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Abstract

We construct a classical dynamical system whose phase space is a certain infinite-dimensional Grassmannian manifold, and propose that it is equivalent to the large N limit of two-dimensional QCD with an $O(2N+1)$ gauge group. In this theory, we find that baryon number is a topological quantity that is conserved only modulo 2. We also relate this theory to the master field approach to matrix models.

Keywords: Two-dimensional QCD; classical dynamical systems; topological solitons; baryon number; Grassmannian manifold; conservation law; 't Hooft equation for mesons; matrix model; master field.

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1 Introduction

An interesting type of a theory of the strong interactions is one in which there is a topological quantity which can be related to hadrons; several theories relate such a quantity to baryon number. The plausibility of such theories may be enhanced by their being consistent with the well-accepted gauge theory of the strong interactions - i.e. with QCD.

One paper by Rajeev proposes such a theory in two dimensions, and shows its equivalence to the large N limit of $SU(N)$ QCD [1]. The theory is a classical one in which the phase space is an infinite dimensional Grassmannian manifold, and baryon number is the topological invariant that corresponds to the $\mathbf{Z}$-many connected components of the manifold. Rajeev's theory is tied into the gauge theory of the strong interactions via the equation for the mass of the mesons, which agrees with the one derived by 't Hooft in the large N limit of $SU(N)$ gauge theory [2].

Here, we propose an analog of Rajeev's theory for the large N limit of $O(2N+1)$ QCD in two dimensions[1], and arrive at an interesting consequence regarding baryon number non-conservation. For our phase space, we construct a different infinite dimensional Grassmannian manifold for which the meson equation is the same as the one arrived at in the $O(2N+1)$ gauge theory. Our phase space has a topological invariant of its own, which also corresponds to baryon number. Unlike the $SU(N)$ case, in this case there are only two connected components; particles with even baryon number are assigned to one component, and those with odd baryon number to the other component. We thereby discover that baryon number is conserved only modulo two, and baryons may annihilate or be created in pairs. This fact, as we will explain, is independently true of QCD with an $O(2N+1)$ gauge group.

[1]A different analog was considered in [3, 4].

We also show that our theory is related to the master field approach to large N matrix models via a master field whose commutation relations match those of our theory.

This paper is organized as follows. In Section 2 we review 't Hooft's derivation of the equation for mesons in two dimensions in the context of the large N limit of $U(N)$ QCD. In Section 3, we review the theory developed by Rajeev, concentrating on the derivation of the meson equation and the topological invariant corresponding to baryons. In Section 4 we present our analog for QCD with an orthogonal gauge group along with the topological invariant corresponding to baryons. In Section 5 we show that our result about baryon number non-conservation agrees with two-dimensional QCD with an odd orthogonal gauge group. In Section 6 we discuss the relation between our model and the master field approach to matrix models for large N QCD. We conclude in Section 7 with several suggestions for further work.

2 Planar diagrams and mesons in two dimensions

The planar diagram theory developed by 't Hooft [5] and elaborated upon by Witten [6] provided a simplification of QCD gauge theory. 't Hooft considered QCD with color gauge group $SU(N)$ in the limit $N \to \infty$ with $g^2 N$ held fixed, and arrived at the notion that in this limit, only planar diagrams need to be considered, all others being suppressed by factors of 1/N. Still, however, calculations remain complicated even when they include only the planar diagrams.

For the case of two-dimensional QCD, 't Hooft showed that a further simplification arises, and derived an equation for the meson spectrum [2]. The derivation is briefly reviewed as follows:

We start with the QCD lagrangian

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \overline{\psi}(i\gamma^\mu D_\mu + m)\psi \tag{1}$$

where $F_{\mu\nu}$ is the field strength, D_μ the covariant derivative, ψ the quark wave function, and m the quark mass. In two dimensions, the index μ runs over $\{0, 1\}$. Switching to light-cone coordinates, where $x^\pm = (x^1 \pm x^0)/\sqrt{2}$, $p_\pm = (p_1 \pm p_0)/\sqrt{2}$, $A_\pm = (A_1 \pm A_0)/\sqrt{2}$, and $g_{ab} = \delta_{ab} - 1$, we impose the gauge $A_- = A^+ = 0$ and the lagrangian simplifies to:

$$\mathcal{L} = -\frac{1}{2} Tr(\partial_- A_+)^2 - \overline{\psi}(i\gamma^\mu \partial_\mu + m + g\gamma_- A_+)\psi. \tag{2}$$

There is only one vertex, the gauge fields do not interact with themselves, and the feynman rules simplify considerably.

