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Abstract

Models of particle physics based on manifolds of G_2 holonomy are in most respects much more complicated than other string-derived models, but as we show here they do have one simplification: threshold corrections to grand unification are particularly simple. We compute these corrections, getting completely explicit results in some simple cases. We estimate the relation between Newton's constant, the GUT scale, and the value of α_{GUT}, and explore the implications for proton decay. In the case of proton decay, there is an interesting mechanism which (relative to four-dimensional SUSY GUT's) enhances the gauge boson contribution to $p \to \pi^0 e_L^+$ compared to other modes such as $p \to \pi^0 e_R^+$ or $p \to \pi^+ \overline{\nu}_R$. Because of numerical uncertainties, we do not know whether to intepret this as an enhancement of the $p \to \pi^0 e_L^+$ mode or a suppression of the others.

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1 Introduction

In the original estimation of the running of coupling constants in grand unified theories [1], it was found that the low energy $SU(3) \times SU(2) \times U(1)$ couplings can be unified in a simple gauge group such as $SU(5)$ [2] at an energy of about 10^{15} GeV. This energy is suggestively close to the Planck mass $M_{Pl} = 1.2 \times 10^{19}$ GeV, at least on a log scale. This hints at a unification that includes gravity as well as the other forces.

Subsequently, with more precise measurements of the low energy couplings, it became clear that coupling unification occurs much more precisely in *supersymmetric* grand unification. Supersymmetry raises the unification scale M_{GUT} to about $2 - 3 \times 10^{16}$ GeV [3], assuming that the running of couplings can be computed using only the known particles plus Higgs bosons and superpartners. Incorporating supersymmetry reduces the discrepancy between the unification scale and the Planck scale, measured logarithmically, by about one third.

In this discussion, it is not clear if the unification mass should be compared precisely to M_{Pl} or, say, to $M_{Pl}/2\pi$. For this question to make sense, one needs a more precise unified theory including gravity. The first sufficiently precise model was the perturbative $E_8 \times E_8$ heterotic string [4]. In this model, the discrepancy between the scales of grand unification and gravity was again reduced, relative to the original estimates, essentially because the string scale is somewhat below the Planck scale. The remaining discrepancy, evaluated at tree level, is about a factor of 20 – that is, the GUT or grand unification scale as inferred from low energy couplings is about 20 times smaller than one would expect based on the tree level of the heterotic string. For a discussion, see [5].

This discrepancy, which is about six percent on a log scale (and thus roughly a third as large as the mismatch originally estimated in [1]), is small enough to raise the question of how much the one-loop threshold corrections might close the gap. The threshold corrections have been calculated [6], generalizing the corresponding computations in field theory [7], [8]. The threshold corrections depend on the detailed choice of a heterotic string compactification. In most simple models, they seem too small to give the desired effect.

An alternative is to assume additional charged particles with masses far below the unification scale, so as to modify the renormalization group estimate of the unification scale. To preserve the usual SUSY GUT prediction for the weak mixing angle $\sin^2 \theta_W$, one might limit oneself to complete $SU(5)$

multiplets; for assessment of possibilities in this framework, see [9],[10],[11].

Another alternative is to consider the *strongly coupled* $E_8 \times E_8$ heterotic string, in which an eleventh dimension opens up and gauge fields propagate on the boundaries [12]. In this case, rather than a prediction for the relation between Newton's constant $G_N = 1/M_{Pl}^2$ and the scale of grand unification, one gets only an inequality [13]. The inequality is difficult to evaluate precisely; the estimate given in [13] is

$$G_N \geq \frac{\alpha_{GUT}^2}{16\pi^2} \left| \int_Z \omega \wedge \frac{\mathrm{tr} F \wedge F - \frac{1}{2} R \wedge R}{8\pi^2} \right|, \tag{1.1}$$

where ω is the Kahler form of the Calabi-Yau manifold Z, and one would expect the integral in (1.1) to be of order $1/M_{GUT}^2$ times a number fairly close to 1; a further estimate was given in [14] , where this bound was lowered by a factor of 2/3. The observed value of G_N is rather close to saturating this inequality and probably does so within the uncertainties.[1] If this inequality, or the one derived in the present paper, is saturated, this is a very interesting statement about nature, but a theoretical reason to expect the inequality to be saturated is not clear.

