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Schwinger Pair Creation of Kaluza-Klein Particles: Pair Creation Without Tunneling

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Schwinger pair creation of Kaluza-Klein particles: Pair creation without tunneling

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We study Schwinger pair creation of charged Kaluza-Klein (KK) particles from a static KK electric field. We find that the gravitational backreaction of the electric field on the geometry—which is incorporated via the electric KK-Melvin solution—prevents the electrostatic potential from overcoming the rest mass of the KK particles, thus impeding the tunneling mechanism which is often thought of as responsible for the pair creation. However, we find that pair creation still occurs with a finite rate formally similar to the classic Schwinger result, but via an apparently different mechanism, involving a combination of the Unruh effect and vacuum polarization due to the E-field.

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I. INTRODUCTION AND SUMMARY

The classic study of the rate of creation of electron-positron pairs in a uniform, constant electric field was done more than 50 years ago in the seminal paper by Schwinger [1]. The concepts and methodology introduced in this work have had a lasting impact on the formal development of quantum field theory, and by now several alternative derivations of the effect have been invented (see, e.g., [2–5]).

Schwinger's predicted rate per unit time and volume is given by [1,5]

$$\mathcal{W}(E) = qE \int \frac{d^2k_i}{(2\pi)^2} \sum_{n=1}^{\infty} \frac{1}{n} \exp\left(-\frac{\pi n(m_e^2 + k_i^2)}{|qE|}\right) \quad (1.1)$$

for a spin $1/2$ particle in 4 flat space-time dimensions, with m_e and q the electron mass and charge, k_i the transverse momenta, and E the electric field.[1]

In an earlier set of equally classic papers [9,10], Kaluza and Klein (KK) introduced their unified description of general relativity and electromagnetism, in which charged particles appear as quanta with nonzero quantized momentum around a compact extra dimension. It has of course always been clear that the charged Kaluza-Klein particles do not have the correct properties to represent electrons; most notably, the mass of the fundamental KK particles is equal to (or bounded below by) their charge, while for the electron the ratio m_e/q is about 10^{-21}. So in comparison with electrons, KK particles are either very heavy or have (in a large extra dimension scenario) an exceedingly small KK electric charge. Nonetheless, or rather, because of this fact, it is an interesting theoretical question whether it is at all possible, via an idealized gedanken experiment, to pair produce KK particles by means of the Schwinger mechanism. As far as we know, this question has not been addressed so far in the literature, and probably for a good reason: it turns out to be a subtle problem. We will show that unlike the standard Schwinger pair creation effect, pair production of KK particles cannot be given the simple and rather intuitive interpretation of a tunneling mechanism.

Imagine setting up our gedanken experiment as in Fig. 1, with two charged plates with a nonzero KK electric field in between. As seen from Eq. (1.1), to turn on the effect by any appreciable amount will require an enormous KK

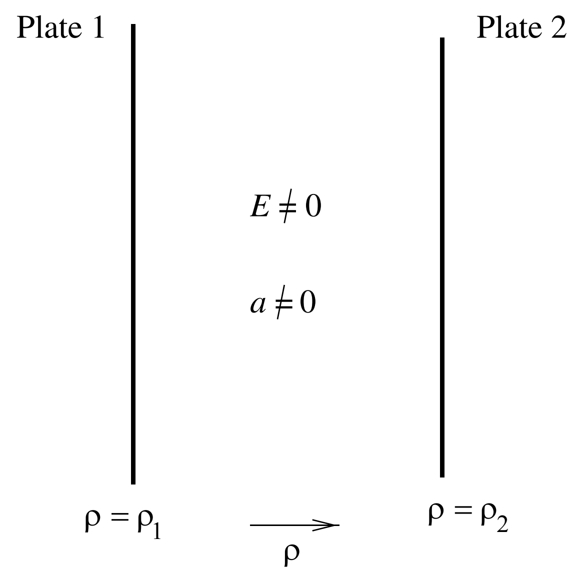

FIG. 1. The gedanken experiment we will imagine in this paper, with two charged plates at $\rho = \rho_1$ and $\rho = \rho_2$, producing a nonzero E-field in the intermediate region. The backreaction and the finite mass density of the plates results in a nonzero gravitational acceleration a.

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[1]The rate (1.1) is still very small for experimentally accessible electric fields. For the rate to be appreciable, the field must be very large, around $E_{\mathrm{crit}} = 10^{16}$ eV/cm. A static field of this magnitude is difficult to obtain in laboratories, largely because it is several orders of magnitude above the electric field that can be sustained by an atom, namely 10^8 eV/cm. See [6] for a recent experiment that has obtained pair creation from oscillating electric fields, which were studied theoretically in [7], and see [8] for an upcoming experiment studying pair production from the low-frequency, Schwinger limit of such fields.

electric field, and since Kaluza-Klein theory automatically includes gravity, the backreaction of the E-field on space-time will need to be taken into account. The best analog of a constant electric field in this setting is the electric version of the Kaluza-Klein Melvin (KKM) background; the magnetic version was studied recently in [11–13][2], and in [13] the electric version also appeared. We will study some of the features of the electric KKM background in Sec. II. For our problem, the relevant properties of this background are that

(i) the background geometry depends on a longitudinal coordinate, which we will call ρ;
(ii) a gravitational acceleration $a(\rho)$, directed along the E-field, is included;
(iii) the total gravitational and electrostatic potential energy remains positive everywhere.

The first two properties are expected backreaction effects. The last property, however, implies that the negative electrostatic potential can never be made large enough to compensate for the positive contribution coming from the rest mass of the particles. The physical reason for this obstruction is that before one reaches the critical electrostatic potential, backreaction effects will cause space-time itself to break down: if one would formally continue the solution beyond this point, the space-time develops closed timelike curves, which are known to be unphysical. In this way, gravity puts an upper limit on the potential difference one can achieve between the two plates in Fig. 1.

This result may look like an insurmountable obstacle for pair creation, which is usually [4,5] thought of as a tunneling effect by which particle pairs can materialize by using their electrostatic energy to overcome their rest mass. The modern instanton method [3,4] of computing the pair creation rate, for example, crucially depends on this intuition. However, as mentioned in point (ii) above, it turns out that the backreaction necessarily implies that the vacuum state of the KK particles needs to be defined in the presence of a nonzero gravitational acceleration. As we will explain in Appendix A, the necessary presence of this acceleration can be thought of as due to the non-zero-mass of the parallel plates that produce the KK electric field. Consequently, the Schwinger effect needs to be studied in conjunction with its direct gravitational analog, the equally famous Hawking-Unruh effect [15–17].

It has been recognized for some time that the Hawking-Unruh effect and Schwinger pair creation are rather closely related (see, for example, [5]); both can be understood via a distortion of the vacuum, which may be parametrized by means of some appropriate Bogolyubov transformation that relates the standard energy eigenmodes to the new energy eigenmodes in the nontrivial background field. Also, like the Schwinger effect, the Hawking-Unruh effect has been thought of as a tunneling mechanism and was derived as such recently [18]; see also [19] for a related study of de Sitter radiation.

By combining both the Schwinger and the Unruh effects we will obtain the following result for the pair creation rate of the Kaluza-Klein particles (which we will assume to be scalar particles) as a function of the electric field E and gravitational acceleration a

$$\mathcal{W}(E, a) = \frac{a^{3/2}}{2\pi^2} \int \frac{\prod^{d-2} dk_i}{(2\pi)^{d-2}} \sum_q (q^2 + \Lambda k_i^2)^{1/4} \times \exp[-2\pi\omega(a, q, k_i)], \tag{1.2}$$

where

$$\omega(a, q, k_i) = \frac{q^2 + k_i^2}{|\frac{1}{2} qE + a\sqrt{q^2 + \Lambda k_i^2}|}, \qquad \Lambda = 1 - \frac{E^2}{4a^2}. \tag{1.3}$$

Here the summation is over the full KK tower of all possible charges $q = n/R$ with n integer and R the radius of the extra dimension, and a is the "bare" acceleration, that the particles would experience with the E-field turned off. While E in this formula is a constant, a in fact depends on the longitudinal coordinate via $1/a = \rho + \text{const.}$ The potential energy in (1.3) is the manifestly positive quantity we referred to in property (iii) above. A more detailed explanation of the result (1.2) will be given in Sec. V.

Since in our case mass equals charge the result (1.2) looks like a reasonable generalization of the classic result (1.1) of Schwinger and of Unruh [5,16]. In particular, if we turn off the E-field, our expression (1.2) reduces to the Boltzmann factor with Hawking-Unruh temperature $\beta = 2\pi/a$. Moreover, if we would allow ourselves to drop all terms containing the acceleration a, the result is indeed very similar to the dominant $n = 1$ term in Schwinger's formula (1.1). However, it turns out that in our case, the gravitational backreaction dictates that the acceleration a cannot be turned off; rather, it is bounded from below by the electric field via

$$a > |E/2|. \tag{1.4}$$

Our formula (1.2) indeed breaks down when a gets below this value. So, in particular, there is no continuous weak field limit in which our result reduces to Schwinger's answer. We will further discuss the physical interpretation of our result in the concluding section, where we will make a more complete comparison with the known rate [20] for Schwinger production in an accelerating frame.

This paper is organized as follows. In Sec. II we describe some properties of the electric Kaluza-Klein Melvin space-time. In Secs. III and IV we study classical particle me-

[2]In [11], pair production of Kaluza-Klein monopoles from the magnetic Kaluza-Klein Melvin solution was studied. See also [14] for a study of other aspects of the magnetic solutions.

chanics and wave mechanics in this background. Finally in Sec. V, we set out to calculate the pair creation rate, using (and comparing) several methods of computation. Sec. VI contains some concluding remarks. We discuss our experimental setup in Appendix A, and in Appendix B we summarize the known result for Schwinger pair production in an accelerating frame.