To derive the equation for the meson, we first need the dressed propagator, $G(k, m)$, for the quark. Let $i\Gamma(k)$ be the amplitude for the (planar) irreducible quark self-energy diagram. Then $G(k, m)$ is a sum of diagrams created from the irreducible self-energy blob, and is equal to

$$G(k, m) = \frac{-ik_-}{m^2 + 2k_+k_- - k_-\Gamma(k) - i\epsilon}. \tag{3}$$

An expression for $\Gamma(k)$ can be derived from a bootstrap equation from which a logarithmic UV divergence is removed and into which an IR cutoff λ is introduced. The expression is

$$\Gamma(k) = \Gamma(k_-) = -\frac{g^2}{\pi}\left(\frac{sgn(k_-)}{\lambda} - \frac{1}{p_-}\right),$$

and using it in equation (3) gives

$$\begin{aligned} G(k, m) &= \frac{-ik_-}{m^2 + 2k_+k_- - g^2/\pi + g^2|k_-|/\pi\lambda - i\epsilon} \\ &= \frac{-ik_-}{M^2 + 2k_+k_- + g^2|k_-|/\pi\lambda - i\epsilon}, \end{aligned} \tag{4}$$

where $M^2 = m^2 - g^2/\pi$. Since λ is small, we see that the poles of $G(k)$ occur at $k_+ \to \infty$, which means there is no physical single quark state - no free quarks.

To find the spectrum for mesons, we consider a blob out of which come a quark and an antiquark.

One can derive a bootstrap equation for such a blob. Let $\psi(p, r)$ represent a blob out of which go a quark with mass m_1 and momentum p and an antiquark of mass m_2 and momentum $p - r$. Then a bootstrap equation for it is given by

$$\psi(p, r) = -\int \frac{d^2k}{(2\pi)^2 i} 4g^2 G(p - r, m_2) G(p, m_1) \frac{1}{k_-^2} \psi(p + k, r). \tag{5}$$

Defining $\phi(p_-, r) = \int dp_+ \psi(p_+, p_-, r)$, and substituting the expressions for $G(k, m)$ from equation (4), we get after some algebra

$$\mu^2 \phi(x) = \left(\frac{\alpha_1}{x} + \frac{\alpha_2}{1 - x} \right) \phi(x) - P \int_0^1 \frac{\phi(y)}{(y - x)^2} dy, \tag{6}$$

where $\mu^2 \pi / g^2 = r \cdot r = -2 r_+ r_-$ (i.e. μ is the meson mass in units of $g/\sqrt{\pi}$), $\alpha_i = \pi m_i^2 / g^2 - 1$, $x = p_-/r_-$, "P" stands for the principal value integral, i.e.

$$P \int \frac{\phi(y)}{(y - x)^2} dy = \frac{1}{2} \int \frac{\phi(y + i\epsilon)}{(y + i\epsilon - x)^2} dy + \frac{1}{2} \int \frac{\phi(y - i\epsilon)}{(y - i\epsilon - x)^2} dy,$$

which is finite, and where the infrared cutoff λ has disappeared. Equation (6) is known as 't Hooft's integral equation for mesons. The spectrum (i.e. the eigenvalues μ^2) can be shown to be a discrete set of positive eigenvalues if $m_i^2 > 0$.

When we replace $U(N)$ with $O(M)$ as the gauge group (where M may be even or odd), the derivation of the integral equation is the same so the same equation holds for mesons in two-dimensional QCD with an $O(M)$ gauge group.

3 $SU(N)$ QCD on a Grassmannian Manifold

Now we review the main ingredients of Rajeev's construction of the large N limit of two-dimensional QCD as a classical dynamical system [1].

There are three necessary ingredients for a classical dynamical system: a phase space, a symplectic form, and a Hamiltonian. Here, the phase space is given by an infinite dimensional Grassmanian manifold, described in two equivalent ways. The first description is the following set of operators on a polarized Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$:

$$Gr_1 = \{\Phi \mid \Phi^\dagger = \Phi \ \ ; \ \Phi^2 = 1 \ \ ; \ \ [\epsilon, \Phi] \text{ is Hilbert-Schmidt}\}, \tag{7}$$

where $\epsilon = \pm 1$ on $\mathcal{H}_\pm$, and an operator A is Hilbert-Schmidt if $tr(A^T A) < \infty$. The Hilbert space is taken to be the space of square integrable complex valued functions on the real line (or the circle, with radius taken to infinity); it is spanned by Fourier modes: $F(\theta) = \sum_{-\infty}^{\infty} F_m e^{im\theta}$; $\mathcal{H}_+$ is defined as the span of $\{e^{im\theta},\, m \geq 0\}$, and $\mathcal{H}_-$ is its orthogonal complement.