The purpose of the present paper is to study this and related questions in the context of another related type of model, namely M-theory compactification on a (singular) manifold of G_2 holonomy. Such models can be dual to heterotic strings; in many (but presumably not all) instances, a compactification of M-theory on a G_2 manifold X is dual to the compactification of a heterotic string on a Calabi-Yau threefold Z. So many models actually have (at least) three different dual descriptions: as a perturbative $E_8 \times E_8$ heterotic string; as a strongly coupled heterotic string with gauge fields on the boundary; and in terms of M-theory on a singular G_2 manifold. As in most examples of duality, when one of these descriptions is useful, the others are strongly coupled and difficult to use.

Apart from duality with the heterotic string, G_2 models that might be relevant to phenomenology can be constructed via duality with certain Type II orientifolds [15]-[18], which in turn may have descriptions via the Type I string or $SO(32)$ heterotic string.

A particular approach to constructing semirealistic particle physics models based on G_2 manifolds was described in [19]. The ability to generate chiral

[1]There is also a special case, with equal instanton numbers in the two boundaries, in which one does not obtain such an inequality, and G_N can be much smaller. This case was not pursued seriously in [13], because of a preference to maintain the successes of grand unification.

fermions depended on somewhat subtle singularities that have been studied in [20] - [23]. We will pursue this approach further in the present paper.

G_2 manifolds are much more difficult and less fully understood than the Calabi-Yau threefolds that can be used in constructing heterotic string compactifications. However, as we will see, they have at least one nice simplification relative to the generic Calabi-Yau models: the threshold corrections are rather simple in the case of compactification on a G_2 manifold. They are in fact given by a topological invariant, the Ray-Singer analytic torsion [24],[25]. As a result, the threshold corrections can be computed explicitly based on topological assumptions, without needing to know the detailed form of the G_2 metric. That is convenient, since explicit G_2 metrics are unknown.

In this paper, we will carry out for G_2 manifolds as much as we can of the usual program of grand unification. We compute the threshold corrections, obtain as precise an estimate as we can for the radius of compactification, derive the inequality which is analogous to (1.1), and attempt to estimate the proton lifetime. Of course, there is some serious model dependence here; we attempt to make statements that apply to classes of interesting models. For example, to discuss proton decay in a sensible fashion, we need to assume a mechanism for doublet-triplet splitting; here, we assume the mechanism of [19] (which in turn is a close cousin of the original mechanism for doublet-triplet splitting in Calabi-Yau compactification of the heterotic string [26]; this mechanism has been reconsidered recently from a bottom-up point of view [27] - [31]). One consequence of the particular doublet-triplet splitting mechanism that we use is that dimension five (or four) operators contributing to proton decay are absent; proton decay is dominated, therefore, by dimension six operators. In four dimensions, those operators are induced by gauge boson and Higgs boson exchange; we will see that in the present context, in a sense, there is a direct M-theory contribution. We will focus in this paper on $SU(5)$ models, partly for simplicity and partly because the doublet-triplet splitting mechanism of [19] has been formulated for this case.

The results we get are as follows:

(1) For $SU(5)$ models, the threshold corrections due to Kaluza-Klein harmonics do not modify the prediction for $\sin^2\theta_W$ obtained from four-dimensional supersymmetric GUT's. (This is somewhat analogous to a similar result for massive gauge bosons in four-dimensional $SU(5)$ models.) But they do modify the unification scale; that is, they modify the relation between the compactification scale and the scale of grand unification as estimated from low energy gauge couplings. This modification is significant in estimating the predictions for Newton's constant and for proton decay.

(2) The inequality analogous to (1.1) is again rather close to being saturated and perhaps even more so. The details depend on the threshold corrections as well as on other factors that will appear in the discussion.

