooth convex top is A, lying in a parallel plane a distance z above the xy-plane. $\mathcal{P}$ is the convex hull of $A \cup B$. We use A and B to represent the curves, and, when convenient, the regions bounded by the curves. Thus the notation $p \in A$ should be read as $p \in \partial A$, for we will not need to consider points interior to the region. A_0 is the orthogonal projection of A onto the xy-plane. We place no restriction on the relationship between A_0 and B, but it will be convenient to assume at first that $A_0 \subset B$.

We parameterize B by a function $b(t)$ such that for each $t \in [0, 2\pi]$, $b(t)$ is a point on B. We choose a parametrization so that $b(t)$ moves at unit velocity, i.e., $|\dot{b}| = 1$.[2] Similarly, A is parameterized by $a(t)$, in concert so that the rib $(a(t), b(t)) = ab$ is a segment of $\mathcal{P}$. (The dependence on t often will be suppressed in contexts where it may be inferred.) Note that the parametrization of A is controlled by that of B and the convex hull construction, and would normally result in $a(t)$ moving at a variable velocity. It is important in what follows to recognize that ab is a segment of the convex hull, and so is the intersection of a supporting plane H with $\mathcal{P}$, where H is tangent to both A and B at a and b respectively. Thus the tangents of A at a, and of B at b, are parallel, for these tangents lie in H, as well as in the planes containing A and B respectively.

One can define a *flat prismatoid* as the shape that is the limit of some prismatoid $\mathcal{P}$ of height z, as $z \to 0$. For flat prismatoids, $A_0 = A$. Flat prismatoids are in a sense the most difficult to unfold without overlap. We will start our investigation with flat prismatoids with $A \subset B$, and eventually remove the nesting condition, and later the flat restriction.

Side Unfolding. We define the unfolding of the side of $\mathcal{P}$ to be the collection of unfolded ribs ab, where each rib is unfolded by rotating it around the tangent at $b \in B$ until it lies in the xy-plane. During this rotation, the angle between the rib $ab = (a(t), b(t))$ and the tangent $\dot{b}(t)$ remains fixed, for this is the angle on the surface of $\mathcal{P}$ at $b(t)$. The unfolding rotation moves a on the rim of the base of a right circular cone whose apex is b and whose axis is parallel to $\dot{b}(t)$; see Fig. 5. For flat prismatoids, ab unfolds to a segment of the same length reflected across $\dot{b}(t)$, which amounts to rotating $180°$ around this cone.

We record the following observation, evident from Fig. 5, for later reference:

Lemma 2.1 *For any smooth prismatoid, $u(t) - a(t)$ is always orthogonal to $\dot{b}(t)$.*

Proof: The segment au lies in the circular base of the cone in Fig. 5. Any chord of the base circle is orthogonal to the axis $\dot{b}$. □

We define the locus of the images of $a(t)$, i.e., the tips of the unfolded ribs, as the *side unfolding* U of $\mathcal{P}$; the qualifier "side" will be dropped when clear from the context. U is parametrized as $u(t)$, such that $a(t)$ unfolds to $u(t)$. The unfolding of a right circular cone (Fig. 1(c)) and a truncated right circular cone are both circles, but in general the unfolding can be more complicated, as Fig. 3 adumbrates. Fig. 6 shows a more complex example.

We now argue that the smoothness of A and B implies smoothness of U.

Lemma 2.2 *$u(t)$ is a smooth function of t.*

Proof: We will not write out an explicit equation for $u(t)$ (except in the flat case, below), but we can describe the form of such an equation without computing it. The rotation shown in Fig. 5 could be written as a matrix multiplication that rotates a through the depicted angle (call it θ) about the line parallel to $\dot{b}$ through b. This would express u as a polynomial function whose terms include $\sin\theta$, $\cos\theta$, and the components of b, $\dot{b}$, and a. For smooth A and B, all these terms are themselves smooth; in particular $\theta(t)$ is smooth. And because there is

[2] The length of a vector v is $|v|$.

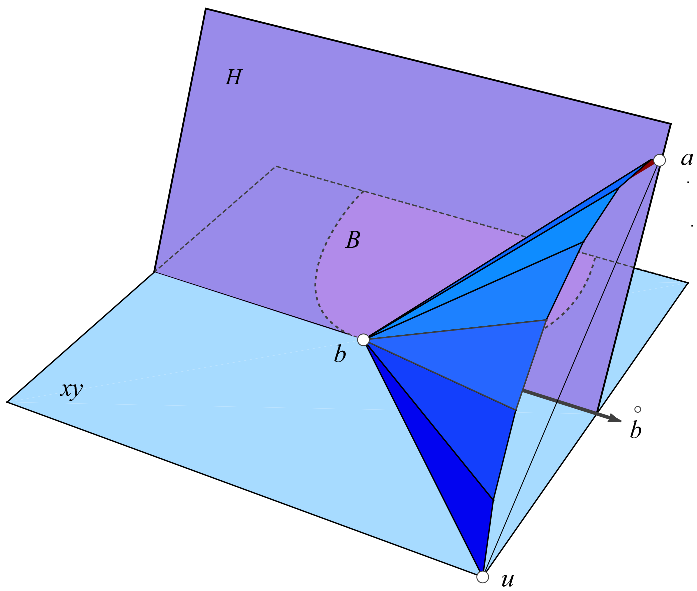


Figure 5: The unfolding of rib ab. H is the supporting plane such that $H \cap \mathcal{P} = ab$.

no division involved in the expression, $u(t)$ is a polynomial of smooth functions, and so is itself smooth. □

In particular, this means that $u(t)$ is differentiable, which is all we need in the sequel.