It will be convenient at times to rewrite this description in terms of $M = \Phi - \epsilon$:

$$Gr_1 = \{M \mid M^\dagger = M \ \ ; \ M^2 + \epsilon M + M\epsilon = 0 \ \ ; \ \ [\epsilon, M] \text{ is Hilbert-Schmidt}\}. \tag{8}$$

The second description of the space, equation (8), comes from the following realization: each $\Phi \in Gr_1$ can be diagonalized to ϵ via the action of a unitary transformation g on the same Hilbert space: $g\phi g^\dagger = \epsilon$, where by "unitary" we mean that g is an element in $U_{res}(\mathcal{H})$ defined by

$$U_{res}(\mathcal{H}) = \{g \,|\, g^\dagger g = 1; [\epsilon, g] \ \text{ is Hilbert-Schmidt}\} \,. \tag{9}$$

The stabilizer of ϵ under this action of the unitary group is $U(\mathcal{H}_+) \times U(\mathcal{H}_-)$. Therefore, we may view Gr_1 as the coset space

$$Gr_1 = \frac{U_{res}(\mathcal{H})}{U(\mathcal{H}_+) \times U(\mathcal{H}_-)} \ , \tag{10}$$

which can be recognized as an infinite dimensional analog of finite dimensional Grassmannian spaces such as $U(n)/[U(r) \times U(n-r)]$.

To derive equations of motion, we need a symplectic form - which would give the Poisson Brackets - and a Hamiltonian. The symplectic form on this space is given by

$$\omega(U,V) = -\frac{i}{8} Tr \Phi [U(\Phi), V(\Phi)] \ , \tag{11}$$

where U, V are tangent vector fields to Gr_1 (they, too, are hermitian operators, and they satisfy $V(\Phi)\Phi + \Phi V(\Phi) = 0$).

While we will not be using this symplectic form for our case, we review here some of its properties: in addition to $\omega(U,V)$ being a closed, non-degenerate two-form, it is invariant under the unitary group action, $\Phi \to g\Phi g^\dagger$ or $M \to gMg^\dagger + g[\epsilon, g^\dagger]$. For the infinitesimal form of this action, $\Phi \to \Phi + V_u = \Phi + i[u, \epsilon + M]$ where $e^{iu} = g$ and u is Hermitian, there is a function f_u associated with each V_u defined by

$$\omega(V_u, \cdot) = df_u \cdot$$

namely

$$f_u(M) = -\frac{1}{2} Tr M u \, .$$

We note here that while in a finite dimensional space, the associated function f_u of any vector field always exists (the symplectic form, which is nondegenerate, is "invertible"), this is not always the case for an infinite dimensional space, but such a function does exist here.

The Poisson Brackets of two such functions are defined in terms of ω by

$$\{f_u, f_v\} = \omega(V_u, V_v) \,.$$

We get the following relation:

$$\{f_u, f_v\}(M) = f_{i[u,v]}(M) - \frac{i}{2} Tr[\epsilon, u]v \,. \tag{12}$$

We will now translate this into integral kernel language: the integral kernel $M(x, y)$ of the operator M, (also known as a master field - see Section 6), for any $F \in \mathcal{H}$, is defined by

$$(MF)(x) = \int M(x, y)F(y)dy \,. \tag{13}$$

The kernel $\epsilon(x, y) = \epsilon(x-y)$ for the operator ϵ is defined similarly (and is known as the Hilbert transform operator). Furthermore, a trace such as $TrMu$ is rewritten as $\int dx\, dy\, M(x, y)u(y, x)$. Now we can rewrite equation (12) in terms of integral kernels:

$$\frac{i}{2}\{M(x, y), M(z, w)\} = \delta(x - w)[M + \epsilon](z, y) - \delta(y - z)[M + \epsilon](x, w) \tag{14}$$

(one can go back from this form to equation (12) by multiplying by $u(y, x)v(w, z)$ and integrating over all four variables.) The Hamiltonian is taken to be

$$H(M) = \int dxdyh(x - y)M(x, y) - \frac{1}{2}g^2 \int dxdyG(x - y)M(x, y)M(y, x) \ , \tag{15}$$

where the first term is the kinetic term, the second is the potential (interaction) term, and g is a constant parameter. The kernels $h(x - y)$ and $G(x - y)$ are given by

$$h(x - y) = \frac{i}{2}(-\delta'(x - y) + \frac{i}{2} sgn(x - y)) \ , \qquad G(x - y) = -\frac{1}{2}|x - y| \ ,$$

which are the kernels of the Fourier transforms of

$$h(p) = \frac{1}{2}(p + \mu^2/p) \ , \qquad G(p) = \frac{1}{p^2} \ .$$

The kinetic term above is derived using 2-dimensional space with quasi-light cone coordinates: let $u = x^0 - x^1$, $x = x^1$, with metric $ds^2 = du(du + 2dx)$. The momenta in these coordinates are $p_u = p_0$ (associated with energy) and $p = p_x = p_0 + p_1$. In these coordinates, the invariant mass μ^2 is given by

$$2pp_u - p^2 = \mu^2 \ ,$$

so

$$p_u = \frac{1}{2}(p + \frac{\mu^2}{p}) \ ,$$

where p_u can be interpreted as the kinetic energy.