(3) In four-dimensional SUSY GUT's, the gauge boson exchange contributions to proton decay, if they dominate, lead to a proton lifetime that has been estimated recently as $5 \times 10^{36\pm1}$ years [32]. (Four-dimensional GUT's also generally have a dimension five contribution that can be phenomenologically troublesome; it is absent in the models we consider. See also [33] for another approach.) Relative to four-dimensional GUT's, we find a mechanism that enhances proton decay modes such as[2] $p \to \pi^0 e_L^+$ relative to $p \to \pi^0 e_R^+$ or $p \to \pi^+ \overline{\nu}_R$; these are typical modes that arise from gauge boson exchange. Given numerical uncertainties that will appear in section 5, it is hard to say if in practice this mechanism enhances the $p \to \pi^0 e_L^+$ modes relative to GUT's or suppresses the others. Because of these issues, to cite a very rough estimate of the proton lifetime, we somewhat arbitrarily keep the central value estimated in the four-dimensional models and double the logarithmic uncertainty, so that the proton lifetime in these models if gauge boson exchange dominates might be $5 \times 10^{36\pm2}$ years.[3] (In practice, the uncertainty could be much larger if because of additional light charged particles, the unification scale is modified; the same statement applies to four-dimensional GUT models.)

This paper is organized as follows. In section 2, we review how semi-realistic models of particle physics can be obtained by compactification of M-theory on a singular manifold of G_2 holonomy. In section 3, we express the threshold corrections in terms of Ray-Singer torsion and make everything completely explicit in the case of a lens space. In section 4, we work out the implications for Newton's constant. In section 5, we discuss proton decay. In section 6, we discuss some unresolved issues about these models. Finally, in the appendices, we do some calculations of torsion to fill in some loose ends in section 3.

2 Review Of Models

As we will now recall, duality with the heterotic string implies that semi-realistic models of particle physics can be derived by compactification of

[2]And similar modes with π^0 replaced by K^0, or e_L^+ by μ_L^+. As in most four-dimensional GUT's, we do not have precise knowledge of the flavor structure.

[3]The current experimental bound on $p \to \pi^0 e^+$ is 4.4×10^{33} years (for a recent report, see [34]) and the next generation of experiments may improve this by a factor of 10 to 20 (for example, see [35]).

M-theory on a manifold of G_2 holonomy. (One could also use Type II orientifolds as the starting point here [15]-[18].) Semi-realistic means that gauge groups and fermion quantum numbers come out correctly, but there is no good understanding of things that depend on supersymmetry breaking, like the correct fixing of moduli, as well as fermion masses (which depend on the moduli).

In fact, semi-realistic models of particle physics can be derived by compactification of the $E_8 \times E_8$ heterotic string on a Calabi-Yau threefold Z. Most Calabi-Yau threefolds participate in mirror symmetry. Mirror symmetry is interpreted to mean [36] that in a suitable region of its moduli space, Z is fibered over a three-manifold Q with fibers that are generically copies of a three-torus $\mathbf{T}^3$.

Now, we take Q to be very large in string units, while keeping the size of the $\mathbf{T}^3$ fixed and taking the string coupling constant to infinity. The strong coupling limit of the heterotic string on $\mathbf{T}^3$ is M-theory on K3. So in this limit, we replace all of the $\mathbf{T}^3$'s by K3's. The heterotic string on Z then turns into M-theory on a seven-manifold X that is fibered over Q with generic fibers being copies of K3. Supersymmetry requires that X actually has G_2 holonomy.

Suppose that the original heterotic string model had an unbroken gauge group $G \subset E_8 \times E_8$. If G is simple, for example $G = SU(5)$, then it is a group of type A, D, or E (as these are the simple gauge groups that occur in heterotic string models at level one), and will appear in M-theory on K3 as an orbifold singularity of the appropriate type. Such a singularity will appear in each fiber of the fibration $X \to Q$. They will fit together into a copy of Q embedded in X along which X has the appropriate A, D, or E orbifold singularity. In the present paper, when we want to be specific, we consider $G = SU(5)$, in which case Q is a locus of $\mathbf{Z}_5$ orbifold singularities. This means, concretely, that the normal space to Q in X can be parametrized locally by complex coordinates w_1, w_2 with the identification

$$(w_1, w_2) \to (e^{2\pi i/5} w_1, e^{-2\pi i/5} w_2). \tag{2.1}$$

Though our focus in this paper is on the case that G is simple, we briefly note that if G is semi-simple, this means that the K3 contains several disjoint singularities, one for each factor in G. These will lead, generally, to different components of the singular set of X. (They might intersect, in which case chiral superfields may be supported at the points of intersection.) Also, $U(1)$ factors in G appear in M-theory as modes of the three-form field C of M-theory.