II. THE ELECTRIC KALUZA-KLEIN-MELVIN SPACE-TIME

We start with describing the classical background of $d+1$-dimensional Kaluza-Klein theory, representing a maximally uniform KK electric field.

A. Definition of the electric KKM space-time

Consider a flat $d+1$ dimensional flat Minkowski space-time, with the metric

$$ds^2 = -dt^2 + dx^2 + dy_i dy^i + dx_{d+1}^2, \tag{2.1}$$

with $i = 2, \ldots, d-1$. From this we obtain the electric Kaluza-Klein-Melvin space-time by making the identification

$$\begin{pmatrix} t \\ x \\ y_i \\ x_{d+1} \end{pmatrix} \to \begin{pmatrix} t' \\ x' \\ y'_i \\ x'_{d+1} \end{pmatrix} = \begin{pmatrix} \gamma(t-\beta x) \\ \gamma(x-\beta t) \\ y_i \\ x_{d+1} + 2\pi R \end{pmatrix} \tag{2.2}$$

with $\gamma^2(1-\beta^2) = 1$. This geometry can be viewed as a nontrivial Kaluza-Klein background in d dimensions, in which the standard periodic identification $x_{d+1} \equiv x_{d+1} + 2\pi R$ of the extra dimension is accompanied by a Lorentz boost in the x-direction. Since the $d+1$ dimensional space-time is flat everywhere, and the identification map (2.2) is an isometry, it is evident that the electric Melvin background solves the equation of motion of the Kaluza-Klein theory. As we will describe momentarily, from the d-dimensional point of view, it looks like a nontrivial background with a constant nonzero electric field E and with, as a result of its nonzero stress-energy, a curved space-time geometry. Here the electric field E is related to the boost parameters β and γ by

$$\beta = \tanh(\pi R E), \qquad \gamma = \cosh(\pi R E). \tag{2.3}$$

The map (2.2) represents a proper spacelike identification, for which

$$\begin{aligned} &-(t'-t)^2 + (x'-x)^2 + (x'_{d+1} - x_{d+1})^2 \\ &= (2\pi R)^2 - (2\gamma - 2)(x^2 - t^2) > 0 \end{aligned} \tag{2.4}$$

provided we restrict to the region

$$\rho < \frac{\pi R}{\sinh\frac{\pi E R}{2}}, \qquad \rho^2 \equiv x^2 - t^2, \tag{2.5}$$

where we used (2.3). Outside of this regime, the electric Melvin space-time contains closed timelike curves. We will exclude this pathological region from our actual physical setup.[3]

B. Classical trajectories

As a first motivation for the identification of E with the KK electric field, it is instructive to consider classical trajectories in this space-time. This is particularly easy, since in flat $d+1$ Minkowski space, freely moving particles move in straight lines:

$$\begin{gathered} x = x_0 + p_1 s, \qquad x^- = t_0 + p_0 s, \qquad y_i = k_i s, \\ x_{d+1} = qs. \end{gathered} \tag{2.6}$$

Assuming the particle is massless in $d+1$-dimensions, we have

$$p_0 = \sqrt{p_1^2 + k_i^2 + q^2}, \tag{2.7}$$

which is the mass-shell relation of a d-dimensional particle with mass equal to q. Let us introduce coordinates ρ and τ via

$$\begin{aligned} x &= \rho \cosh\left(\tau - \frac{1}{2} E x_{d+1}\right), \\ t &= \rho \sinh\left(\tau - \frac{1}{2} E x_{d+1}\right), \end{aligned} \tag{2.8}$$

and coordinates X and T by

$$X = \rho \cosh\tau, \qquad T = \rho \sinh\tau. \tag{2.9}$$

The identification (2.2) in the new coordinates becomes

$$\begin{pmatrix} T \\ X \\ y_i \\ x_{d+1} \end{pmatrix} \to \begin{pmatrix} T \\ X \\ y_i \\ x_{d+1} + 2\pi R \end{pmatrix}, \tag{2.10}$$

which is the standard Kaluza-Klein identification. The trajectory in terms of these is

[3] There is also a different notion of the electric version of the KK-Melvin space-time, which is obtained by applying an electromagnetic duality transformation $F \to e^{2\sqrt{3}\phi} * F$, $\phi \to -\phi$ to the magnetic KK-Melvin space-time [21]. This background looks like an electric flux-tube in a $U(1)$ gauge theory with an electric coupling constant e that diverges at large transverse distance from the flux-tube (due to the fact that the size of the extra dimension shrinks at large distance). Putting a reasonable physical upper bound on the size of e restricts the maximal allowed length of the flux-tube, suggesting that the obstruction against creating an arbitrarily large electrostatic potential may be more general than only for the type of backgrounds studied in this paper.

$$X = (x_0 + p_1 s)\cosh\left(\frac{1}{2}Eqs\right) + (t_0 + p_0 s)\sinh\left(\frac{1}{2}Eqs\right),$$
$$T = (t_0 + p_0 s)\cosh\left(\frac{1}{2}Eqs\right) + (x_0 + p_1 s)\sinh\left(\frac{1}{2}Eqs\right). \tag{2.11}$$

Considering a particle at rest at the origin $x_0 = 0$ and $t_0 = 0$, we find

$$\frac{d^2X}{dT^2} = \frac{qE}{p_0}. \tag{2.12}$$

This is the expected acceleration of a particle with charge and rest-mass q.

C. Kaluza-Klein reduction

Let us now perform the dimensional reduction to d dimensions. Using the coordinates ρ and τ defined in (2.8) the $d+1$ dimensional metric becomes

$$ds^2 = -\rho^2\left(d\tau + \frac{1}{2}Edx_{d+1}\right)^2 + d\rho^2 + dy_i dy^i + Sdx_{d+1}^2, \tag{2.13}$$

while the identification (2.2) simplifies to a direct periodicity in x_{d+1} with period $2\pi R$, leaving (ρ, τ, y_i) unchanged. We may rewrite the metric (2.13) as

$$ds^2 = -\frac{\rho^2}{\Lambda}d\tau^2 + d\rho^2 + dy_i dy^i + \Lambda\left(dx_{d+1} - \frac{E\rho^2}{2\Lambda}d\tau\right)^2 \tag{2.14}$$

with

$$\Lambda \equiv 1 - \frac{1}{4}E^2\rho^2. \tag{2.15}$$

In this form, we can readily perform the dimensional reduction.

The d dimensional low energy effective theory is described by the Einstein-Maxwell theory coupled to the Kaluza-Klein scalar V via

$$S = \int \sqrt{-g_d}\left(V^{1/2}R_d + \frac{1}{4}V^{3/2}F_{\mu\nu}F^{\mu\nu}\right). \tag{2.16}$$

Here the d-dimensional fields are obtained from the $d+1$ metric via the decomposition

$$ds_{d+1}^2 = ds_d^2 + V(dx_{d+1} + A_\mu dx^\mu)^2. \tag{2.17}$$

Comparing (2.14) and (2.17) gives the dimensionally reduced form of the electric Melvin background

$$ds_d^2 = -\frac{\rho^2}{\Lambda}d\tau^2 + d\rho^2 + dy_i dy^i \tag{2.18}$$

$$A_0 = \frac{E\rho^2}{2\Lambda}, \qquad V = \Lambda \equiv 1 - \frac{1}{4}\rho^2E^2. \tag{2.19}$$

It describes a curved space-time, together with an electric field in the ρ-direction given by

$$E_\rho \equiv \sqrt{g^{00}}\partial_\rho A_0 = \frac{E}{\Lambda^{3/2}}. \tag{2.20}$$

This electric field is equal to E at $\rho = 0$, but diverges at $\rho = 2/E$; this singular behavior is related to the mentioned fact that outside the region (2.5), the identification map (2.2) becomes timelike and produces closed timelike curves. Note, however, that the location of the divergence in E_ρ slightly differs from the critical value noted in (2.5), but coincides with it in the limit of small ER.

The d dimensional metric in (2.18) reduces for $E = 0$ to the standard Rindler space-time metric. For finite E there is a nonzero gravitational acceleration

$$a_\rho(\rho) \equiv g^{00}\partial_\rho g_{00} = \frac{1}{\rho\Lambda}, \tag{2.21}$$

which includes the gravitational backreaction due to the stress-energy contained in the electric field. Notice that $a(\rho)$ diverges at $\rho = 2/E$.

The above static Rindler type coordinate system will be most useful for the purpose of providing a background with a static KK electric field. To obtain a more global perspective of the full electric KK-Melvin space-time, we can use the coordinates X and T defined in Eq. (2.9). In this coordinate system, the solution looks like

$$ds^2 = -dT^2 + dX^2 - \frac{E^2}{4\Lambda}(XdT - TdX)^2 + dy_i dy^i,$$
$$A_0 = \frac{EX}{2\Lambda}, \qquad A_1 = -\frac{ET}{2\Lambda}, \qquad V = \Lambda,$$
$$\Lambda \equiv 1 - \frac{1}{4}E^2(X^2 - T^2). \tag{2.22}$$

In this coordinate system we can distinguish four different regions:

$$\text{Region I}: \quad X > |T|, \qquad \text{Region II}: \quad X < -|T|,$$
$$\text{Region III}: \quad T > |X|, \qquad \text{Region IV}: \quad T < -|X|.$$

Regions I and II are static regions (that is, they admit a timelike Killing vector) and are analogous to the left and right wedges of Rindler space. They are separated by a "horizon" (as seen only by static observers at $\rho =$ const.) at $X^2 = T^2$ from two time-dependent regions III and IV (see Fig. 2). We will mostly dealing with the physics of region I. For a discussion of the physics in region III, see [13].

