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Computational Geometry Column 43

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Abstract

The concept of pointed pseudo-triangulations is defined and a few of its applications described.

A *pseudo-triangle* is a planar polygon with exactly three convex vertices. Each pair of convex vertices is connected by a reflex chain, which may be just one segment. Thus, a triangle is a pseudo-triangle. A *pseudo-triangulation* of a set S of n points in the plane is a partition of the convex hull of S into pseudo-triangles using S as a vertex set. Under the name of "geodesic triangulations," these found use in ray shooting [CEG$^+$94], and then for visibility algorithms by Pocchiola and Vegter [PV95], who named them (after a dual relationship to pseudoline arrangements) and studied many of their properties [PV96]. Recently the identification of *minimum pseudo-triangulations* by Streinu [Str00] has generated a flurry of new applications that we selectively sample here.

Minimum pseudo-triangulations have the fewest possible number of edges for a given set S of points. The term *pointed pseudo-triangulation* is gaining prominence because of the following theorem[1] (see Fig. 1):

Theorem 1 *A pseudo-triangulation of S is minimum iff every point $p \in S$ is* pointed *in the sense that its incident edges span less than π (i.e., they fall within a cone apexed at p with aperture angle $< \pi$).*

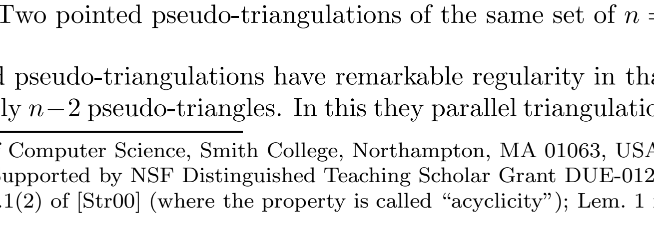

Figure 1: Two pointed pseudo-triangulations of the same set of $n = 10$ points.

Pointed pseudo-triangulations have remarkable regularity in that each contains exactly $n-2$ pseudo-triangles. In this they parallel triangulations of simple

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[1] Thm. 3.1(2) of [Str00] (where the property is called "acyclicity"); Lem. 1 in [KKM$^+$01].

polygons and are unlike triangulations of point sets, which may have from $n-2$ to $3(n-2)$ triangles. Another indication of their well-behavedness is the recent result [KKM$^+$01] that every set of n points in general position has a pointed pseudo-triangulation with maximum vertex degree of 5, a property not enjoyed by triangulations (of either polygons or point sets). This bounded degree may prove useful for designing data structures employing pseudo-triangulations, such as those used for collision detection [KSS02].

We mention two disparate applications of pointed pseudo-triangulations. Two years ago a long-standing open problem was settled by Connelly, Demaine, and Rote when they proved that polygonal linkages cannot lock in the plane: open polygonal chains can be straightened, and closed polygonal chains can be convexified [CDR00] [O'R00]. Their unlocking motions are *expansive*: no distance between any pair of vertices decreases. Their proof relies on proving the existence of a solution to differential equations describing this motion. Although not difficult to solve numerically in practice,[2] theirs does not constitute an algebraic proof. This was provided by Streinu [Str00]. She proved that a pointed pseudo-triangulation is rigid when viewed as a bar and joint framework, and that removal of a convex hull edge yields a mechanism with one degree of freedom (1-dof) that executes an expansive motion. Adding bars to a given collection of polygonal chains to produce a pointed pseudo-triangulation, and using a useful "flipping" property of pseudo-triangulations, leads ultimately to a piecewise-algebraic straightening/convexifying motion. These 1-dof expansive motions have a rich structure that is just now being elucidated [RSS01].

Second, pointed pseudo-triangulations were used to make an advance on an open problem posed by Urrutia [Urr00]: Can a simple polygon of n vertices be illuminated by cn interior π-floodlights placed at vertices, with $c < 1$ (for sufficiently large n)? For several years the lower bound on c was $3/5$, but no one could prove that c was smaller than 1. Starting from a pointed pseudo-triangulation of the polygon, Speckmann and Tóth have recently established [ST01] that $(2/3)n$ lights suffice, finally breaking the $c = 1$ barrier.[3]

We close with an open problem, posed in [RRSS01]. For a given set S of points, let $\#T$ be the number of triangulations of S, and $\#PPT$ be the number of pointed pseudo-triangulations of S. They conjecture that $\#T \le \#PPT$ for all S, with equality only when the points of S are in convex position. This has been established for all sets of at most 9 points [BKPS01].

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[3] See `http://cs.smith.edu/~orourke/TOPP/`, Problem 23, for further developments.

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