We can abstract what we have learned from the way that we have learned it and say that in M-theory on a G_2 manifold X, gauge theory of type A, D, or E arises from the existence on X of a three-dimensional locus Q of A, D, or E orbifold singularities. Many such examples arise by duality with the heterotic string, but there may also be examples that have this structure but are not dual in this way to heterotic string compactifications. This mechanism of generating gauge symmetry from G_2 manifolds has been discussed in [37] and elsewhere.

In analyzing the four-dimensional effective gauge and gravitational couplings, the volumes of Q and of X, which we denote as V_Q and V_X, will be important. We will meet the dimensionless number

$$a = \frac{V_X}{V_Q^{7/3}}. \tag{2.2}$$

In models with K3 fibrations, we have $V_X = V_{\mathrm{K3}} \cdot V_Q$. (Volumes are multiplicative in fibrations such as $X \to Q$.) In the region in which one can see the duality with the heterotic string (so that the manifold X definitely exists), Q is very large compared to the $\mathbf{T}^3$ or K3, and hence $a << 1$. Another class of models arises from Type II orientifolds [15] -[18]; in this case, the radius of the M-theory circle is a factor in the volume of X but not of Q, so again $a << 1$ when the duality can be used to deduce the existence of X. It seems reasonable to guess that at least for many or most G_2 manifolds obtained by these dualities and useful for phenomenology, there is an upper bound of order one on the possible values of a, with a singularity developing if one tries to make a too large. This would be analogous to what happens in the strongly coupled heterotic string if, for most values of the instanton numbers at the two ends, one tries to make the length of the eleventh dimension too long compared to the scale of the other compact dimensions. Unfortunately, we have no way to prove our conjecture that a is generally bounded above or to determine the precise bound.

One can also construct G_2 orbifolds in which it is possible to have any value of a. (They lack the singularities, discussed momentarily, that generate chiral fermions.) Also, in a local picture , explicit G_2 metrics with $Q = \mathbf{S}^3$ or a finite quotient of $\mathbf{S}^3$ are known [38],[39], and have been studied in a number of recent papers [40],[41],[20],[42]. (These can be used to get gauge symmetry but not chiral fermions; symmetry breaking by Wilson lines was incorporated in the last of those papers.) Those papers focused on the behavior as Q shrinks to a point, whereupon topology-changing transitions may occur, as first proposed in [40]. When Q shrinks to a point, a singularity develops that is a cone on $\mathbf{S}^3 \times \mathbf{S}^3$, or a finite quotient. It is not known that such a singularity can develop in a compact G_2 manifold X, but presumably this is possible. In an example in which this can occur, there is no upper bound on a, but there may instead be a lower bound on a!

Since the heterotic string on a Calabi-Yau manifold (or a Type II orientifold) can readily have chiral fermions, it must also be possible to get chiral fermions in M-theory on a G_2 manifold. The precise mechanism was determined in [22]: chiral multiplets are supported at points on Q at which X has a singularity that is worse than an orbifold singularity. At these points, Q itself is smooth but the normal directions to Q in X have a singularity more complicated than the orbifold singularity (2.1). The relevant singularities have been studied in [20] - [23]. Their details will not be important in the present paper.

Model Of Grand Unification

In constructing a model similar to a four-dimensional grand unified model, we start with M-theory on $\mathbf{R}^4 \times X$, where $\mathbf{R}^4$ is four-dimensional Minkowski space and X has G_2 holonomy. We assume that X contains a three-manifold Q with a $\mathbf{Z}_5$ orbifold singularity (described locally by (2.1)) in the normal direction. This means that $SU(5)$ gauge fields propagate on $\mathbf{R}^4 \times Q$.