D. Physical boundary conditions

In order to have in mind a physical picture of the part of this space-time that we will be studying, we recall the gedanken experiment as shown in Fig. 1, in which two charged plates produces a *static* KKM electric field between them. As explained in detail in Appendix A, the space-time between the two plates will correspond to a

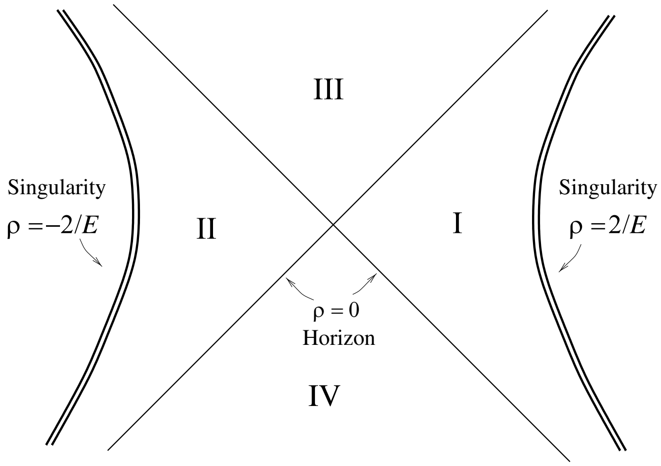

FIG. 2. The electric KK-Melvin space-time divides up into four regions. Regions I and II are static regions, while regions III and IV are time-dependent.

finite interval within region I:

$$\rho_1 < \rho < \rho_2, \quad \text{with } 0 < \rho_1 < \rho_2 < |2/E|. \tag{2.23}$$

By concentrating on the physics within this region, our physical setup will automatically exclude the unphysical regime with the closed timelike curves, as well as the horizon at $\rho = 0$. The details of this setup are given in Appendix A.

III. PARTICLE MECHANICS

In this section we consider the classical mechanics of charged particles in the electric KK-Melvin space-time, deriving the expression for the total gravitational and electrostatic potential energy. This discussion will be useful later on when we consider the quantum mechanical pair production.

A. Classical action

The classical action for a massless particle in $d+1$ dimensions is

$$S_{d+1} = \int ds[p_M \dot{x}^M + \lambda(G^{MN} p_M p_N)], \tag{3.1}$$

where $M, N = 0, \ldots, d$ and λ denotes the Lagrange multiplier imposing the zero-mass-shell condition $G^{MN} p_M p_N = 0$. Upon reduction to d dimensions, using the general Kaluza-Klein ansatz (2.17), for which

$$G^{MN} = \begin{pmatrix} g^{\mu\nu} & -A^\nu \\ -A^\mu & V^{-1} + A_\mu A^\mu \end{pmatrix}, \tag{3.2}$$

where $\mu, \nu = 0, \ldots, d-1$, the action (3.1) attains the form (here we drop the x_{d+1}-dependence)

$$S_d = \int ds\bigg[p_\mu \dot{x}^\mu + \lambda\bigg(g^{\mu\nu}(p_\mu - qA_\mu)(p_\nu - qA_\nu) + \frac{q^2}{V}\bigg)\bigg]. \tag{3.3}$$

Here we identified $q = p_{d+1}$. The λ equation of motion gives

$$g^{\mu\nu}(p_\mu - qA_\mu)(p_\nu - qA_\nu) + \frac{q^2}{V} = 0. \tag{3.4}$$

This is the constraint equation of motion of a particle with charge q and a (space-time-dependent) mass $m = q/\sqrt{V}$. For the electric KK-Melvin background (2.18), the constraint (3.4) takes the form

$$-\frac{\Lambda}{\rho^2}\bigg(p_\tau + \frac{qE\rho^2}{2\Lambda}\bigg)^2 + p_\rho^2 + p_i^2 + \frac{q^2}{\Lambda} = 0, \tag{3.5}$$

or

$$-\frac{p_\tau^2}{\rho^2} + \bigg(q - \frac{1}{2}Ep_\tau\bigg)^2 + p_\rho^2 + p_i^2 = 0. \tag{3.6}$$

Since the background is independent of all coordinates except ρ, all momenta are conserved except p_ρ. Let us denote these conserved quantities by

$$p_\tau = \omega, \qquad p_i = k_i. \tag{3.7}$$

The constraint (3.6) allows us to solve for p_ρ in terms of the conserved quantities as

$$p_\rho = \pm\sqrt{(\omega^2/\rho^2) - \mu^2}, \qquad \mu^2 \equiv k_i^2 + \bigg(q - \frac{1}{2}E\omega\bigg)^2. \tag{3.8}$$

Using this expression for p_ρ, we can write the total action of a given classical trajectory purely in terms of its beginning and end points as

$$S_\pm(x_2, x_1) = \omega\tau_{21} + k_i y_{21}^i \pm \int_{\rho_1}^{\rho_2} d\rho\sqrt{(\omega^2/\rho^2) - \mu^2}. \tag{3.9}$$

Performing the integral gives

$$S_\pm(x_2, x_1) = S_\pm(x_2) - S_\pm(x_1), \tag{3.10}$$

with

$$S_\pm(\{\rho, \tau, y\}) = k_i y^i + \omega(\tau \pm \tau_0(\rho, k_i, \omega)), \tag{3.11}$$

where

$$\tau_0(\rho, k_i, \omega) = \sqrt{1 - \bigg(\frac{\mu\rho}{\omega}\bigg)^2} - \log\bigg[\frac{\omega}{\mu\rho}\bigg(1 + \sqrt{1 - \bigg(\frac{\mu\rho}{\omega}\bigg)^2}\bigg)\bigg]. \tag{3.12}$$

This result will become useful in the following.

Notice that, for given radial location ρ, the classical trajectory only crosses this location provided the energy ω satisfies $\omega \geq \mu\rho$ with μ as defined in (3.8). The physical meaning of the quantity τ_0 in (3.12) is that it specifies the (time difference between the) instances $\tau = \pm\tau_0$ at which the trajectory passes through this radial location. Notice that indeed $\tau_0 = 0$ when $\omega = \mu\rho$, indicating that at this energy, ρ is the turning point of the trajectory.

B. Potential energy

We can use the mass-shell constraint (3.5) to solve for the total energy

$$H \equiv p_\tau = \frac{\rho}{\Lambda}\sqrt{\Lambda(p_\rho^2 + k_i^2) + q^2} - \frac{qE\rho^2}{2\Lambda}. \tag{3.13}$$

The corresponding Hamilton equations

$$\partial_\tau \rho = \frac{\partial H}{\partial p_\rho}, \qquad \partial_\tau p_\rho = -\frac{\partial H}{\partial \rho}, \tag{3.14}$$

determine the classical trajectory $\rho(\tau)$. An important quantity in the following will be the potential energy $\omega(\rho, q, k_i)$, defined via

$$\omega(\rho, q, k_i) \equiv H(p_\rho = 0) = \frac{\rho}{\Lambda}\sqrt{q^2 + \Lambda k_i^2} - \frac{qE\rho^2}{2\Lambda}. \tag{3.15}$$

That is, $\omega(\rho, q, k_i)$ is the energy of particles, with $p_{d+1} = q$ and transverse momentum $p_i = k_i$, that have their turning point at ρ.

As expected, the potential energy $\omega(\rho)$ contains two contributions: the first term is the gravitational energy due to the rest mass and momentum of the particle, and the second term represents the electrostatic potential. If qE is positive, this last term makes the particle effectively lighter than its gravitational energy. The total energy for any qE, however, never becomes negative. For qE negative, the expression (3.15) is manifestly positive. For qE either positive or negative, it can be rewritten as

$$\omega(\rho, q, k_i) = \frac{q^2 + k_i^2}{|\frac{1}{2}qE + \rho^{-1}\sqrt{q^2 + \Lambda k_i^2}|}, \tag{3.16}$$

which is again manifestly positive. We have plotted this function for $k_i^2 = 0$ in Fig. 3.

This behavior of the potential energy $\omega(\rho, q, k_i)$ should be contrasted with the classical electrostatic case, where $V(\rho, q) = m - qE\rho$ with m the rest mass, in which case the particle *can* get a negative total energy. When going to single particle wave mechanics, this negative energy leads to the famous Klein paradox, and upon second quantization, to the Schwinger pair creation effect. Since in our case the potential remains positive, there is no Klein paradox and no immediate reason to expect a vacuum instability. Nonetheless, as we will see shortly, pair creation will take place.

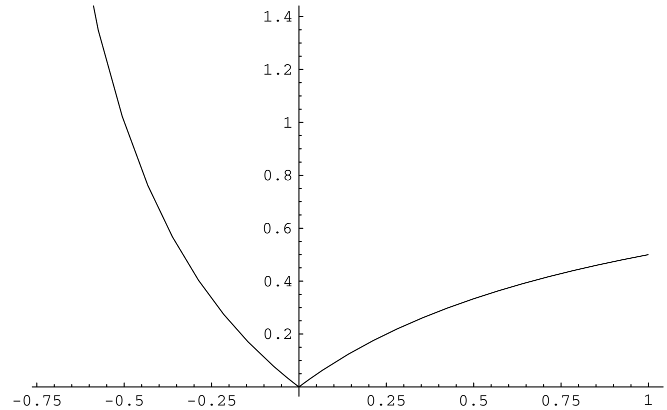

FIG. 3. The effective potential $\omega(\rho)$ defined in Eq. (3.15) (for $k_i = 0$, and multiplied by E) as a function of $x = \frac{1}{2}qE\rho$, with $q = \pm 1$.