Now, the equation of motion, with t as the time variable, is

$$\begin{aligned} \frac{i}{2}\frac{\partial M(x,y,t)}{\partial t} &= \frac{i}{2}\{H, M(x,y)\} \\ &= \int dz \, [h(x-z)M(z,y) - M(x,z)h(z-y)] \\ &\quad + g^2 \int dz \, G(y-z)\epsilon(x,z)M(z,y) - G(z-x)\epsilon(z,y)M(x,z) \\ &\quad + g^2 \int dz \, M(x,z)M(z,y)[G(y-z) - G(z-x)] \ . \end{aligned}$$

To arrive at the meson equation, we take a linear approximation around the vacuum $\Phi = \epsilon$ ($M = 0$), so we neglect the terms quadratic in M:

$$\begin{aligned} \frac{i}{2}\frac{\partial M(x,y,t)}{\partial t} &= \frac{i}{2}\{H, M(x,y)\} \\ &= \int dz \, [h(x-z)M(z,y) - M(x,z)h(z-y)] \\ &\quad + g^2 \int dz \, G(y-z)\epsilon(x,z)M(z,y) - G(z-x)\epsilon(z,y)M(x,z) \ . \end{aligned} \tag{16}$$

Translating to momentum space, where

$$M(p,q) = \int dx\, dy\, e^{i(-px+qy)} M(x,y) \ , \tag{17}$$

the linearized equation becomes

$$\frac{i}{2}\frac{\partial M(p,q,t)}{\partial t} = [h(p)-h(q)]M(p,q) \ + \ g^2(sgn(p)-sgn(q))\int \frac{dr}{r^2}M(p-r,q-r) \ . \tag{18}$$

Now, define

$$P = p-q \ , \qquad \xi = p/P \ , \qquad \chi(P,\xi) = PM(P\xi, -(1-\xi)P) \ . \tag{19}$$

The quantity P has the meaning of momentum, since the translation $M(x,y) \to M(x+a, y+a)$ yields in momentum representation $M(p,q) \to e^{i(-pa+qa)}M(p,q)$. Furthermore, the constraint (see equation (8)) $M^2 + M\epsilon + \epsilon M = 0$, which to first order in M is $M\epsilon + \epsilon M = 0$ or in kernel language $\int dy\, M(x,y)\epsilon(y,z) + \epsilon(x,y)M(y,z) = 0$, has the Fourier transform

$$(sgn(p) + sgn(q))M(p,q) = 0 \, .$$

We also have $M^\dagger = M$, or $M^*(y,x) = M(x,y)$, which translates to $M^*(p,q) = M(q,p)$ in momentum space. Putting these together we note that $M(p,q)$ may be non-zero only when the signs of p and q are opposite, and due to the hermiticity condition, it is sufficient to consider the case where $p > 0, q < 0$. These conditions also imply that ξ of equation (19) ranges between 0 and 1. Now, changing from $p, q, M(p,q)$ to $P, \xi, \chi(P,\xi)$, and using the ansatz $M(p,q,t) = e^{-ip_u u}M(p,q)$ we get

$$(2p_u P - P^2)\chi(\xi) = \mathcal{M}^2\chi(\xi) = (\frac{\mu^2}{\xi} + \frac{\mu^2}{1-\xi})\chi(\xi) + 4g^2\int_0^1 \frac{d\xi'}{(\xi-\xi')^2}\chi(\xi') \ , \tag{20}$$

which is 't Hooft's integral equation for mesons (see equation (6)).

Now we turn to baryons, which are topological solitons in this classical theory, and discuss the relevant topological properties of Gr_1: for an element $\Phi \in Gr_1$, let V_Φ^+ be the subspace of $\mathcal{H}$ which is the $+1$ eigenspace of Φ, and let V_Φ^- be the subspace of $\mathcal{H}$ which is the -1 eigenspace of Φ (the condition $\Phi^2 = 1$ leads to the fact that Φ has eigenvalues $\lambda = \pm 1$); also, let $M_\Phi = \Phi - \epsilon$ as before. The fact that $[\epsilon, \Phi]$ is Hilbert-Schmidt implies (see [7]) that the number of independent vectors in $\mathcal{H}$ which have eigenvalue $+1$ under the action of ϵ but become -1 eigenvectors under Φ is finite, i.e. $dim(V_\Phi^- \cap \mathcal{H}_+)$ is finite; similarly, $dim(V_\Phi^+ \cap \mathcal{H}_-)$ is finite. Define

$$J(\Phi) = I_+(\Phi) - I_-(\Phi) = dim(V_\Phi^- \cap \mathcal{H}_+) - dim(V_\Phi^+ \cap \mathcal{H}_-) \; , \tag{21}$$

which can also be written

$$\begin{aligned} J(\Phi) &= \left(-\frac{1}{2}Tr(\Phi - \epsilon)|_{\mathcal{H}_+}\right) - \left(\frac{1}{2}Tr(\Phi - \epsilon)|_{\mathcal{H}_-}\right) \\ &= -\frac{1}{2}TrM_\Phi = -\frac{1}{2}\int dx M(x,x) \; . \end{aligned} \tag{22}$$

The quantity $J(\Phi)$ can take on any integer value and is known as the virtual rank, or index, of the operator Φ. It is a topological invariant: smoothly varying Φ leaves the index fixed. This index divides Gr_1 into $\mathbf{Z}$ connected components.