We further assume that the first Betti number of Q vanishes, $b_1(Q) = 0$. This is actually true in all known examples (and notably it is true for examples arising by duality with the heterotic string; in that context, $b_1(Q) = 0$, since the first Betti number of the Calabi-Yau threefold Z is actually zero). This means that, in expanding around the trivial $SU(5)$ connection on Q, there are no zero modes for gauge fields. Such zero modes would lead to massless chiral superfields in the adjoint representation of $SU(5)$.

Instead, we assume that that there is a nontrivial finite fundamental group $\pi_1(Q)$ and first homology group $H_1(Q)$. A typical example (which can arise in duality with the heterotic string) is a lens space, $Q = \mathbf{S}^3/\mathbf{Z}_q$ for some q. We will keep this in mind as a concrete example, for which we will give a completely explicit formula for the threshold corrections. Having a finite and nontrivial first homology makes it possible to break $SU(5)$ to the standard model subgroup $SU(3) \times SU(2) \times U(1)$ by a *discrete* choice of flat connection in the vacuum. Just as in the case of the heterotic string on a Calabi-Yau manifold [26], the discreteness leads to interesting possibilities for physics, including options for solving the doublet-triplet splitting problem.

For example, if $\pi_1(Q) = \mathbf{Z}_q$, we take the holonomy around a generator of $\pi_1(Q)$ to be

$$U = \begin{pmatrix} e^{4\pi iw/q} & & & & \\ & e^{4\pi iw/q} & & & \\ & & e^{4\pi iw/q} & & \\ & & & e^{-6\pi iw/q} & \\ & & & & e^{-6\pi iw/q} \end{pmatrix} \tag{2.3}$$

with some integer q. This breaks $SU(5)$ to the standard model subgroup as long as $5w$ is not a multiple of q.

For a more realistic GUT-like model, we also need various chiral superfields. These include Higgs bosons transforming in the $\mathbf{5}$ and $\overline{\mathbf{5}}$ of $SU(5)$, as well as quarks and leptons transforming as three copies of $\overline{\mathbf{5}} \oplus \mathbf{10}$, and possibly additional fields in real representations of $SU(5)$. Such chiral superfields are localized at points P_i on Q at which the singularity in the normal direction is more severe than the orbifold singularity (2.1).

Doublet-triplet splitting can be incorporated by assuming suitable discrete symmetries. In the example considered in [19], Q was the lens space $\mathbf{S}^3/\mathbf{Z}_q$, and the discrete symmetry group was a copy of $F = \mathbf{Z}_n$ acting on Q. The fixed point set of F consisted of two circles, and by suitable assignments of the points P_i to the two circles, doublet-triplet splitting was ensured. We will not recall the details here as we will not make explicit use of them. (For another approach to making the proton sufficiently long-lived, see [43].)

Kaluza-Klein Reduction

Now we consider the Kaluza-Klein reduction of the vector supermultiplet on $\mathbf{R}^4 \times Q$ to give massive particles on $\mathbf{R}^4$.

We begin with the gauge field A. It can be expanded around the Wilson loop background A_{cl} as

$$A = A_{cl} + a_\mu dx^\mu + \phi_\alpha dy^\alpha, \tag{2.4}$$

where x^μ, $\mu = 1, \ldots, 4$ are coordinates on $\mathbf{R}^4$, and y^α, $\alpha = 1, \ldots, 3$ are coordinates on Q. Here a_μ transforms as a gauge field on $\mathbf{R}^4$ and a scalar field on Y, and ϕ_α transforms as a scalar on $\mathbf{R}^4$ and a one-form on Y. Both a_μ and ϕ_α are functions of both the x's and the y's, and thus can be expanded

$$\begin{aligned} a_\mu(x,y) &= \sum_n a_\mu^{(n)}(x)\chi^{(n)}(y) \\ \phi_\alpha(x,y) &= \sum_m \phi^{(m)}(x)\psi_\alpha^{(m)}(y). \end{aligned} \tag{2.5}$$