Finally, we note that in the concrete setup of situation I of our gedanken apparatus in Appendix A, the particles are in fact restricted to move within the region $\rho_1 < \rho < \rho_2$ between the two plates. To complete the dynamical rules of the model, we need to specify what happens when the particle reaches the plates; we will simply assume reflecting boundary conditions.

IV. WAVE MECHANICS

In this section we write the solutions to the wave equations in the electric KK-Melvin background, and illustrate the semiclassical correspondence with the classical mechanics.

4.1 Wave equations

The $d + 1$-dimensional wave equation in the background (2.13) is

$$\frac{1}{\sqrt{-G}}\partial_M(\sqrt{-G}G^{MN}\partial_N\Phi) = \left[\frac{1}{\rho}\partial_\rho(\rho\partial_\rho) - \frac{1}{\rho^2}\partial_\tau^2 + \partial_i^2 + \left(\partial_{d+1} + \frac{1}{2}E\partial_\tau\right)^2\right]\Phi = 0, \tag{4.1}$$

subject to the perioding boundary condition in the x_{d+1} direction with period $2\pi R$. For a given eigenmode with $q \equiv p_{d+1} = \frac{n}{R}$, we can reduce the wave equation to d dimensions, where it can be written in the form

$$\left(\frac{\sqrt{\Lambda}}{\rho}\partial_\rho\left(\frac{\rho}{\sqrt{\Lambda}}\partial_\rho\right) - \frac{\Lambda}{\rho^2}\left(\partial_\tau + \frac{iqE\rho^2}{2\Lambda}\right)^2 + \partial_i^2 - \frac{q^2}{\Lambda}\right)\Phi = 0. \tag{4.2}$$

Here we recognize the conventional wave equation

$$\frac{1}{\sqrt{-g}}D_\mu(\sqrt{-g}g^{\mu\nu}D_\nu\Phi) - M^2\Phi = 0 \tag{4.3}$$

of a d dimensional charged particle with charge q in the background (2.18) and with a position dependent mass equal to $M^2 = q^2/\Lambda$. Note the direct correspondence of the above wave equations with the classical equations (3.5) and (3.6). They need to be solved subject to the boundary conditions imposed by our physical setup. In the case of situation I, see Fig. 5 in Appendix A, we will choose to impose Dirichlet boundary conditions at the two plates

$$\Phi|_{\rho=\rho_1} = \Phi|_{\rho=\rho_2} = 0. \tag{4.4}$$

B. Mode solutions

The $d + 1$-dimensional wave equation is solved by

$$\Phi_{qk\omega} = e^{ix_{d+1}(q-(1/2)E\omega)+ik_iy^i+i\omega\tau}K(\omega, \mu\rho), \tag{4.5}$$

with μ as defined in (3.8), and where $K(\omega, \mu\rho)$ solves the differential equation

$$((\rho\partial_\rho)^2 + \omega^2 + \mu^2\rho^2)K(\omega, \mu\rho) = 0. \tag{4.6}$$

The solution K has the integral representation

$$K(\omega, \mu\rho) = \int_{-\infty}^{\infty} d\sigma e^{i\omega\sigma - i\mu\rho\sinh\sigma}, \tag{4.7}$$

and can be expressed in terms of standard Bessel and Hankel functions [22,23]. The functions $K(\omega, \mu\rho)$ are defined for arbitrary real ω. However, upon imposing the boundary conditions that $K(\omega, \mu\rho_i) = 0$ at the location of the two plates, we are left with only a discrete set of allowed frequencies ω_ℓ. Since the corresponding mode functions (4.7) form a complete basis of solutions to (4.6), they satisfy an orthogonality relation of the form

$$\int_{\rho_1}^{\rho_2} d\rho\rho K^*(\omega_\ell, \mu_\ell\rho)K(\omega_j, \mu_j\rho) = f(\omega_\ell)\delta_{\ell,j}, \tag{4.8}$$

where $f(\omega)$ some given function that depends on ρ_1 and ρ_2.

For large ω and $\mu\rho$, we can approximate the integral in (4.7) using the stationary phase approximation. The stationary phase condition $\omega = \mu\rho\cosh\sigma$ has two solutions

$$\sigma_\pm = \pm\log\left[\frac{\omega}{\mu\rho}\left(1 + \sqrt{1 - \left(\frac{\mu\rho}{\omega}\right)^2}\right)\right] \tag{4.9}$$

provided $|\omega| > \mu\rho$, leading to

$$K(\omega, \mu\rho) \simeq \frac{\sqrt{2\pi}\cos[\omega\tau_0(\rho) + \frac{\pi}{4}]}{\sqrt{w}\sqrt[4]{1 - (\frac{\mu\rho}{\omega})^2}}, \qquad |\omega| > \mu\rho, \tag{4.10}$$

with τ_0 as given in Eq. (3.12). This formula is accurate for energies ω larger than the potential energy $\omega(\rho)$. For smaller energies there is no saddle point and the function $K(\omega, \mu\rho)$ is exponentially small

$$K(\omega, \mu\rho) \simeq \sqrt{\frac{\pi}{\mu\rho}}e^{-\mu\rho}, \qquad |\omega| < < \mu\rho, \tag{4.11}$$

reflecting the fact that the corresponding classical trajectory has its turning point before reaching ρ.

Notice that, upon inserting (4.10), the full mode function $\Phi_{qk\omega}$ in (4.5) can be written as a sum of two semiclassical contributions

$$\Phi_{qk\omega}(x) \sim \sum_\pm e^{ix_{d+1}(q+(1/2)E\omega)+ik_iy^i+i\omega(\tau\pm\tau_0(\rho,k,\omega))} \sim \sum_\pm e^{iS_\pm(y,\tau,\rho)}, \tag{4.12}$$

corresponding to the left- and right-moving part of the trajectory, respectively.

V. PAIR CREATION

In this section we will compute the pair creation rate of the Kaluza-Klein particles, following three different (though related) methods. We will start with the simplest method, by looking for Euclidean "bounce" solutions. We then proceed with a more refined method of computation, more along the lines of Schwinger's original calculation, producing the nontrivial result quoted in the introductory section. Finally, we show that the obtained result can naturally be interpreted by considering the Hawking-Unruh effect, and we use the method of Bogolyubov transformations to compute the expectation value of the charge current.

A. Classical Euclidean trajectories

Assuming that, in spite of the fact that the effective potential (3.15) seems to suggest otherwise, the nucleation of the charged particle pairs can be viewed as the result of a quantum mechanical tunneling process, we compute the rate by considering the corresponding Euclidean classical trajectory. The analytic continuation of the electric KK-Melvin space-time to Euclidean space is

$$ds_{\mathrm{E}}^2 = \frac{\rho^2}{\Lambda_{\mathrm{E}}}d\theta^2 + d\rho^2 + dy_idy^i + \Lambda_{\mathrm{E}}\left(dx_{d+1} - \frac{E\rho^2}{2\Lambda_{\mathrm{E}}}d\theta\right)^2,$$
$$A_\theta = \frac{E\rho^2}{2\Lambda_{\mathrm{E}}}, \qquad V = \Lambda_{\mathrm{E}}, \tag{5.1}$$

with θ a periodic variable with period 2π, and

$$\Lambda_{\mathrm{E}} \equiv 1 + E^2\rho^2/4. \tag{5.2}$$

This Euclidean geometry is obtained from the Lorentzian electric KK-Melvin solution via the replacement

$$E \rightarrow iE, \qquad t \rightarrow -i\theta, \tag{5.3}$$

and coincides with the spacelike section of the magnetic KK-Melvin space-time. Unlike the Lorentzian version, this Euclidean space-time extends over the whole range of positive ρ values and ends smoothly at $\rho = 0$, by virtue of the periodicity in θ. This is standard for Euclidean cousins of space-times with event horizons, and a first indication that quantum field theory in the space-time naturally involves physics at a specific finite temperature.

The Euclidean action of a point-particle, with charge (momentum in the $d + 1$-direction) equal to $p_{d+1} = q$ and mass $M = |q|/\sqrt{\Lambda_{\rm E}}$ moving in this background reads

$$S_{\rm E} = \int ds \mathcal{L}_E, \qquad \mathcal{L}_E = \frac{|q|}{\sqrt{\Lambda_{\rm E}}}\sqrt{\frac{\rho^2\dot{\theta}^2}{\Lambda_{\rm E}} + \dot{\rho}^2 + \dot{y}_i^2} - \frac{qE\rho^2}{2\Lambda_{\rm E}}\dot{\theta}. \tag{5.4}$$

As a first step, let us look for closed circular classical trajectories at constant ρ and y_i. The above point-particle action then reduces to

$$S_{\rm E}(\rho) = \frac{\pi}{\Lambda_{\rm E}}(2|q|\rho - qE\rho^2). \tag{5.5}$$

The first term is the energy of a static particle times the length of the orbit, and the second term is the interaction with the background field times the area of the loop. Looking for an extremum yields one real and positive solution

$$|E|\rho = 2\sqrt{2} - 2 \cdot \mathrm{sign}(qE) \tag{5.6}$$

with total action

$$S_{\rm E} = \frac{2\pi|q|}{|E|}(\sqrt{2} - \mathrm{sign}(qE)). \tag{5.7}$$

The existence of these solutions with finite Euclidean action is a first encouraging sign that pair creation may take place after all. The answer (5.5) for the Euclidean action also looks like a rather direct generalization of the standard semiclassical action for the Schwinger effect, and it is therefore tempting to conclude at this point that the total pair creation rate is proportional to

$$e^{-S_{\rm E}} = e^{-2\pi|q|(\sqrt{2}-\mathrm{sign}(qE))/|E|}, \tag{5.8}$$

which looks only like a numerical modification of the classic result (1.1). This conclusion is somewhat premature, however, since, in particular, the pair creation rate should depend on ρ. We would like to determine this ρ-dependence.