3 Flat Prismatoids

Throughout this entire section, we assume $\mathcal{P}$ is flat, so that $A = A_0$. We also start by assuming, in addition, that $A \subset B$.

3.1 Nonoverlap of the Unfolding

We first show that U itself does not self-overlap. (We have only found a somewhat cumbersome proof of this nearly obvious fact.)

Lemma 3.1 *For flat prismatoids, U does not self-overlap.*

Proof: Suppose to the contrary that it did. That means that there are two ribs a_1b_1 and a_2b_2 whose corresponding unfoldings u_1b_1 and u_2b_2, intersect. (Here we are using a_1 as an abbreviation for $a(t_1)$, etc.) As noted above, we must have $\dot{a}_1$ parallel to $\dot{b}_1$. Orient so that these two tangents are horizontal. By relabeling and/or reflection if necessary, we can arrange that a_2 is right of a_1. By convexity of A, a_2 must be below a_1.

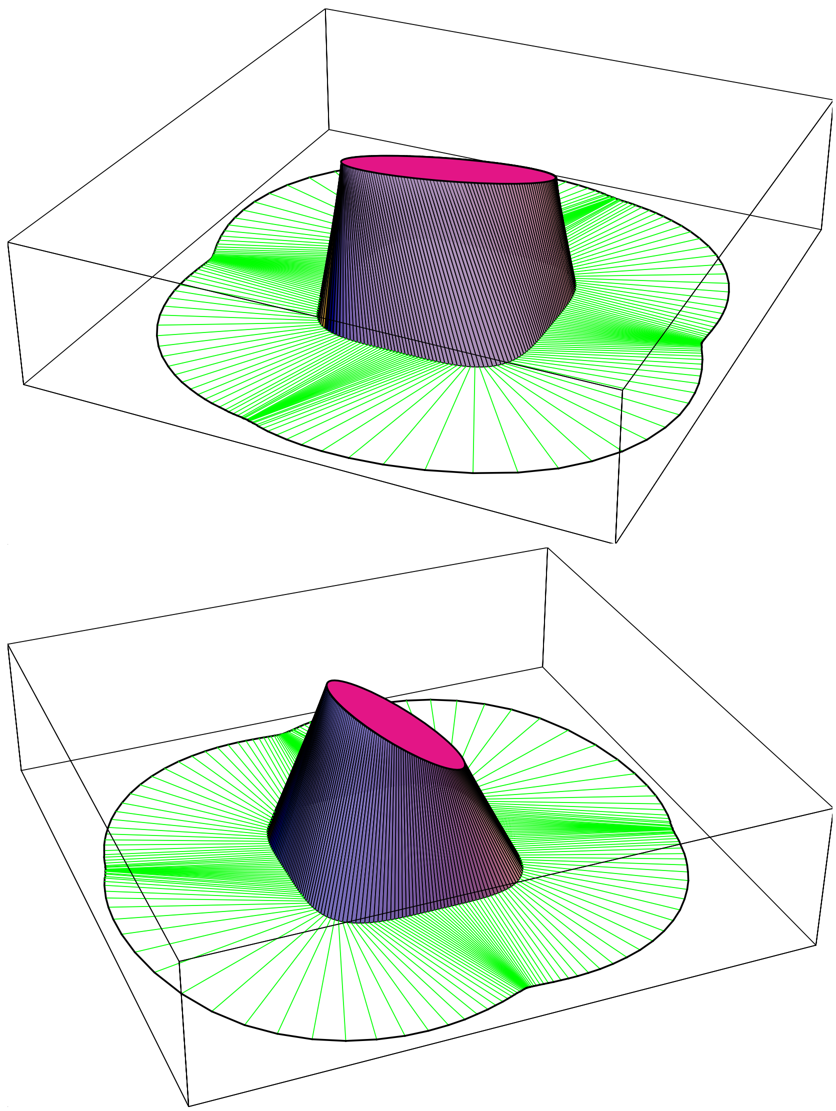


Figure 6: Two views of the side unfolding of a 3D prismatoid. The top A is an ellipse in a plane parallel to the base. (See Fig. 9a for an overhead view of a flat version of this prismatoid.)

We distinguish two cases, depending on whether b_2 is beyond the vertical clockwise around B, or not. In terms of t, the cases depend on the difference $t_1 - t_2$ (where the t angles are measured counterclockwise):

1. Case $t_1 - t_2 \in [0, \frac{\pi}{2}]$.

   See Fig. 7. Draw the line $a_2 u_2$, under the assumption that $u_1 b_1$ and $u_2 b_2$,

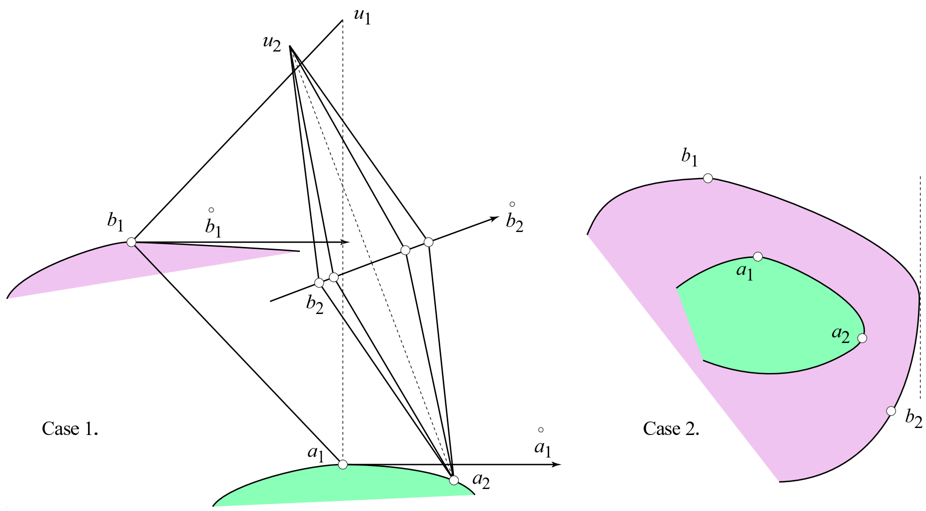


Figure 7: Construction for proof of nonoverlap of U.