Here $\chi^{(n)}$ and $\psi^{(m)}$ are eigenfunctions, respectively, of Δ_0 and Δ_1, where by Δ_k we mean the Laplacian acting on k-forms with values in the adjoint representation of $SU(5)$:

$$\begin{aligned}\Delta_0\chi^{(n)} &= \lambda_n^{(0)}\chi^{(n)}\\ \Delta_1\psi^{(m)} &= \lambda_m^{(1)}\psi^{(m)}.\end{aligned} \tag{2.6}$$

The k-form eigenvalues $\lambda_n^{(k)}$, $k=0,1$, are interpreted in four dimensions as the mass squared. The $a_\mu^{(n)}$ are vector fields in four dimensions, and form part of a vector multiplet, while the $\phi^{(n)}$ are scalars and form part of a chiral multiplet.

Along with the gauge field, the seven-dimensional vector multiplet contains three additional scalar fields, which, in compactification on $\mathbf{R}^4\times Q$, behave as another one-form $\widetilde{\phi}_\alpha$ on Q. As explained in [37], this happens because of the twisting of the normal bundle. $\widetilde{\phi}_\alpha$ has a similar Kaluza-Klein expansion:

$$\widetilde{\phi}_\alpha(x,y)=\sum_m \widetilde{\phi}^{(m)}(x)\psi_\alpha^{(m)}(y). \tag{2.7}$$

The fields $\phi^{(m)}$ and $\widetilde{\phi}^{(m)}$ together make up the bosonic part of a chiral multiplet.

In sum, then, each zero-form eigenfunction $\chi^{(n)}$ leads to a vector multiplet, with helicities $1,-1,1/2,-1/2$, and each one-form eigenfunction $\psi^{(m)}$ leads to a chiral multiplet, with helicities $0,0,1/2,-1/2$. (The origin of the fermionic states in these multiplets is described in [37] and depends again on the twisting of the normal bundle.) This is all the information we need to compute the threshold corrections in section 3. However, we pause to describe in more detail the Higgs mechanism, by virtue of which some chiral and vector multiplets combine into massive vector multiplets.

If $\chi^{(n)}$ is a zero mode of the Laplacian, then $a_\mu^{(n)}$ is a generator of the unbroken $SU(3)\times SU(2)\times U(1)$. If $a_\mu^{(n)}$ is one of these 12 modes, then it is part of a massless vector multiplet. For all the infinitely many other values of n, $a_\mu^{(n)}$ is part of a massive vector multiplet, obtained by combining a vector multiplet and a chiral multiplet, via a Higgs mechanism.

Let us see how this comes about. Since we have assumed that $b_1(Q)=0$, there are no harmonic one-forms. Thus, any one-form ψ on Q is a linear combination of a closed one-form $d_A\chi$ and a co-closed one-form $d_A^*\lambda$ (d_A and d_A^* are the gauge-covariant exterior derivative and its adjoint, defined using the background gauge field A_{cl}; χ is a zero-form and λ is a two-form).

In particular, the eigenfunctions $\psi^{(m)}$ are either closed or co-closed. The co-closed eigenfunctions yield massive chiral multiplets in four dimensions that do not participate in any Higgs mechanism. But a closed one-form eigenfunction is the exterior derivative of a zero-form eigenfunction,

$$\psi^{(m)} = d_A \chi^{(n)}, \tag{2.8}$$

for some n. Here $\chi^{(n)}$ is a zero-form eigenfunction of the Laplacian with the same eigenvalue as $\psi^{(m)}$. In this situation, the vector multiplet derived from $\chi^{(n)}$ and the chiral multiplet derived from $\psi^{(m)}$ combine via a Higgs mechanism into a massive vector multiplet. The Higgs mechanism is most directly seen using the transformation of the gauge field A under a gauge transformation generated by the zero-form $\chi^{(n)}$. This is $\delta A = d_A \chi^{(n)}$. In terms of four-dimensional fields, it becomes

$$\delta a_\mu^{(n)} = \frac{D}{Dx^\mu} \chi^{(n)}, \quad \delta\phi^{(m)} = \chi^{(n)}, \tag{2.9}$$

where in the last formula, the shift in $\phi^{(m)}$ under a gauge transformation demonstrates the Higgs mechanism.