For this, we take a second step and consider closed Euclidean trajectories that are not necessarily circular. As in Sec. III, we now go to a Hamiltonian formulation. To transform the formulas in Sec. III to the Euclidean setup, we need to make, in addition to (5.3), the following replacements

$$s \to -is, \qquad p_\rho \to ip_\rho, \qquad p_i \to ip_i, \qquad q \to iq. \tag{5.9}$$

In this way we obtain from (3.13) a Euclidean Hamiltonian

$$H_{\rm E} \equiv p_\theta = \frac{\rho}{\Lambda_{\rm E}}\sqrt{|q|^2 - \Lambda_{\rm E}(p_\rho^2 + k_i^2)} - \frac{qE\rho^2}{2\Lambda_{\rm E}}, \tag{5.10}$$

that generates the motion of particle as a function of the Euclidean time θ, and a corresponding potential energy

$$\omega_{\rm E}(\rho, k_i, q) \equiv -H_{\rm E}(p_\rho = 0) = -\frac{\rho}{\Lambda_{\rm E}}\sqrt{|q|^2 - \Lambda_{\rm E}k_i^2} + \frac{qE\rho^2}{2\Lambda_{\rm E}}. \tag{5.11}$$

In addition to a change in sign, which is the standard way in which a potential changes when going to Euclidean space, this Euclidean potential differs from (3.15) via the replacement $\Lambda \to \Lambda_{\rm E}$. We have drawn $\omega_{\rm E}$ for $k_i = 0$ in Fig. 4. Note that $\omega_{\rm E}$ for $k_i = 0$ is proportional to the reduced effective action (5.5) for circular trajectories, and the critical radii (5.6) reside at the two minima in Fig. 4.

Our goal is to obtain semiclassical estimate for the pair creation rate at some given ρ. How should we use this Euclidean potential for this purpose? As seen from Fig. 4, there is a range of Euclidean energies $H_{\rm E}$ around the two minima (5.6) for which there exist stable, compact orbits. These orbits have a maximal and minimal radius, ρ_+ and ρ_-, at which $H_{\rm E} = \omega_{\rm E}(\rho_\pm)$. The idea now is to associate to a given ρ the corresponding Euclidean trajectory for which ρ equals one of these extrema $\rho_\pm$, and then use the total action $S_{\rm E}(\rho)$ for this trajectory to get a semiclassical estimate of the pair creation rate via

$$\mathcal{W}(\rho) \simeq e^{-S_{\rm E}(\rho)}. \tag{5.12}$$

Here it is understood that in $S_{\rm E}(\rho)$ we undo the rotation $E \to iE$, so that $\Lambda_{\rm E} \to \Lambda$. Equation (5.12) is then a clear and unambiguous formula, provided the classical orbit is closed.

In general, however, the orbits need not be closed: the period of oscillation does not need to be 2π or even a

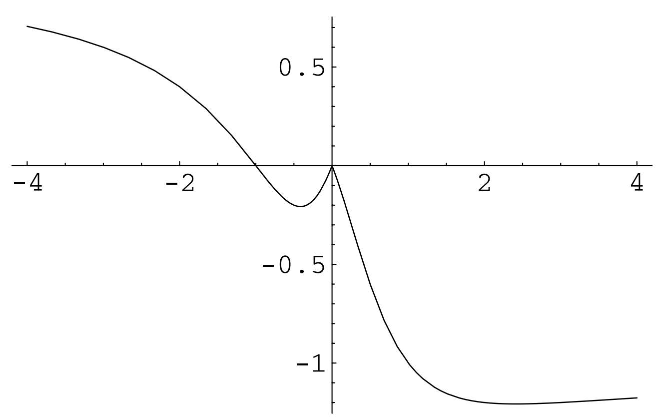

FIG. 4. The Euclidean effective potential $\omega_E(\rho)$ defined in Eq. (5.11) (for $k_i = 0$, and multiplied by E) as a function of $x = \frac{1}{2}qE\rho$, with $q = \pm 1$.

fraction or multiple thereof. How should we *define* the total classical action, to be used in (5.12) for such a trajectory?

Our proposal, that perhaps may look *ad hoc* at this point but will be confirmed and justified in the subsequent subsections, is to take for $S_{\rm E}$ the total action averaged over one full rotation period of 2π. Concretely, suppose that the compact trajectory has an "oscillation period" θ_0, in which it goes through a full oscillation starting and returning to its maximal radial position $\rho = \rho_+$. We then define $S_{\rm E}(\rho)$ as

$$S_{\rm E}(\rho) \equiv \lim_{\theta\to\infty} \frac{2\pi}{\theta} \int_0^\theta d\theta \mathcal{L}_{\rm E}(\theta, \rho) = \frac{2\pi}{\theta_0} \int_0^{\theta_0} d\theta \mathcal{L}_{\rm E}(\theta, \rho). \tag{5.13}$$

With this definition, and using the results in Sec. III A, we can now easily evaluate $S_{\rm E}(\rho)$. From (the Euclidean analog of) Eq. (3.11), while noting that $\tau_0(\rho) = 0$ since ρ is the turnaround point, we obtain

$$S_{\rm E}(\rho) = 2\pi\omega(\rho, q, k_i), \tag{5.14}$$

with ω as given in (3.15). Here we made the replacement $E \to iE$, as prescribed.

The result (5.14) together with (5.12) gives our proposed semiclassical estimate of the pair creation rate as a function of ρ. Clearly, the derivation as presented thus far needs some independent justification. It also leaves several open questions. In particular, it is not clear how we should interpret the Euclidean bounce solutions, given the fact that the real effective potential (3.15) does not seem to lead to any tunneling. A better understanding of the physics that leads to the pair creation seems needed. In the next two subsections we will present two slightly more refined derivations of the rate, which will help answer some of these questions.

B. Sum over Euclidean trajectories

We will now evaluate the pair creation rate, per unit time and volume, by means of the path-integral. Since we expect that this rate will be a function of longitudinal position ρ, we would like to express the final result as an integral over ρ. We start from the sum over all Euclidean trajectories

$$\mathcal{W} = \int \mathcal{D}p\mathcal{D}x \exp\left(-\frac{1}{\hbar} S[p, x]\right) \tag{5.15}$$

defined on flat $d+1$-dimensional space with metric and periodicity condition

$$ds_E^2 = dx^* dx + dy_i dy^i + dx_{d+1}^2, \tag{5.16}$$

$$(x, x^*, y_i, x_{d+1}) \equiv (e^{i\pi ER}x, e^{-i\pi ER}x^*, y_i, x_{d+1} + 2\pi R). \tag{5.17}$$

In the end we intend to rotate back to Lorentzian signature, replacing $E \to iE$.

We can read the expression (5.15) as a trace over the quantum mechanical Hilbert space of the single particle described by the action (3.1) or (3.3). The idea of the computation is to write this as a sum over winding sectors around the 11th direction. For each winding number w, the closed path is such that the end points are related via a rotation in the (x, x^*)-plane over an angle $w\pi ER$. Using this insight, we can write (5.15) as

$$\int d^d x \mathcal{W}(x) = R \int_0^\infty \frac{dT}{T} \sqrt{\frac{2\pi}{T}} \sum_w e^{-(1/2T)(2\pi Rw)^2} \times \mathrm{Tr}[e^{\pi iwERJ} e^{-(T/2)(p^* p + p_i^2)}], \tag{5.18}$$

where T denotes the Schwinger proper time variable, and where J denotes the rotation generator in the (x, x^*) plane. The exponent in front of the trace is the $d+1$-dimensional part of the classical action of the trajectory with winding number w. To compute the trace, we write it as an integral over mixed position and momentum eigenstates

$$\mathrm{Tr}\, A = \int d^2 x \int \frac{\prod^{d-2} dk_i}{(2\pi)^{d-2}} \langle x, k_i | A | x, k_i \rangle. \tag{5.19}$$

Next we evaluate

$$\langle x | e^{\pi iwERJ} e^{-(T/2)p^* p} | x \rangle = \frac{1}{\pi T} e^{-(xx^*/2T)(e^{\pi iERw} - 1)(e^{-\pi iERw} - 1)}, \tag{5.20}$$

where we used the standard formula for the heat kernel in two dimensions. Inserting this into (5.18), we can write the production rate as an integral over ρ of

$$\mathcal{W}(\rho) = R \int_0^\infty \frac{dT}{T} \sqrt{\frac{2\pi}{T}} \int \frac{\prod^{d-2} dk_i}{(2\pi)^{d-2}} \times \frac{e^{-(T/2)k_i^2}}{\pi T} \sum_w e^{-(1/2T)((2\pi Rw)^2 + 4\rho^2 \sin^2(\pi ERw/2))}, \tag{5.21}$$

which we will interpret as the pair production rate at the location ρ.