   intersect. We must have u_2 left of u_1 in order to obtain this intersection. But then, regardless of where b_2 lies, the tangent $\dot{b}_2$ is turned upward (counterclockwise) with respect to $\dot{b}_1$, whereas convexity of B demands that it turn downward (clockwise). Thus, intersection of these reflected ribs is incompatible with the convexity of A and B.

2. Case $t_1 - t_2 \in [\frac{\pi}{2}, \pi]$. a_2 is also placed clockwise beyond the vertical in this case. By continuity of the tangents to the base, we need only look at the endpoints of the angular interval. If the angle between t_1 and t_2 is $\frac{\pi}{2}$, then $\dot{b}_2$ is vertical and the segment $b_2 u_2$ is restricted to the halfplane to the right of $\dot{b}_2$. But the segment $b_1 u_1$ is left of this halfplane, so there is no possibility for overlap. If the angle between t_1 and t_2 is π, then $\dot{b}_2$ is horizontal and the segment $b_2 u_2$ is restricted to the halfplane below $\dot{b}_2$. But the segment $b_1 u_1$ is above this halfplane, so there is no possibility for overlap

□

Note that this proof relies on convexity. Were either A or B nonconvex, U might well overlap. We will see later (Lemma 4.1) the lemma remains true for nonflat prismatoids.

3.2 Tangency and Overlap

Lemma 3.1 shows that the only concern for volcano unfoldings is the placement of A. If one thinks of a smooth prismatoid as a limit of approximating polyhedral prismatoids, it should be clear that A should be attached to one rib, retaining its tangent angle. More precisely, let the tangent to A at a be $\dot{a}$. In the flat case, if ab unfolds to au, then the attached A unfolds to a reflected image A' of A tangent to the reflection of $\dot{a}$. This is illustrated for two ribs in Fig. 3. In the nonflat case, we imagine a rotation of A about $\dot{a}$ until it lies in the supporting plane H that includes ab, and then a rigid rotation about the cone (cf. Fig. 5), which again places a reflected copy A' tangent to the rotated $\dot{a}$.

Because we assume A is smooth, it is arbitrarily close to its tangent in a neighborhood of any point. As we argued above, U is also smooth. Therefore, overlap between the flipout of A and U can only be avoided when the reflected tangent coincides with the tangent to U. Because the reflection of $\dot{a}$ is a reflection over $\dot{b}$, which is parallel to $\dot{a}$, we conclude that

Lemma 3.2 *A volcano unfolding avoids overlap only if A is attached to a rib ab that enjoys* mutual tangency*: $\dot{a}$ is parallel to $\dot{u}$.*

The reason this necessary condition is not sufficient to avoid overlap is that it only avoids overlap locally, in a neighborhood of the attachment point $a = u$.

3.3 Reflection Geometry.

In the flat case, the rotation illustrated in Fig. 5 becomes reflection. Because we assumed the parametrization of b is chosen such that $|\dot{b}(t)| = 1$ for all t, the lenght of the projection of $(b - a)$ onto $\dot{b}$ is just $(b - a) \cdot \dot{b}$. We can then find a vector from a to the $\dot{b}$ line as $(b - a) + \dot{b}[(b - a) \cdot \dot{b}]$, as illustrated in Fig. 8, and from this obtain u:

$$u(t) = a(t) + 2\{(b(t) - a(t)) + \dot{b}(t)[(b(t) - a(t)) \cdot \dot{b}(t)]\}.$$

This explicit equation for u makes Lemma 2.2's claim of smoothness obvious in the flat case.

We have already seen in Fig. 3 that U is not necessarily convex. To give more sense of this function, we display four examples of unfoldings of flat prismatoids in Figs. 9 and 10. The numerical computations of the derivatives we used to create these figures lead to noise which produces jagged curves and just-intersecting ribs; but with infinite precision the curves would be smooth and the ribs nonintersecting.

3.4 Mutual Tangency

3.4.1 $A \subset B$.

In light of Lemma 3.2, our goal is to find mutual tangency between $a(t)$ and $u(t)$. We find this tangency at the maximum distance (in a sense) between A and U.