The integer $J(\Phi)$ corresponding to each Φ is the topological invariant which corresponds to the baryon number.

4 $O(2N+1)$ QCD on a(nother) Grassmannian

Here we construct a manifold $\mathcal{S}$, a real analog of Gr_1, as the phase space for a classical dynamical system, which we propose to be equivalent to the large N limit of $O(2N+1)$ QCD. The hamiltonian will be taken to be the analog of the one in Section 3; as for the symplectic form, we take it to be the canonical one on the loop group LO_{2n}, which is closely related to $\mathcal{S}$. We will show that mesons in this theory satisfy 't Hooft's integral equation. In the next section we will discuss baryons in this theory.

As in Section 3, the phase space can be described in two ways. The definition analogous to equation (7) is

$$\mathcal{S} = \{\Phi \mid \Phi^T = \Phi \;\; ; \;\; \Phi^2 = 1 \;\; ; \;\; [\epsilon, \Phi] \text{ is Hilbert-Schmidt}\} \;, \tag{23}$$

where here Φ are operators with real matrix elements, and we use the transpose instead of hermitian conjugate. Again, there is a description in terms of $M = \Phi - \epsilon$:

$$\mathcal{S} = \{M \mid M^T = M\,; \;\; M^2 + \epsilon M + M\epsilon = 0; \;\; [\epsilon, M] \text{ is Hilbert-Schmidt}\} \;. \tag{24}$$

Following a reasoning similar to that in Section 3, we get the analog of equation (10):

$$\mathcal{S} = \frac{O_{res}(\mathcal{H})}{O(\mathcal{H}_+) \times O(\mathcal{H}_-)} \;, \tag{25}$$

where

$$O_{res}(\mathcal{H}) = \{g \mid g^T g = 1; [\epsilon, g] \quad \text{is Hilbert-Schmidt}\} \;. \tag{26}$$

Turning to the symplectic form, we note that for $\mathcal{S}$, the form analogous to the one defined in equation (11) is not nondegenerate[2]. Furthermore, the Poisson Brackets of equation (14) are inconsistent with the constraint $\Phi^T = \Phi$, which translates into $M(x, y) = M(y, x)$. Therefore, we shall describe a different space, the loop group LO_{2n}, which is closely related to $\mathcal{S}$ and on which there is a canonical symplectic form.

[2]Proving nondegeneracy for the form ω in equation (11) (see [1]) involves looking at ω at the chosen point $\Phi = \epsilon$, where we have $\omega_\epsilon(U, V) \propto tr(u^\dagger v - v^\dagger u)$ for appropriate u, v. At this chosen point, ω_ϵ is non-degenerate, but for $\mathcal{S}$, we would have $tr(u^T v - v^T u)$ which is identically zero. Unlike here, in [4], a complexified version of equation (25) was used, and their analog of equation (11) is still non-degenerate.

The group LO_{2n}, which is the loop group of the orthogonal group $O(2n)$, is defined as the group of smooth maps from the circle to a closed curve in $O(2n)$:

$$LO_{2n} = \{\gamma : S^1 \longrightarrow O(2n)\} \ , \tag{27}$$

where the group multiplication is given by pointwise multiplication in $O(2n)$. Since the fundamental group of $O(2n)$ is $\mathbf{Z}_2$:

$$\pi_1(O(2n)) = \mathbf{Z}_2 \ , \tag{28}$$

we see by definition that LO_{2n} has two connected components. We shall use them below in connection with baryon number.

The group LO_{2n} is related to $\mathcal{S}$ because it is embedded in $O_{res}(\mathcal{H})$: let $\mathcal{H}$ be the Hilbert space of square integrable functions from the circle to $\mathbf{R}^{2n}$.[3] Then $\gamma \in LO_{2n}$ acts on $F \in \mathcal{H}$ by pointwise multiplication given by an operator M_γ:

$$(M_\gamma \cdot F)(\theta) = \gamma(\theta)F(\theta) \ . \tag{29}$$

If we write $\gamma(\theta) = \sum \gamma_k e^{ik\theta}$, $\gamma_k \in O(2n)$, and $F(\theta) = \sum F_l e^{il\theta}$, $F_l \in \mathbf{R}^{2n}$, then in terms of the standard basis for $\mathcal{H}$, M_γ is a $\mathbf{Z} \times \mathbf{Z}$ matrix, with each entry M_{pq} being a $2n \times 2n$ matrix, given by $M_{pq} = \gamma_{p-q}$. It can be shown ([7]) that this action is in $O_{res}(\mathcal{H})$, so that LO_{2n} is a subgroup of $O_{res}(\mathcal{H})$.