Equation (5.21) is an exact evaluation of the Euclidean functional determinant. To put it in a more useful form, we will assume that we are in the regime $\rho^2 \gg T$ (an assumption that we will be able to justify momentarily), so that we can simplify the expression by means of the Villain approximation

$$\sum_w e^{-(1/2T)((2\pi Rw)^2 + 4\rho^2 \sin^2(\pi ERw/2))} \simeq \sum_{w,n} e^{-(1/2T)((2\pi Rw)^2 + \rho^2(\pi ERw - 2\pi n)^2)}. \tag{5.22}$$

This replacement essentially amounts to a semiclassical approximation. The right-hand side can be reexpressed via the Poisson resummation formula (note here that the left-hand side below is just a trivial rewriting of the right-hand side above)

$$\sum_w e^{-(1/2T)[\Lambda_E(2\pi R w-(\pi E\rho^2 n^2/\Lambda_E))^2+((4\pi^2\rho^2 n^2)/\Lambda_E)]} = \sqrt{\frac{T}{2\pi\Lambda_E R^2}}\sum_m e^{-(1/\Lambda_E)[(T/2)(m^2/R^2)+(2\pi^2\rho^2 n^2/T)+(i\pi E m n\rho^2/R)]}. \tag{5.23}$$

As the final step, we may now evaluate the integral over the Schwinger parameter T via the saddle-point approximation. The saddle points are at[4]

$$T_0 = \frac{2\pi\rho|n|}{\sqrt{\frac{m^2}{R^2}+\Lambda_E k_i^2}}, \qquad n = \pm 1, \pm 2, \ldots. \tag{5.24}$$

Before plugging this back in to obtain our final result, let us first briefly check our assumption that $\rho^2 \gg T$: setting $k_i = 0$, we find $\rho^2/T_0 = m\rho/2\pi|n|R$. So as long as the spatial distance scale ρ is much larger than the KK compactification radius R, we are safe to use (5.22).

With this reassurance, we proceed and find our final answer for the pair creation rate per unit time and volume where we made the replacement $E \to iE$.[5]

$$\mathcal{W}(\rho) = \frac{1}{2\pi^2\rho^{3/2}}\int\frac{\prod^{d-2} dk_i}{(2\pi)^{d-2}}\sum_m\left(\frac{m^2}{R^2}+\Lambda k_i^2\right)^{1/4}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}} \times \exp[-2\pi n\omega(\rho, q, k_i)], \tag{5.25}$$

where $\omega(\rho, q, k_i)$ is the potential energy introduced in Eq. (3.15). The summation over n in (5.25) can be seen to correspond to the "winding number" of the Euclidean trajectory around the periodic Euclidean time direction. The $n = 1$ term dominates, and is the result announced in the Introduction. Before discussing it further, we will now proceed with a second method of derivation.

C. The Hawking-Unruh effect

The result (5.25) looks like a thermal partition function, indicating that it can be understood as produced via the Hawking-Unruh effect. We will now make this relation more explicit.

The functional integral (5.15) over all Euclidean paths represent the one-loop partition function of a scalar field Φ in the $d+1$-dimensional electric KK-Melvin space-time. We can compute this determinant also directly via canonical quantization of this field. The full expansion of Φ into modes starts with a decomposition over wave numbers along the extra dimension (in this section we restrict q to be positive)

$$\Phi = \Phi_0 + \sum_{q=1}^{\infty}(e^{iqx_{d+1}}\Phi_q + e^{-iqx_{d+1}}\Phi_q^*), \tag{5.26}$$

where Φ_0 is massless and real, and Φ_q are complex and have mass $m = q$. Let us define μ_+ and μ_- via

$$\mu_+^2 = \left(q+\frac{1}{2}\omega E\right)^2 + k_i^2, \qquad \mu_-^2 = \left(-q+\frac{1}{2}\omega E\right)^2 + k_i^2, \tag{5.27}$$

so that now $\mu_\pm$ are quantities related to positively or negatively charged particles.

To proceed, we now need to expand the field Φ_q in creation and annihilation modes, allowing only modes that satisfy the boundary conditions (4.4) that $\Phi_q(\rho_i) = 0$ at the location of the two charged plates.

$$\Phi_q = \sum_{\omega>0}\int\frac{\prod^{d-2} dk_i}{(2\pi)^{d-2}}\frac{e^{-i\omega\tau+ik_iy^i}}{\sqrt{\omega f(\omega)}}(K(\omega, \mu_+\rho)a_q(k_i, \omega) + K^*(\omega, \mu_-\rho)a^\dagger_{-q}(k_i, \omega)), \tag{5.28}$$

with $K(\omega, \mu\rho)$ and $f(\omega)$ as defined in (4.7) and (4.8). The creation and annihilation modes then satisfy the usual commutation relations.

$$[a_{\pm q}(k_1, \omega_1), a^\dagger_{\pm q}(k_2, \omega_2)] = \delta(k_1 - k_2)\delta_{\omega_1\omega_2}. \tag{5.29}$$

Our goal is to determine what the natural vacuum state of the Φ field looks like, as determined by the initial conditions. In the far past, we imagine that the KK electric field was completely turned off. The electric KK-Melvin background then reduced to Rindler or Minkowski space—depending on which coordinate system one introduces. The most reasonable initial condition is that the quantum state of all Φ quanta starts out in the vacuum as defined in the Minkowski coordinate system. Let us denote this Minkowski vacuum by $|\Omega\rangle$.

To determine the expression for $|\Omega\rangle$ in terms of our mode basis, we can follow the standard procedure [16,17]. We will not go into the details of this calculation here, except to mention one key ingredient: the mode functions, when extended over the full range of ρ values, have a branch-cut at the horizon at $\rho = 0$, such that

$$K(\omega, -\mu\rho) = e^{\pm 2\pi\omega}K^*(\omega, \mu\rho), \tag{5.30}$$

depending on whether the branch-cut lies in the upper or lower-half plane. This behavior of $K(\omega, \mu\rho)$ near $\rho = 0$ is sufficient to deduce the form of the Bogolyubov transformation relating the modes $a(\omega, k)$ to the Minkowski

[4] We drop the term with $n = 0$, since it corresponds to the vacuum contribution.

[5] Note that the same result can be obtained by replacing the sum in (5.21) by (5.23), integrating over T exactly, and then using Eq. (4.11) to approximate the resulting Bessel function.

creation and annihilation modes (see, e.g., [17]). As a result, one finds that the Minkowski vacuum, $|\Omega\rangle$, behaves like a thermal density matrix for the observable creation and annihilation modes in (5.28). In particular, the number operator for each mode has the expectation value

$$\langle\Omega|a_q^\dagger(k,\omega)a_q(k,\omega)|\Omega\rangle = \frac{1}{e^{2\pi\omega}-1}, \tag{5.31}$$

while the overlap of $|\Omega\rangle$ with the empty vacuum state, defined via $a_q(k,\omega)|0\rangle = 0$, becomes

$$|\langle 0|\Omega\rangle|^2 = \exp\left[\int \frac{\prod^{d-2} dk_i}{(2\pi)^{d-2}} \sum_{q,\omega} |\log(1-e^{-2\pi\omega})|\right] \tag{5.32}$$

$$= \exp\left[-\int \frac{\prod^{d-2} dk_i}{(2\pi)^{d-2}} \sum_{q,\omega,n} \frac{1}{n} e^{-2n\pi\omega}\right]. \tag{5.33}$$

This expression represents the probability that the state Ω does not contain any particles—and its dominant $n = 1$ term looks indeed closely related to the result (5.25) obtained in the previous subsection.

The difference between the two equations is that (5.25) is defined at a particular location ρ, while (5.32) contains a summation over all frequencies. To make the relation more explicit, imagine placing some measuring device at a location ρ. As mentioned before, only modes with a sufficiently large frequency will reach this location with any appreciable probability, and the probability attains a maximum for frequencies equal to the potential energy at ρ, since for those frequencies, ρ is the turning point. Via this observation, we can view the position ρ as a parametrization of the space of frequencies, via the insertion of

$$1 = \int d\rho\, \delta(\omega - \omega(\rho))|\partial_\rho \omega(\rho)|, \tag{5.34}$$

with $\omega(\rho)$ as given in Eq. (3.15), thus replacing the summation over ω in (5.32) by an integral over ρ. The integrand at given ρ is then naturally interpreted as the production rate (5.25) at the corresponding location. This procedure is a good approximation provided the distance $d = \rho_2 - \rho_1$ between the plates is large enough, so that many frequencies contribute in the sum.

This same condition is also important for a second reason [5]. Since we would like to imagine that the pair production takes place at a constant rate per unit time, we would like to see that the overlap (5.32) in fact decays exponentially with time. This comes about as follows [5]. Suppose we restrict the field modes to be supported over a finite time interval $0 < \tau < T$. This translates into a discreteness of the frequencies. Ignoring at first the other discreteness due to the reflecting boundary condition at the two parallel plates, it is clear that the density of frequencies allowed by the time restriction grows linearly with T. The sum over the frequencies thus produces an overall factor of T. In this way, we recover the expected exponential decay of the overlap (5.32).

This exponential behavior breaks down, however, as soon as the time interval T becomes of the same order as the distance d between the plates, or more precisely, when $1/T$ approaches the distance between the discrete energy levels allowed by the reflecting boundary conditions at the plates. At this time scale, the situation gradually enters into a steady state, in which the pair creation rate gets balanced by an equally large annihilation rate. The system then reaches a thermal equilibrium, specified by the thermal expectation value (5.31). The physical temperature of the final state depends on the location ρ via

$$\beta = 2\pi\sqrt{g_{00}} = \frac{2\pi\rho}{\sqrt{\Lambda}}. \tag{5.35}$$

Note that this temperature diverges at $\rho = 0$ and $\rho = |2/E|$; neither location is within our physical region, however.