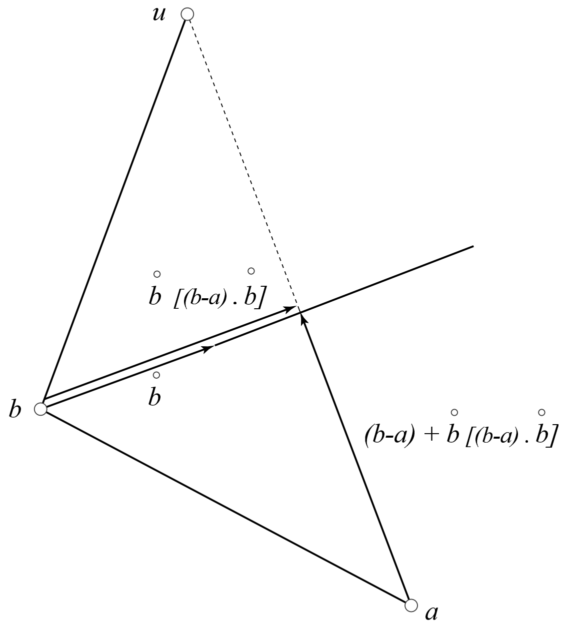
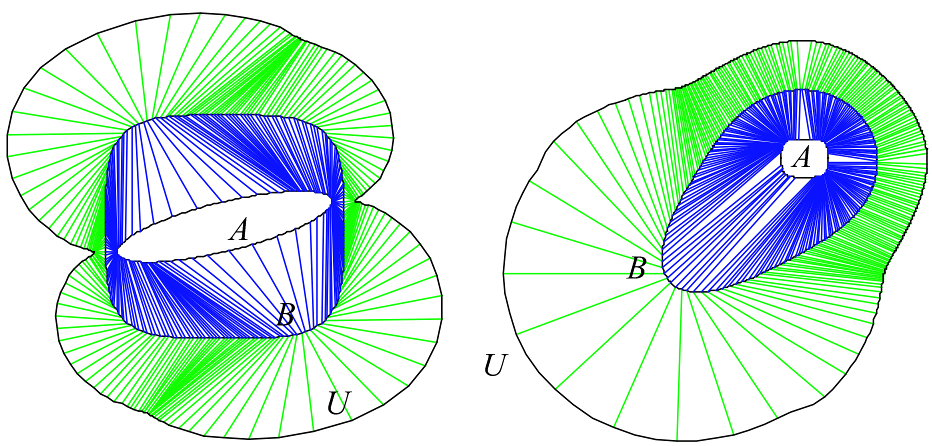


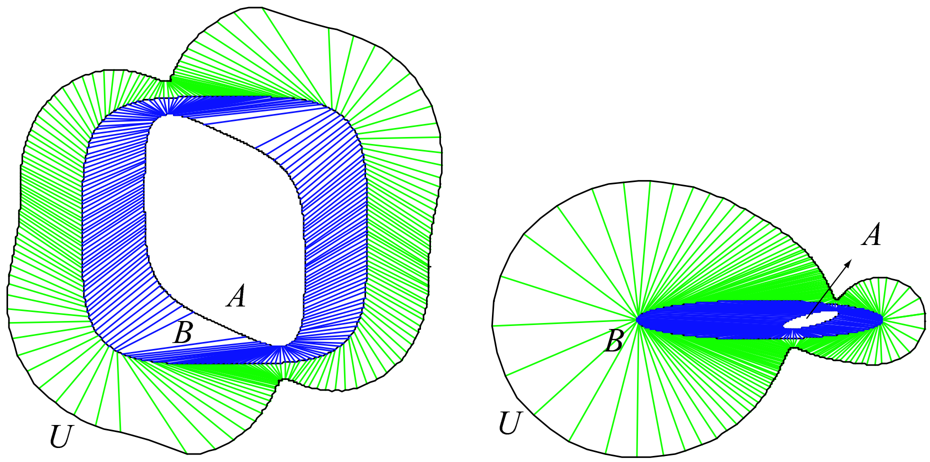
Figure 8: a reflects over $\dot{b}$ to u.



(a) Rotated ellipse inside rounded square.

(b) Rounded square inside mouse-shape.

Figure 9: Flat prismatoids.



(a) Rounded parallelogram inside rounded square.

(b) Rotated ellipse inside ellipse

Figure 10: More flat prismatoids.

Lemma 3.3 *Let $\mathcal{P}$ be a flat, smooth prismatoid whose top is nested inside its base. If $|u(\hat{t}) - a(\hat{t})|$ is at a maximum at $t = \hat{t}$, then mutual tangency occurs at $\hat{t}$.*

Proof: Let $R(t) = |u(t) - a(t)|$ be at a maximum. Then $\dot{R}(t) = 0$. Note that $|u(t) - a(t)| \neq 0$, since the top does not completely enclose the base.

$$R(t) = \sqrt{(u(t) - a(t)) \cdot (u(t) - a(t))}$$

$$\dot{R}(t) = \frac{(u(t) - a(t)) \cdot (\dot{u}(t) - \dot{a}(t))}{\sqrt{(u(t) - a(t)) \cdot (u(t) - a(t))}}$$

So,

$$\dot{R}(t) = 0 \Longleftrightarrow (u(t) - a(t)) \cdot (\dot{u}(t) - \dot{a}(t)) = 0$$

$$\begin{aligned} 0 &= (u(t) - a(t)) \cdot (\dot{u}(t) - \dot{a}(t)) \\ &= (u(t) - a(t)) \cdot \dot{u}(t) + (a(t) - u(t)) \cdot \dot{a}(t) \end{aligned} \tag{1}$$

Thus,

$$(u(t) - a(t)) \cdot \dot{u}(t) = (u(t) - a(t)) \cdot \dot{a}(t).$$

Now recall that $\dot{a}(t)$ and $\dot{b}(t)$ are always parallel by our choice of parametrization. Second, we know from Lemma 2.1 that $\dot{b}(t)$ is orthogonal to $u(t) - a(t)$ (cf. Fig. 8). Therefore, $(u(t) - a(t)) \cdot \dot{a}(t) = 0$. So we have

$$(u(t) - a(t)) \cdot \dot{u}(t) = 0$$

Because $u(t) - a(t) \neq 0$, this implies that $\dot{u}(t)$ is orthogonal to $u(t) - a(t)$. So $\dot{u}(t)$ and $\dot{a}(t)$ are both orthogonal to the same vector, and so parallel, which is our definition of mutual tangency. □

3.4.2 A crosses B.

We now remove the restriction that $A \subset B$, and permit A and B to "cross." For flat prismatoids where A and B cross, some side faces face upward, and some downward. The upward faces reflect just as before, but the downward faces do not move in the unfolding. The next lemma shows the mutual tangency established in the previous lemma still holds in this case.