Another fact relating LO_{2n} and $\mathcal{S}$ is their number of connected components: it is a property of $\mathcal{S}$ that

$$dim(V_\Phi^- \cap \mathcal{H}_+) = dim(V_\Phi^+ \cap \mathcal{H}_-) \ , \tag{30}$$

[3]Note that this Hilbert space, $L^2(S^1; \mathbf{R}^{2n})$, may be identified with the Hilbert space $L^2(\mathbf{R}; \mathbf{C})$ defined as $\mathcal{H}$ at the beginning of Section 3. See [7].

i.e. $I_+(\Phi) = I_-(\Phi) = I(\Phi)$ (the definitions of $V_\Phi^\pm$, $I_\pm(\Phi)$, and $J(\Phi)$ are analogous to the ones corresponding to U_{res} given in Section 3). From this follows that $J(\Phi) = 0$ always; however, another property of $\mathcal{S}$ is that smoothly varying Φ may change $I(\Phi)$ only by multiples of 2. This means that the parity of the dimension $I(\Phi)$ is a topological invariant; it divides $\mathcal{S}$ into two connected components, namely $I(\Phi)$ even and $I(\Phi)$ odd. This can be written as follows:

$$\Xi(\Phi) \equiv I(\Phi) \bmod 2 \; = -\frac{1}{2} tr(M)|_{\mathcal{H}_+} \bmod 2 \; , \tag{31}$$

so $\Xi(\Phi)$ is either 0 or 1, and it tells us explicitly to which connected component $\Phi \in \mathcal{S}$ belongs. So LO_{2n} and $\mathcal{S}$ both have two connected components. We shall use this in section 5.

Now we construct the symplectic form ω on LO_{2n} in terms of the Killing form of $\mathfrak{o}(2n)$, the Lie algebra of $O(2n)$. We begin by defining the Lie algebra $L\mathfrak{o}_{2n}$ associated with LO_{2n}. It is the set of smooth maps from the circle to $\mathfrak{o}(2n)$:

$$L\mathfrak{o}_{2n} = \{\eta : S^1 \longrightarrow \mathfrak{o}(2n)\} \; , \tag{32}$$

where the commutator is defined by pointwise commutators in $\mathfrak{o}(2n)$. The Killing form on $\mathfrak{o}(2n)$ - which is a symmetric, nondegenerate, invariant bilinear form - is defined by

$$K(X, Y) = Tr(adX \, adY) \; , \tag{33}$$

where $X, Y \in \mathfrak{o}(2n)$, Tr denotes the trace of a matrix, and adX denotes the operator in the adjoint representation of $\mathfrak{o}(2n)$ corresponding to X. The calculation of the Killing form on $\mathfrak{o}(2n)$ is straightforward and gives

$$K(X, Y) = (2n - 2) Tr(XY) \; . \tag{34}$$

Now, a symplectic form on LO_{2n} is a two-form, namely a map which takes two vector fields and gives a real number. Since a Lie algebra in fact consists of vector fields at the identity of the group manifold, we can define a symplectic form on the identity component ΩO_{2n} of the loop group as follows (see [7]):

$$\omega(\eta, \xi) = \frac{1}{2\pi} \int_0^{2\pi} K(\eta(\theta), \xi'(\theta)) d\theta \ . \tag{35}$$

The form $\omega(\eta, \xi)$ is a map from $L\mathfrak{o}_{2n} \otimes L\mathfrak{o}_{2n}$ to the real numbers, and $\eta(\theta)$ and $\xi(\theta)$ are elements of $L\mathfrak{o}_{2n}$. The properties of the Killing form, as well as the decomposition of $\eta(\theta)$ and $\xi(\theta)$ into Fourier modes, ensure that ω satisfies the conditions of a symplectic form, i.e. it is an antisymmetric, nondegenerate, closed two-form. Note that this equation defines a symplectic form for the loop group of any Lie group, with K standing for the Killing form of the corresponding Lie algebra.

The Hamiltonian on $\mathcal{S}$ can be taken to be the same one as equation (15), and it is also a Hamiltonian on LO_{2n} via the embedding $LO_{2n} \subset O_{res}(\mathcal{H})$ (equation (29)): we replace $M(x, y)$ by $M_\gamma(x, y)$ which is the integral kernel of $M_\gamma(x)$ defined the same way as in (13). To get equations of motion for mesons, we would normally need to derive the Poisson Brackets corresponding to the symplectic form given in equation (35). However, we argue below that we can simply rely on the result for the $U(N)$ case, i.e. that mesons on Gr_1 satisfy 't Hooft's integral equation, and deduce that on the space $\mathcal{S}$ the equation for mesons will again be 't Hooft's integral equation, as is also true for QCD with an $O(M)$ gauge group, to which we claim that the theory on $\mathcal{S}$ is equivalent:

The relationship between LO_{2n} and $\mathcal{S}$, i.e. the fact that $LO_{2n} \subset O_{res}(\mathcal{H})$ where $\mathcal{S}$ is the coset space $O_{res}(\mathcal{H})/[O(\mathcal{H}_+) \times O(\mathcal{H}_-)]$ (equation (25)), has an analog in the unitary case: it is true as well (see [7]) that LU_n, the loop group of the unitary group, is a subgroup of $U_{res}(\mathcal{H})$, where we remember that Gr_1 is the coset space

$U_{res}(\mathcal{H})/[U(\mathcal{H}_+) \times U(\mathcal{H}_-)]$ (equation (10)). The symplectic form on Gr_1 given in equation (11) is invariant under the unitary group action on Gr_1 and is the unique such form (see [1]); similarly, the form in equation (35), when it is taken on LU_n, is invariant under the corresponding action in LU_n (in fact, it is invariant under any translations in the group). Therefore, the meson equation derived on LU_n using equation (35) and the meson equation obtained on Gr_1 using equation (11) appear to be the same (or closely related), i.e they both are the 't Hooft integral equation. Similarly, equations obtained directly on $\mathcal{S}$ using a symplectic form invariant under the orthogonal group action, and the equations derived from equation (35) considered on LO_{2n} appear to be the same (or closely related). Since $U_{res}(\mathcal{H})$ and $O_{res}(\mathcal{H})$ are analogous and the embeddings $LU_n \subset U_{res}(\mathcal{H})$ and $LO_{2n} \subset O_{res}(\mathcal{H})$ have the same structure, the resulting meson equations on $\mathcal{S}$ and on LO_{2n} are analogs of those on Gr_1 and LU_n, and we conclude that we get 't Hooft's equations on $\mathcal{S}$ (and on LO_{2n}) as we did on Gr_1.

Now that we established that our theory is consistent with 2DQCD through properties of mesons, it makes sense to consider the topological invariant of the space $\mathcal{S}$, which is analogous to the virtual rank of Gr_1, to be related to baryon number, i.e. to consider baryons to be solitons, and we do so below.

5 Baryon Number Non-Conservation

As we have established, our phase space has two connected components (see equations (28) and (31)), and it is this topological property which we propose here to be related to the baryon number B. We will show in this section that the properties of baryon number given by our theory are consistent with QCD in two dimensions with an odd orthogonal gauge group.

We propose that in the space $\mathcal{S}$, the topological invariant $\Xi(\Phi)$ given in equation (31) corresponds to the parity of baryon number, i.e. states Φ with $\Xi(\Phi) = 0$ have even baryon number, and states with $\Xi(\Phi) = 1$ have odd baryon number. Correspondingly, in the space LO_{2n}, shrinkable loops correspond to even baryon number, and non-shrinkable loops to odd baryon number. The physical meaning of relating a quantity to a topological invariant is that the quantity is conserved. So in this case, the quantity Q_B defined by

$$Q_B = B \bmod 2 \ , \tag{36}$$

where B is the baryon number, is conserved, i.e. $\triangle Q_B = 0$, which means that baryon number can change by multiples of two. In other words, baryon number is conserved only modulo two. Physically, this means that baryon-baryon pairs can annihilate.[4]

The equivalence of the theory on $\mathcal{S}$ to QCD in two dimensions with an orthogonal gauge group is supported by the following argument: first, baryon-baryon annihilation can be shown directly to occur in $O(M)$ QCD, where M here may be even or odd [9]: unlike $SU(N)$, whose fundamental N-dimensional representation is complex, $O(M)$ has a real fundamental representation, which is the same as its dual. Since quarks are described by the fundamental representation while antiquarks are described by the dual of that representation, this implies that QCD with an $O(M)$ gauge group does not distinguish quarks from antiquarks. Therefore, it does not distinguish baryons from antibaryons either, and just as a baryon-antibaryon pair can annihilate, so can a baryon-baryon pair. This is the same result we arrived at above!

[4]The idea that baryon number would be conserved only mod 2 appeared in a different context in [8].

The question of baryon number conservation is a bit more subtle; baryon-baryon annihilation does not necessarily imply the conservation of baryon number modulo two: it means only that baryon number can be conserved at most modulo 2, i.e. single baryons might still appear or disappear, in which case there would be no conservation at all. We now show that baryon conservation is determined by the parity of M, i.e. it depends on whether the gauge group $O(M)$ is an even orthogonal group, $M = 2N$, or an odd orthogonal group, $M = 2N + 1$.

The lagrangian of QCD with an orthogonal gauge group has the same form as the one given in equation (1), except that the gauge fields A_μ are now the generators of the orthogonal rather than the unitary group (they are real anti-symmetric rather than complex anti-hermitian) and the quark field ψ is real. It is clear that the lagrangian is invariant under the transformation $\psi \to -\psi$, the only phase transformation allowed for real fields. This means that the only expectation values which can be non-zero are those which contain an even number of quarks, so that quarks are conserved modulo two.