For our purposes, a vector multiplet, massless or not, has four states of helicities $\pm 1, \pm 1/2$, and a chiral multiplet has four states of helicities $0, 0, \pm 1/2$. A massive vector multiplet is the combination of a vector multiplet and a chiral multiplet via the Higgs mechanism to a multiplet with eight helicity states $1, 1/2, 1/2, 0, 0, -1/2, -1/2, -1$. We will not use the concept of a massive vector multiplet in the rest of the paper, since for computing the threshold corrections, one need not explicitly take account of the way the Higgs effect combines a vector multiplet and a chiral multiplet into a single massive vector multiplet.

For more detail, with also some information that will be useful later, see the table. As we note in the table, massless states come from harmonic forms (in the above discussion and in the rest of the paper, we make topological restrictions on Q so that there are no harmonic one-forms leading to massless chiral multiplets, but in the table we include them for completeness and possible future generalizations). We have also indicated in the table which massive multiplets are Higgsed and become part of massive vector multiplets, and which are not.

3 The Threshold Corrections

In this section, we will get down to business and compute the threshold corrections to gauge couplings in grand unification. In doing the compu-

Table 1: Field content in four dimensions.

	Form on Q	Type	Supermultiplet	χ	Str $(\frac{1}{12} - \chi^2)$
massive	0–forms		vector Higgsed	$1, 1/2, -1/2, -1$	$-3/2$
”	1–forms	closed	chiral Higgsed	$1/2, 0, 0, -1/2$	$1/2$
”	”	co-closed	chiral un-Higgsed	$1/2, 0, 0, -1/2$	$1/2$
massless	0–forms	harmonic	vector	$1, 1/2, -1/2, -1$	$-3/2$
”	1–forms	harmonic	chiral	$1/2, 0, 0, -1/2$	$1/2$

tation, it is necessary to assume that the radius R_Q of Q is much greater than the eleven-dimensional Planck length $1/M_{11}$, since otherwise M-theory is strongly coupled at the scale $1/R_Q$. Having $R_Q >> 1/M_{11}$ also means that the Kaluza-Klein harmonics, with masses of order $M_{GUT} \sim 1/R_Q$, are much lighter than generic M-theory excitations, whose masses are of order M_{11}. So it makes sense to compute their effects without knowing exactly what happens at the Planck scale. In section 4, we show that $R_Q >> 1/M_{11}$ to the extent that α_{GUT} is small.

In most of this section, we consider only the case that the normal bundle to Q has only the standard orbifold singularity, so that the only charged light fields are those of the seven-dimensional vector multiplet, compactified on Q. After analyzing this case thoroughly, we consider in section 3.5 the case in which singularities of the normal bundle generate additional light charged fields.

A priori, one would expect the one-loop threshold corrections to gauge couplings to depend on the chiral multiplets that describe the moduli of X. However, the imaginary parts of those multiplets are axion-like components of the C-field and manifestly decouple from the computation of one-loop threshold corrections.[4] Those computations depend only on the particle masses – that is, the eigenvalues of the Laplacian. Hence the perturbative threshold corrections are actually constants, independent of the moduli of X.

The threshold corrections are given by a sum over contributions of different eigenvalues. The sum makes sense for any metric on Q, but we want to evaluate it for metrics that are induced by G_2 metrics on X. From the argument in the last paragraph, the threshold correction is independent of the metric on Q as long as it comes from a G_2 metric on X. We do not know any useful property of such metrics on Q. So the most obvious way

[4]Nonperturbatively small threshold corrections due to membrane instantons will depend on the C-field. These effects, though extremely small for G_2 manifolds, can be substantial in a dual heterotic string or Type II orientifold description.

for the threshold correction to be independent of the moduli of X is for it to be entirely independent of the metric of Q, that is, to be a topological invariant.

The most obvious topological invariant of a three-manifold Q, endowed with a background flat gauge field A, that can be computed from the spectrum of the Laplacian is the Ray-Singer analytic torsion [25]. We will show that torsion does indeed give the right answer.