Lemma 3.4 *Let $\mathcal{P}$ be a flat smooth prismatoid. If A is partially outside of B, then the global maximum of $|u(t) - a(t)|$ is on a reflected portion of the unfolding U.*

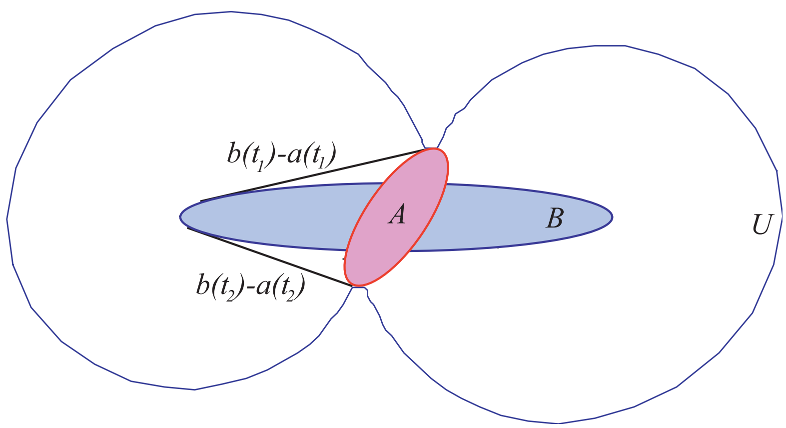


Figure 11: Bi-Tangents.

Proof: Let A be partially outside of the B. Then there is a bi-tangent to A and B at each transition from a reflected portion of the unfolding U to a nonreflected portion. See Fig.11. At any bi-tangent, we have $b(t) - a(t)$ coinciding with $\dot{a}(t)$ and $\dot{b}(t)$. Thus, $a(t)$ is reflected onto itself, so $u(t) - a(t) = 0$ at any bi-tangent. Furthermore, all nonreflected ribs have $|u(t) - a(t)| = 0$. Let U_r denote a reflected portion of the unfolding U. Then U_r is bounded by two bi-tangents, say at t_1 and t_2. The unfolding U is continuous on $[t_1, t_2]$ and differentiable on (t_1, t_2). Thus, there is a local maximum on (t_1, t_2), since $|u(t_1) - a(t_1)| = 0 = |u(t_2) - a(t_2)|$ and $|u(t) - a(t)| \geq 0$ for all $t \in [t_1, t_2]$. Comparing the lengths of the local maxima obtained from the reflected portions of the unfolding U, there is a longest such maximum which is the global maximum of $|u(t) - a(t)|$. □

Corollary 3.5 *Mutual tangency occurs at the global maximum of $|u(t) - a(t)|$, which is always on the reflected portion of the U.*

3.4.3 $A \supset B$.

Finally, we remove all restrictions on the relationship between A and B, permitting A to enclose B.

Lemma 3.6 *Let $\mathcal{P}$ be a flat prismatoid. If A completely encloses B, then the tangents to $u(t)$ match the tangents to $a(t)$ and $b(t)$ for all values of t.*
Proof: Let A completely enclose B and let $z = 0$. Then all ribs $b(t) - a(t)$ are nonreflected. So, $u(t) - a(t) = 0$ for all t. Therefore, $\dot{u}(t) = \dot{a}(t)$. This implies that the tangents to $u(t)$ match the tangents to $a(t)$ and $b(t)$ for all values of t, since $\dot{a}(t)$ and $\dot{b}(t)$ are always parallel. □

3.5 Offset Curve

We have now established that the global maximum of $|u(t) - a(t)|$ yields mutual tangency for all flat prismatoids. But, as we observed earlier, this only means that the flip-out A' does not locally overlap U. To achieve global nonoverlap, we need to prove that the global maximum is achieved at a point on the convex hull of U, for then the tangent $\dot{u}(\hat{t})$ provides a supporting line for U, separating A' from U. We establish this via an *offset curve* (or "parallel curve") for A, one displaced from A by a constant offset along the curve's normal. We first establish the nearly obvious claim that the offset of a convex curve is convex.

Lemma 3.7 *The normals to $a(t)$ are normal to any offset curve $o(t)$ of $a(t)$.*
Proof: Let $a(t)$ be parameterized by arc length. (Note, this is not the parametrization we employed before.) Let $n(t)$ be the unit normal vectors to $a(t)$. Then $n(t) = (\dot{a_2}(t), -\dot{a_1}(t))$. Let $o(t)$ be a parallel curve of A. Then $o(t) = a(t) + kn(t)$ for some constant k. So,

$$\begin{aligned}
\dot{o}(t) \cdot n(t) &= (\dot{a}(t) + k\dot{n}(t)) \cdot n(t) \\
&= \dot{a}(t) \cdot n(t) + kn(t) \cdot \dot{n}(t) \\
&= k(\dot{a_2}(t), -\dot{a_1}(t)) \cdot (\ddot{a_2}(t), -\ddot{a_1}(t)) \\
&= k(\dot{a}(t) \cdot \ddot{a}(t)) \\
&= 0, \text{ since } a(t) \text{ is parameterized by arclength.}
\end{aligned} \tag{2}$$

Thus the normals to $a(t)$ are normal to $o(t)$. □

Corollary 3.8 *If $a(t)$ is convex, then $o(t)$ is convex.*
Proof: The normals to $a(t)$ are normal to $o(t)$. So the tangents to $a(t)$ are parallel to the tangents to $o(t)$. Therefore, $o(t)$ is convex, since $a(t)$ is convex.□

Finally, we prove the global maximum is achieved on the hull of U.