We can now see the difference between the even and odd cases. To form a baryon, we must take a product of M quarks. When M is even, the baryon is the product of an even number of quarks and therefore does not change under $\psi \to -\psi$, which means that baryons are not conserved at all: Green's functions containing an odd number of baryons can be non-zero, and processes such as $2B \to 3B$, which both begin and end with an even number of quarks, may occur. However, the situation when M is odd is different: each baryon consists of the product of an odd number of quarks so the baryon is odd under $\psi \to -\psi$. Therefore, non-zero correlation functions must necessarily contain an even number of baryons, which means that baryons are indeed conserved modulo 2, consistent with the theory on $\mathcal{S}$.

6 Relation to master fields

A classical field, such as the field $M(x, y)$ which we have discussed, representing the large N limit of QCD is also known as a *master field*. There is another approach to master fields given in [10, 11], where the mathematical formalism of non-commutative probability theory is used within a matrix model. A master field is in general a classical configuration such that in the large N limit, the values of gauge-invariant Green's functions are given simply by their value at the master field - i.e., no functional integral needs to be done. In [10, 11], the Hilbert space on which the master fields act is given by states generated from the vacuum by creation operators which satisfy the Cuntz algebra, i.e.

$$a(x)|\Omega >= 0, \qquad a(x)a^{\dagger}(y) = \delta(x - y),$$

with no further relations. Master fields are given in terms of these creation and annihilation operators, i.e. $M = M(a, a^{\dagger})$. For the case of matrix fields which have independent distributions and are decoupled, there is the form

$$M(a, a^{\dagger}, x) = a(x) + \sum_{n} M_{n+1} a^{\dagger n}(x) \tag{37}$$

(the M_{n+1} turn out to be connected Green's functions). For the more general case in which the matrices are coupled, the form for the master field becomes

$$M(x) = a(x) + \sum_{k=1}^{\infty} \psi_{x,y_1,\ldots,y_k} a^{\dagger}(y_1) \cdots a^{\dagger}(y_k) \ . \tag{38}$$

An interesting problem would be to make a connection between this framework and the master fields $M(x, y)$ on the Grassmannian manifold described in Sections 3 and 4, i.e. to write $M(x, y)$ in terms of these creation and annihilation operators.

A construction of the master field for 2DQCD is suggested in [10], but we propose a different form for $M(a, a^\dagger)$:

$$M(x, y) = a^\dagger(x)a(y) \ . \tag{39}$$

This form is motivated by the intriguing fact that its commutation relations are identical to equation (14), except for the ϵ term which does not have an analog here, and a sign:

$$[M(x, y), M(z, w)] = \delta(y - z)M(x, w) - \delta(w - x)M(z, y) \ . \tag{40}$$

Therefore, it appears plausible that a matrix model with this master field could very well provide further insight into 2DQCD. A full investigation of this possibility is beyond the scope of this paper.

7 Open Problems

We have extended Rajeev's work to construct a classical dynamical system which is equivalent to the large N limit of two dimensional QCD with an $O(2N + 1)$ gauge group. We argued that the same equation of motion for mesons would result for the theory on $\mathcal{S}$ as for the theory on Gr_1, which furthermore is the same equation derived within 2DQCD with $O(2N + 1)$ and $U(N)$ gauge groups. Considering baryons as topological solitons in our theory, we showed that baryon number is conserved only modulo 2. We also showed that our model is related to the master field approach to matrix models by providing an explicit formula for the master field that has the same commutation relations as in our model.

There are several directions for further investigations which may now be pursued. We suggest a few below:

It would be interesting to see what would happen when the symplectic groups or the exceptional groups serve as the QCD gauge groups. For $Sp(2N)$, we can see immediately that a direct analog of our work would need to be modified. Suppose we propose an equivalence between a classical dynamical system on an infinite-dimensional Grassmannian manifold, or the loop group of the symplectic group, LSp_{2N}, and the large N limit of 2DQCD with $Sp(2N)$ gauge group. Since $\pi_1(Sp(2N)) = \mathbf{Z}$, our construction would lead to conservation of baryon number; however, baryons should not even exist in $Sp(2N)$ QCD (they decay into mesons; see [9]).

Another direction one might pursue is quantizing the classical dynamical system we have described, i.e. promoting the classical observables, which are just functions on the phase space, to operators. Commutation relations would be given by the Poisson Brackets. This has been done for the unitary case in [1], resulting in a theory of QCD at finite N with a correspondence between $\hbar$ and $1/N$.

In addition, one may calculate the mass for a baryon by minimizing the Hamiltonian subject to the constraint of odd baryon number. One would then interpret the resulting minimal energy configuration of odd baryon number to correspond to a stable state of one baryon.

Another interesting investigation would be to study chiral symmetry breaking in this system and its relation to the non-conservation of baryon number. See [12] for a recent investigation of chiral symmetry breaking and stability of topological solitons in $SU(N)$ and $SO(N)$ Yang-Mills theories.

Finally, a full-fledged investigation of the master field approach using the proposed master field (equation (39)) has the potential to shed new light on 2DQCD.

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