D. Charge current

It is edifying to consider the vacuum expectation value of the charge current, since this is a clear physical, observer-independent quantity and a sensitive measure of the local profile of the pair creation rate. For given q, the charge current is given by

$$j_\mu = iq(\Phi_q^*(\rho)\partial_\mu\Phi_q(\rho) - \Phi_q(\rho)\partial_\mu\Phi_q^*(\rho)). \tag{5.36}$$

Using the result (5.31) for the expectation value of the number operator, one finds that the time component of the current, the charge density, is nonzero and equal to[6]

$$\langle\Omega|j_\tau(\rho)|\Omega\rangle = J_+(\rho) - J_-(\rho) \tag{5.37}$$

with

$$J_\pm(\rho) = q\sum_{\omega>0}\int \frac{\prod^{d-2} dk_i}{(2\pi)^{d-2}} \frac{|K(\omega,\mu_\pm\rho)|^2}{f(\omega)(e^{2\pi\omega}-1)} \tag{5.38}$$

the positive and negative charge contributions, respectively. Given the thermal nature of the state $|\Omega\rangle$, the physical origin of this charge density is clear: the presence of the electric field reduces the potential energy of one of the two charge sectors, thereby reducing its Boltzmann suppression, relative to the oppositely charged.

To obtain a rough estimate for the behavior of $J_\pm(\rho)$, it is useful to divide the frequency sum into three regions: (i) ω comparable to the potential energy (3.15), (ii) ω much larger, or (iii) ω much smaller. By comparing the respective suppression factors, we find that the leading semiclas-

[6]Instead of the expectation value (5.37), one could also consider the mixed in-out expectation value $\langle 0|J_\tau(\rho)|\Omega\rangle$, which is related to the derivative of the in-out matrix element $\langle 0|\Omega\rangle$ with respect to E. This relation was in fact used by Schwinger in his original derivation of the pair creation rate [1].

sical contribution comes from regime (i); this is also reasonable from a physical perspective, since these are the particles that spend most time near ρ. Regime (ii) is strongly Boltzmann suppressed and clearly negligible compared to contribution (i), while regime (iii) is suppressed because the corresponding mode functions $K(\omega, \mu\rho)$ are exponentially small at the location ρ, via (4.11). The leading contribution of region (i) is of order $e^{-2\pi\omega(\rho,q,k_i)}$, in accordance with the result (5.25) for the pair creation rate $\mathcal{W}(\rho)$.

Since the mode functions $K(\omega, \mu\rho)$ are real (they are the sum of an incoming and reflected wave), the current in the ρ direction appears to vanish. The result (5.38) for the charge density indeed looks static. This static answer, however, cannot describe the time-dependent pair creation process. Recalling our discussion above, however, we can recover this time-dependence by restricting the sum over only those frequencies necessary to cover the finite time interval $0 < \tau < T$. This is a T dependent subset, thus leading to a T dependent (initially linearly growing) charge density. However, when $1/T$ becomes much smaller than the step-size in the allowed frequency spectrum, the steady state sets in and the charge density indeed becomes a static thermal distribution given by (5.37) and (5.38).

VI. DISCUSSION

In this paper we have tried to make a systematic study of the Schwinger pair production of charged Kaluza-Klein particles. Because of their characteristic property that their mass is of the same order as their charge q, the pair creation requires such strong KK electric fields that gravitational backreaction cannot be ignored. We have included this backreaction by means of the electric KK-Melvin solution, and shown that, in spite of the fact that the electrostatic potential cannot be made to exceed the rest mass of the KK particles, pair production takes place at a rate given by (1.2).

What is the physical mechanism that is responsible for the pair creation? Our final answer (1.2) includes both the KK electric field and a gravitational acceleration a. It is instructive to compare this result with the known rate [20] for Schwinger pair production in an accelerated frame, as quoted in Eq. (B4) in the appendix. Since in our case a is bounded below by $E/2$, we can only directly compare the two answers in the limit of small electric field. In that limit, if we expand the log in Eq. (B4) and take the dominant $n = 1$ term there, both answers become

$$\mathcal{W}(E,a) \simeq \sum_{q=\pm|q|} \int \frac{\prod^{d-2} dk_i}{(2\pi)^d} \exp\left(-2\pi\left(\frac{1}{a}\sqrt{q^2+k_i^2} - \frac{qE}{2a^2}\right)\right). \tag{6.1}$$

In this regime, however, one cannot honestly separate the Schwinger pair creation effect from the pair creation effect due to the acceleration. Electric charge is being produced, but it is just a simple consequence of the fact that the electrostatic potential reduces the Boltzmann factor for one type of charge, while increasing it for the other. Rather than producing the charge "on its own," the electric field just polarizes the thermal atmosphere produced by the Unruh effect.

In fact, if we write the potential $\omega(\rho, q, k_i)$ as in (3.15) instead of (3.16), our final answer (1.2) appears to be just a small modification of (6.1) and the physics that leads to it indeed seems quite identical. So depending on taste, one can either interpret our result (1.2) as pair creation due to a combination of the Schwinger and Unruh effect, or as the result of the Unruh effect only. There is no definite way to decide between the two, since the gravitational acceleration cannot be turned off independently. Either way, what is clear is that the mechanism for pair creation cannot be given a tunneling interpretation.

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APPENDIX A: GEDANKEN APPARATUS

For a good understanding of the situation we wish to study, it will be useful to investigate how, via a concrete gedanken experiment, one may in fact attempt to create a large static Kaluza-Klein electric field. Without taking into account gravitational backreaction, we imagine taking two parallel plates with opposite KK charge density per unit area σ and perpendicular distance d, thus creating an electric field $E = 4\pi\sigma$ in the region between the plates. It turns out, however, that when we include the gravitational backreaction of both the plates and the electric field, there are some restrictions on how symmetric, or static, we can choose our experimental setup.

Consider two charged, infinitesimally thin, parallel plates at positions ρ_1 and ρ_2, separated by a distance

$$d = \rho_2 - \rho_1. \tag{A1}$$

The two plates divide space into three regions: Region A left of the first plate, given by $\rho < \rho_1$, region B in between the two plates, $\rho_1 < \rho < \rho_2$, and region C right of the second plate $\rho > \rho_2$.

Let the mass densities of the plates be given by μ_1 and μ_2, so that

$$T_0^0 = \mu_1 \delta(\rho - \rho_1) + \mu_2 \delta(\rho - \rho_2). \tag{A2}$$

In addition, the two plates have charge densities σ_1 and σ_2

$$\sqrt{g_{00}}T^0_{d+1} = \sigma_1\delta(\rho - \rho_1) + \sigma_2\delta(\rho - \rho_2). \tag{A3}$$

We will assume that the charge densities are opposite, $\sigma_1 = -\sigma_2$, and tuned so that there is a Kaluza-Klein electric field in region B between the plates, but none in regions A or C outside the plates. The region between the plates therefore takes the form of a static slice $\rho_1 < \rho < \rho_2$ of the electric KK-Melvin space-time. The two regions outside the plates, on the other hand, are just flat. More precisely, since the parallel plates in effect produce an attractive gravitational force on freely falling particles in the two outside regions, the regions A and C should correspond to static subregions in Rindler space.

Both Rindler space and the electric KK-Melvin solution differ from Minkowski space only via the g_{00} component. Imposing continuity at $\rho = \rho_i$, this leads us to the following ansatz for the g_{00} component of the metric in the three regions

$$\begin{aligned} g^A_{00}(\rho) &= (1 - a_1(\rho - \rho_1))^2 g^B_{00}(\rho_1), \\ g^B_{00}(\rho) &= \frac{\rho^2}{1 - E^2\rho^2/4}, \\ g^C_{00}(\rho) &= (1 + a_2(\rho - \rho_2))^2 g^B_{00}(\rho_2). \end{aligned} \tag{A4}$$

Here a_1 and a_2 are both positive, and represent the respective free fall accelerations of freely moving particles just outside of the two plates. In other words, via the equivalence principle, a_1 and a_2 are the accelerations (to the left and right, respectively) of the two plates as viewed from the outside Minkowski observers. The quantities ρ_1 and ρ_2 play a similar role, and can be both positive and negative. A physical restriction, however, is that the denominator in the expression (A4) for g^B_{00} remains positive.

In addition there is a nontrivial electric potential $g^B_{0,d+1}$ in the region between the plates, while $g^{A,C}_{0,d+1}$ are constants determined by continuity:

$$g^A_{0,d+1}(\rho) = g^B_{0,d+1}(\rho_1), \qquad g^B_{0,d+1}(\rho) = \frac{\rho^2 E/2}{1 - E^2\rho^2/4}, \qquad g^C_{0,d+1}(\rho) = g^B_{0,d+1}(\rho_2). \tag{A5}$$

The $d + 1$-dimensional Einstein equations of motion result in the following jump conditions for the normal variations of g_{00} and $g_{0,d+1}$ at the location of the plates[7]

$$4\pi\mu_i = g^{00}(\partial_{\rho_+}g_{00} - \partial_{\rho_-}g_{00})|_{\rho=\rho_i}, \tag{A6}$$

$$4\pi\sigma_i = (g^{00})^{1/2}(\partial_{\rho_+}g_{0,d+1} - \partial_{\rho_-}g_{0,d+1})|_{\rho=\rho_i}. \tag{A7}$$

[7]Note that while the expressions $g_{0,d+1}$ are not gauge invariant, the Gauss equation is, as long as λ in $A_\mu dx^\mu \to A_\mu dx^\mu + d\lambda$ is smooth across ρ_1 and ρ_2, i.e., $(\partial_{\rho_+} - \partial_{\rho_-})\lambda|_{\rho=\rho_i} = 0$.