Lemma 3.9 *If $|u(t) - a(t)|$ is a global maximum, $u(t)$ is on the convex hull of U.*

Proof: Let $|u(t) - a(t)|$ be a global maximum at $\hat{t}$, and let $M = |u(\hat{t}) - a(\hat{t})|$. Then there is mutual tangency at $\hat{t}$ by Lemma 3.3. Let $n(t)$ be the unit normal

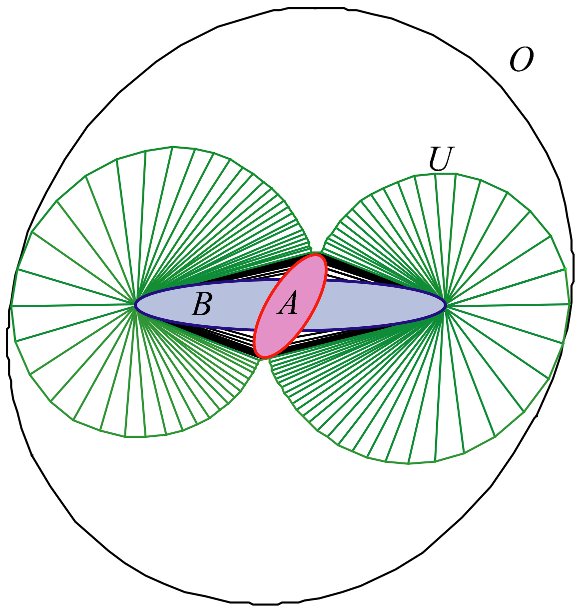


Figure 12: $u(\hat{t})$ is on the hull of U, where O touches U.

vectors to A. Then the offset curve, $o(t) = a(t) + Mn(t)$, touches U at $u(\hat{t})$, because $|u(\hat{t}) - a(\hat{t})|$ is in fact M, and $u(\hat{t}) - a(\hat{t})$ is orthogonal to $\dot{a}(\hat{t})$ by Lemma 3.7 . Therefore the curve $o(t)$ must enclose U. For suppose some point of U were outside $o(t)$. Then its orthogonal distance from A would be greater than M, contradicting M being the maximum value of $|u(t) - a(t)|$. By 3.8 $o(t)$ is convex just as A is. Take a line ℓ tangent to $o(t)$ at $\hat{t}$, i.e., a line with $o(t)$ wholly to one side, since $\dot{u}(\hat{t})$ is orthogonal to $u(\hat{t}) - a(\hat{t})$. Then ℓ is a supporting line to U. So, $u(\hat{t})$ is on the hull. □

Corollary 3.10 *There is a nonoverlapping volcano unfolding of any flat, smooth prismatoid.*

Proof: Flip out A attached to the rib $(a(\hat{t}), b(\hat{t}))$ for the $\hat{t}$ that achieves the global maximum of $|u(t) - a(t)|$. Lemma 3.2 guarantees mutual tangency, and Lemma 3.9 guarantees this tangency occurs on the hull of U. □

4 Nonflat Smooth Prismatoids

We have concentrated on flat prismatoids for two reasons: the geometric relationships are clearer, and in some sense, flat prismatoids are the most difficult. Roughly speaking, lifting A of a flat prismatoid to $z > 0$ rounds out the unfolding U, and maintains all the key relationships we need. If we imagine a continuous lifting from $z = 0$, then initially $u = u_0$, and then u moves out along the line through a_0 and u_0. See Fig. 13. This is because the lifting can be seen as a widening of the cone shown in Fig 5, while maintaining the cone base cutting the xy-plane along the same line. These relationships permit the extension of Lemma 3.1:

Lemma 4.1 *For nonflat prismatoids, U does not self-overlap.*
Proof: The collinearity illustrated in Fig. 13 implies that the geometric situation illustrated in Fig. 7 for the flat case still holds with only minor variation. In particular, we still have triangles $\triangle a_1 b_1 u_1$ etc. as in that figure, but now the triangles are extended out to the true u_1 (i.e., $u(t_1)$ rather than the projected/flat $u_0(t_1)$; cf. Fig. 13), etc. Because our reasoning in Lemma 3.1 only depended on the triangles, and not that they were isosceles, all else remains the same, and the assumption of overlap again contradicts convexity of A and B.□

4.1 Distance Maximum and Mutual Tangency

The central burden of this section is to show that the global maximum of $|u-a|$ for the nonflat case is achieved at the same $\hat{t}$ as in the flat case. We prove this in two steps: from $|u_0 - a_0|$ to $|u - a_0|$ (Lemma 4.2), and from $|u - a_0|$ to $|u - a|$ (Lemma 4.3).

Lemma 4.2 *Let $\mathcal{P}$ be a nonflat smooth prismatoid with the projection of A nested in B. Let $u_0(\hat{t})$ denote the unfolding obtained from the projection of $a(\hat{t})$ onto the xy-plane of the base at time $\hat{t}$. This projection will be denoted by $a_0(\hat{t})$. If $|u_0(\hat{t}) - a_0(\hat{t})|$ is a global maximum, then $|u(\hat{t}) - a_0(\hat{t})|$ is a global maximum.*

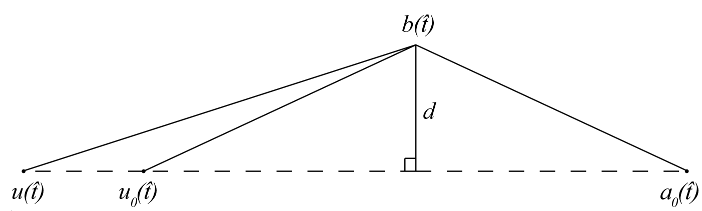


Figure 13: Lifting A to $z > 0$ maintains collinearity of u, u_0, and a_0.

Proof: Let $|u_0(\hat{t}) - a_0(\hat{t})|$ be a global maximum at $\hat{t}$. Let ℓ denote the line through $u(\hat{t})$, $u_0(\hat{t})$, and $a_0(\hat{t})$ and let d denote the altitude displayed in Fig. 13. Then $d = \sqrt{|b(\hat{t}) - a_0(\hat{t})|^2 - (\frac{1}{2}|u_0(\hat{t}) - a_0(\hat{t})|)^2}$.
By Figure 13, we see that $|u(\hat{t}) - u_0(\hat{t})| = |proj_\ell(u(\hat{t}) - b(\hat{t}))| - |proj_\ell(u_0(\hat{t}) - b(\hat{t}))|$. We can further expand this expression by using the following two equalities:

$$|proj_\ell(u(\hat{t}) - b(\hat{t}))| = \sqrt{|u(\hat{t}) - b(\hat{t})|^2 - d^2} \tag{3}$$

$$= \sqrt{|b(\hat{t}) - a(\hat{t})|^2 - d^2} = \sqrt{|b(\hat{t}) - a_0(\hat{t})|^2 + z^2 - d^2} \tag{4}$$

$$|proj_\ell(u_0(\hat{t}) - b(\hat{t}))| = \sqrt{|u_0(\hat{t}) - b(\hat{t})|^2 - d^2} = \sqrt{|b(\hat{t}) - a_0(\hat{t})|^2 - d^2}. \tag{5}$$

So,

$$
\begin{aligned}
|u(\hat{t}) - u_0(\hat{t})| &= \sqrt{|b(\hat{t}) - a_0(\hat{t})|^2 + z^2 - d^2} - \sqrt{|b(\hat{t}) - a_0(\hat{t})|^2 - d^2} \\
&= \sqrt{|b(\hat{t}) - a_0(\hat{t})|^2 + z^2 - (|b(\hat{t}) - a_0(\hat{t})|^2 - (\frac{1}{2}|u_0(\hat{t}) - a_0(\hat{t})|)^2)} \\
&- \sqrt{|b(\hat{t}) - a_0(\hat{t})|^2 - (|b(\hat{t}) - a_0(\hat{t})|^2 - (\frac{1}{2}|u_0(\hat{t}) - a_0(\hat{t})|)^2)} \\
&= \sqrt{z^2 + \frac{1}{4}|u_0(\hat{t}) - a_0(\hat{t})|^2} - \sqrt{\frac{1}{4}|u_0(\hat{t}) - a_0(\hat{t})|^2}
\end{aligned}
\tag{6}
$$

Thus,

$$
\begin{aligned}
|u(\hat{t}) - a_0(\hat{t})| &= |u(\hat{t}) - u_0(\hat{t})| + |u_0(\hat{t}) - a_0(\hat{t})| \\
&= \sqrt{z^2 + \frac{1}{4}|u_0(\hat{t}) - a_0(\hat{t})|^2} + \frac{1}{2}|u_0(\hat{t}) - a_0(\hat{t})|
\end{aligned}
\tag{7}
$$

Because z is fixed independent of t, $|u(\hat{t}) - a_0(\hat{t})|$ is a global maximum. □

Lemma 4.3 *If $|u(\hat{t}) - a_0(\hat{t})|$ is a global maximum, then $|u(\hat{t}) - a(\hat{t})|$ is a global maximum.*

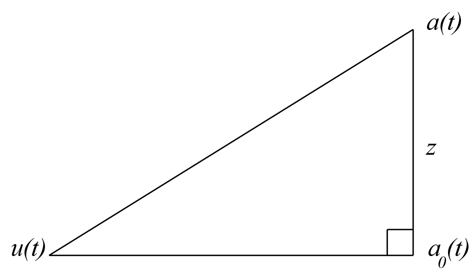


Figure 14: A lifted by z.

Proof: Let $|u(\hat{t}) - a_0(\hat{t})|$ be a global maximum at $\hat{t}$. Recall that (see Fig. 14)

$$|u(\hat{t}) - a(\hat{t})| = \sqrt{|u(\hat{t}) - a_0(\hat{t})|^2 + z^2},$$

which is a global maximum since z is fixed for all t. □

Lemma 4.4 *Consider a nonflat smooth prismatoid of height $z > 0$. If A is partially outside of B, then there is a global maximum of $|u(t) - a_0(t)|$ on the reflected portion of the unfolding U.*

Proof: Let A be partially outside of B. Then the projection A_0 is partially outside of B. By Lemma 3.4, the global maximum of $|u_0(t) - a_0(t)|$ is on a reflected portion of U_0 say at $\hat{t}$. So, $|u(\hat{t}) - a_0(\hat{t})|$ is a global maximum. (Note that this implies that $|u(\hat{t}) - a(\hat{t})|$ is a global maximum by Lemmas 4.2 and 4.3.) □