The first of these equations is known as the Israel equation, while the second is equivalent to Gauss's law in electromagnetism. Inserting our ansatz, the Israel jump conditions become

$$2\pi\mu_1 = a_1 + \frac{1}{\rho_1\Lambda_1}, \qquad 2\pi\mu_2 = a_2 - \frac{1}{\rho_2\Lambda_2}, \qquad \Lambda_i = 1 - \frac{1}{4}E^2\rho_i^2 \tag{A8}$$

while Gauss's law takes to the form

$$4\pi\sigma_1 = E/\Lambda_1^{3/2}, \qquad 4\pi\sigma_2 = -E/\Lambda_2^{3/2}. \tag{A9}$$

Equation (A8) relates the mass density of the two plates to the jump in the surface acceleration when moving from one to the other side, while (A9) relates the charge density to the jump in the KK electric field.

Let us briefly check these formulas by considering some special cases. If $E = 0$, then we can choose the symmetric situation $\mu_1 = \mu_2$ and $a_1 = a_2$. Via (A8) this implies that we should take the limit $\rho_i \to \infty$ keeping the distance (A1) fixed. The intermediate region B then simply reduces to flat Minkowski space. This is as expected, since the two plates lead to an equal and opposite gravitational force, which exactly cancels in the intermediate region. For nonzero E, yet small electrostatic potential $V_{12} = Ed$ between the plates, we can choose parameters such that $E\rho_i \ll 1$ and $\rho_i \gg d$. Equation (A9) then reduces to the standard Gauss law of Maxwell theory.

Let us now consider the general case. There are four equations, and (for given interplate distance d, and densities μ_i and σ_i) four unknowns: a_1, a_2, ρ_1 and E. The second equation in (A9), however, is not really independent from the first, since we should rather read it as a fine-tuning condition on σ_2 (relative to σ_1) ensuring that the E-field vanishes outside the two plates. Discarding this equation, we are thus left with one overall freedom, namely, the overall acceleration of the center of mass of our apparatus.

For practical purposes, we would have preferred to restrict ourselves to the simplest and most symmetric case in which the two plates have equal mass density $\mu_1 = \mu_2$ and equal surface acceleration $a_1 = a_2$. This would, in particular, ensure that our apparatus is at rest. As seen from Eq. (A8), this symmetric situation could be reached if we could take the limit $\rho_i \to \infty$. However, for nonzero E, this limit is forbidden via the restriction $1 - \frac{1}{4}E^2\rho^2 > 0$. Thus we are basically forced to consider the general situation with μ_i and a_i arbitrary, and ρ_i both positive. We call this:

$$\text{Situation I}: \qquad 0 < \rho_1 < \rho_2 < 2/E, \qquad \mu_i \text{ arbitrary}. \tag{A10}$$

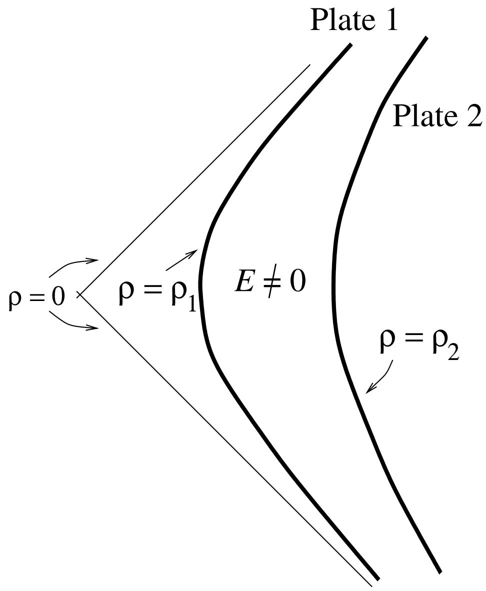

FIG. 5. Situation I describes the static situation with two charged plates at $\rho = \rho_1$ and $\rho = \rho_2$, with $0 < \rho_1 < \rho_2 < 2/E$. The region of interest, in between the two plates, is a static slice of region I of the electric KK-Melvin solution.

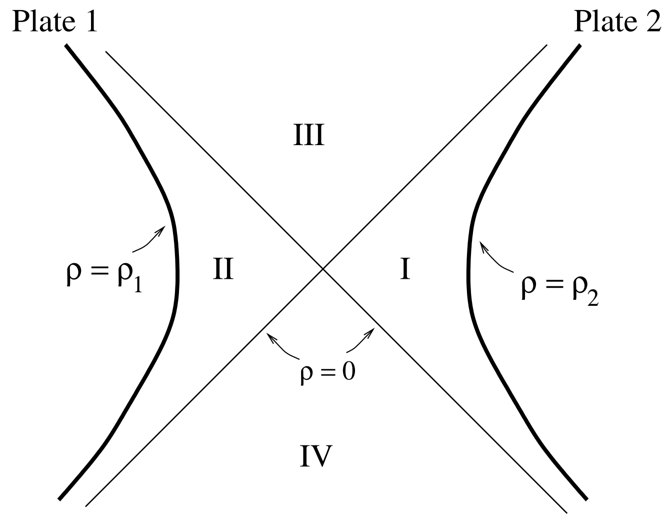

FIG. 6. Situation II describes the time-dependent situation with two accelerating charged plates at $\rho = \rho_1 < 0$ and $\rho = \rho_2 > 0$. The region of interest, in between the two plates, includes the time-dependent regions III and IV of the electric KK-Melvin solution.

In this case, the region of interest, region B, represents a static slice in the right wedge of the electric KK-Melvin solution. This situation I is the natural generalization of a constant, static electric field, and is our starting point for studying the possible Schwinger pair creation of charged KK particles. We sketch it in Fig. 5.

There is, however, another situation we could consider, which does allow for a symmetric solution. Namely we can choose:

$$\text{Situation II:} \qquad \mu_1 = \mu_2, \qquad a_1 = a_2, \qquad \rho_1 = -\rho_2. \tag{A11}$$

In this case the region B includes the special position $\rho = 0$ at which $g_{00} = 0$, the location of the event horizon of the electric KK-Melvin geometry (see Fig. 6).

To better understand the experimental conditions leading to situation II, consider the special case $\mu_1 = \mu_2 = 0$, and $\sigma_1 = \sigma_2 = 0$. This describes two plates with zero-mass and charge, accelerating away from each other with equal but opposite acceleration $a_i = 1/\rho_i$. It is now easy to imagine that one can gradually add mass and charge to the plates, and reach the general situation II. It must be noted that this experimental setup does not lead to a static background, since the geometry now includes the time-dependent regions III and IV enclosed by the Rindler horizon (see Fig. 1). This setup is therefore not a direct analog of the static electric field considered by Schwinger. For a discussion of situation II see [13]; our main focus is situation I.

APPENDIX B: SCHWINGER MEETS RINDLER

In this appendix we summarize the known result for the Schwinger pair creation rate in an accelerating frame [20] of charged particles with mass q and mass m with $m \ll q$. In this regime, pair creation starts to occur while the gravitational backreaction of the electric field is still negligible.

The closest analog of a constant, uniform gravitation field is Rindler space

$$ds^2 = -\rho^2 d\tau^2 + d\rho^2 + dy_i^2. \tag{B1}$$

Particles, or detectors, located at a given ρ undergo a uniform acceleration $a = 1/\rho$. Consider a charged field propagating in this space in the presence of a uniform electric field, described by

$$A_\tau = \frac{1}{2} E\rho^2. \tag{B2}$$

The resulting scalar wave equation reads

$$\left(\frac{1}{\rho}\partial_\rho(\rho\partial_\rho) - \frac{1}{\rho^2}\left(\partial_\tau + \frac{i}{2}qE\rho^2\right)^2 + \partial_i^2 - m^2\right)\Phi = 0. \tag{B3}$$

The above three equations are connected to the ones in Sec. IV by setting $\Lambda = 1$ (which indeed amounts to turning off the backreaction) and by setting $m = q$ above.

Equation (B3) has known mode solutions with given Rindler frequency, in terms of Whittaker functions [20,22]. These functions have a relatively intricate, but known (Eq. 9.233 in [22]), branch-cut structure at $\rho = 0$, from which one can straightforwardly extract the linear combination of (left and right wedge) Rindler creation and annihilation modes that annihilate the Minkowski vacuum $|\Omega\rangle$. One obtains the following result for the total pair creation rate per unit time and (transverse) volume [20]

$$\mathcal{W} \simeq \sum_{q=\pm|q|} \int d\omega \int \frac{\prod^{d-2} dk_i}{(2\pi)^d} \log\left(\frac{(1 - e^{-2\pi\omega})(1 - e^{-\frac{\pi(m^2+k_i^2)}{|qE|}})}{1 - e^{-2\pi(\omega + \frac{m^2+k_i^2}{2|qE|})}}\right). \tag{B4}$$

As explained in Sec. V C, we can extract from this result the pair creation rate at a given radial location ρ, or equivalently, given acceleration $a = 1/\rho$, by equating the frequency ω with the classical potential energy at this location

$$\omega(q, k_i) = \frac{1}{a}\sqrt{q^2 + k_i^2} - \frac{qE}{2a^2}. \tag{B5}$$

As discussed in Sec. VI, in the limit where the electric field is small, the expression (B4) reduces to our result.

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