Lemma 4.5 *If A completely encloses B and $z > 0$, then mutual tangency occurs at the global maximum of $|u(t) - a_0(t)|$.*
Proof: Let A completely enclose B and let $R(t) = |u(t) - a_0(t)|$ obtain its global maximum at $\hat{t}$. Then $\dot{R}(\hat{t}) = 0$. Note that $|u(t) - a_0(t)| \neq 0$, since A_0 and B are noncrossing.

$$R(t) = \sqrt{(u(t) - a_0(t)) \cdot (u(t) - a_0(t))}$$

$$\dot{R}(t) = \frac{(u(t) - a_0(t)) \cdot (\dot{u}(t) - \dot{a}_0(t))}{\sqrt{(u(t) - a_0(t)) \cdot (u(t) - a_0(t))}}$$

So,

$$\dot{R}(\hat{t}) = 0 \Longleftrightarrow (u(\hat{t}) - a_0(\hat{t})) \cdot (\dot{u}(\hat{t}) - \dot{a}_0(\hat{t})) = 0$$

$$\begin{aligned} 0 &= (u(\hat{t}) - a_0(\hat{t})) \cdot (\dot{u}(\hat{t}) - \dot{a}_0(\hat{t})) \\ &= (u(\hat{t}) - a_0(\hat{t})) \cdot \dot{u}(\hat{t}) + (a_0(\hat{t}) - u(\hat{t})) \cdot \dot{a}_0(\hat{t}) \end{aligned} \tag{8}$$

Thus,

$$(u(\hat{t}) - a_0(\hat{t})) \cdot \dot{u}(\hat{t}) = (u(\hat{t}) - a_0(\hat{t})) \cdot \dot{a}_0(\hat{t}).$$

But since $u(\hat{t}) - a_0(\hat{t}) \neq 0$,

$$\dot{u}(\hat{t}) = \dot{a}_0(\hat{t}).$$

But $\dot{a}_0(\hat{t})$ is parallel to $\dot{a}(\hat{t})$, so mutual tangency must occur here at $\hat{t}$. □

Fig. 15 illustrates that $\hat{t}$ achieving the maximum for the flat case corresponds to maximum in the nonflat case.

4.2 Offset Curve

We again use an offset of A to prove that the above identified maximum occurs on the convex hull of U.

Lemma 4.6 *If $|u(t) - a_0(t)|$ is a global maximum at $\hat{t}$, then $u(\hat{t})$ is on the convex hull of U.*

Proof: Let $o(t) = a_0(t) + |u(\hat{t}) - a_0(\hat{t})| n(t)$. Because $u(\hat{t}), u_0(\hat{t}), a_0(\hat{t})$ are collinear, and since $u_0(t) - a_0(t)$ is orthogonal to $\dot{a}_0(t)$, we have $u(\hat{t})$ orthogonal to $\dot{a}_0(\hat{t})$. As in Lemma 3.9, $o(t)$ touches U at $u(\hat{t})$ and must enclose $u(t)$. We again take a line ℓ tangent to $o(t)$ at $\hat{t}$, i.e., a line with $o(t)$ wholly to one side. Then ℓ is a supporting line to U. So, $u(\hat{t})$ is on the hull of U. □

Theorem 4.7 *There is a nonoverlapping volcano unfolding of any smooth prismatoid.*

The proof is identical to that of Corollary 3.10, relying on the corresponding nonflat lemmas.

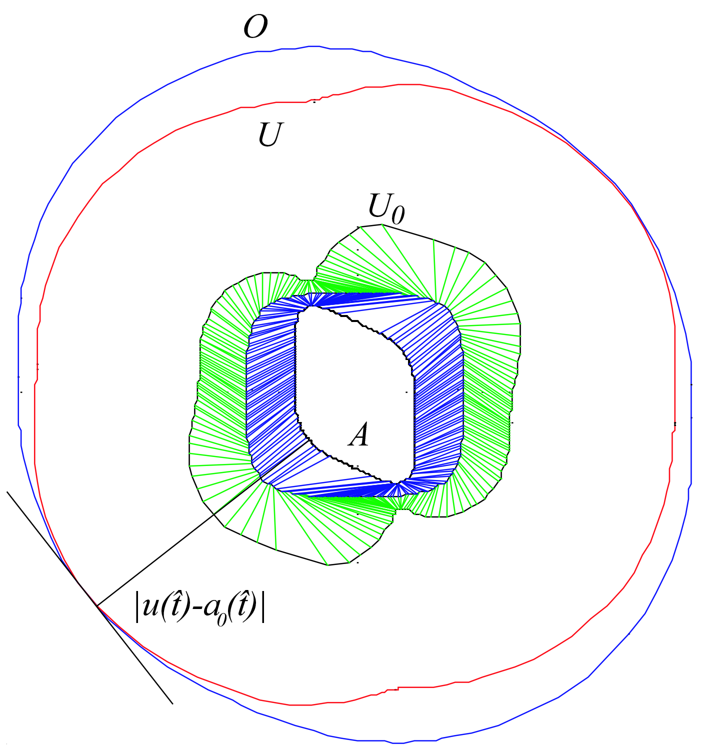


Figure 15: U_0 is the unfolding of the flat prismatoid, U the unfolding of a lifting of A to form a nonflat prismatoid. O is the offset of A.

5 Discussion

We have established that, for any smooth prismatoid $\mathcal{P}$, there is always at least one spot to flip out the top A so that it does not overlap with U, thus producing a nonoverlapping volcano unfolding. We know of examples where there are only two such "safe" flip-out spots, symmetrically placed equal global maxima of $|u - a|$. We hope to use our analysis of smooth prismatoids to answer the question of whether or not every polyhedral prismatoid has a nonoverlapping volcano edge-unfolding.

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