From a physical point of view, the torsion can be represented by a very simple topological field theory [44]. Another hint that torsion is relevant is that analytic torsion (in this case, $\overline{\partial}$ analytic torsion, which varies holomorphically and is not a topological invariant) governs the threshold corrections for the heterotic string, as shown in section 8.2 of [45]. The relation of the threshold corrections for G_2 manifolds to torsion can possibly also be argued using methods in [45], at least in the case of gauge groups of type A or D, plus duality of M-theory with Type IIA superstrings and D-branes. We will establish this relation directly. (Torsion also enters in the theory of membrane instantons [46].)

3.1 The Analytic Torsion

Let H be the subgroup of the unified group G that commutes with the standard model group $SU(3) \times SU(2) \times U(1)$. For $G = SU(5)$, H is just the $U(1)$ of the standard model. Let $\mathcal{R}_i$ be the standard model representations appearing in the adjoint representation of G, and suppose that the part of the adjoint representation of G transforming as $\mathcal{R}_i$ under the standard model transforms as ω_i under H. (Some of the ω_i may be the same for different $\mathcal{R}_i$; also, the ω_i are not necessarily irreducible, though they are irreducible for $G = SU(5)$.)

In vacuum, we suppose that G is broken to the standard model by a choice of a flat H-bundle on Q, that is, by a choice of Wilson lines. Each ω_i determines a flat bundle that we will denote by the same name. We want to define the analytic torsion $\mathcal{T}_i$ for the representation ω_i. Let $\Delta_{k,i}$, $k = 0, \ldots, 3$, be the Laplacian acting on k-forms with values in ω_i. If there is no cohomology with values in ω_i, that is, if the $\Delta_{k,i}$ have no zero modes, then the torsion is defined by the formula[5]

$$\mathcal{T}_i = \frac{1}{2}\sum_{k=0}^{3}(-1)^{k+1}k\log\det\left(\Delta_{k,i}/\Lambda^2\right), \tag{3.1}$$

where Λ is an arbitrary constant (which in our physical application will be the gauge theory cutoff). The determinants in (3.6) are defined using zeta function regularization (for more detail on this, see Appendix A where a simple example is computed). By the theorem of Ray and Singer, $\mathcal{T}_i$ is independent of the metric of Q, and hence also independent of Λ, which can be eliminated by scaling the metric of Q. (In the mathematical theory, Λ is usually just set to 1, but physically, since $\Delta_{k,i}$ is naturally understood to have dimensions of mass squared, we introduce a mass parameter Λ and use the dimensionless ratio $\Delta_{k,i}/\Lambda^2$.) Because the Laplacian commutes with the Hodge $*$ operator which maps k-forms to $(3-k)$-forms, $\Delta_{k,i}$ and $\Delta_{3-k,i}$ have the same spectrum. So we can simplify (3.6):

$$\mathcal{T}_i = \frac{3}{2}\log\det\left(\Delta_{0,i}/\Lambda^2\right) - \frac{1}{2}\log\det\left(\Delta_{1,i}/\Lambda^2\right). \tag{3.2}$$

If ω_i is such that the $\Delta_{k,i}$ do have zero modes, then the $\log\det$'s in (3.6) are not well-defined. In this more general case, we define objects $\mathcal{K}_i$ by replacing $\det\Delta_{k,i}$ (which vanishes when there are zero modes) with $\det{}'\Delta_{k,i}$, the product (regularized via zeta functions) of the non-zero eigenvalues of $\Delta_{k,i}$:

$$\mathcal{K}_i = \frac{3}{2}\log\det{}'\left(\Delta_{0,i}/\Lambda^2\right) - \frac{1}{2}\log\det{}'\left(\Delta_{1,i}/\Lambda^2\right). \tag{3.3}$$

Since $\log\det$ is the same as $\mathrm{Tr}\log$, we can equally well write

$$\mathcal{K}_i = \frac{3}{2}\mathrm{Tr}'\log\left(\Delta_{0,i}/\Lambda^2\right) - \frac{1}{2}\mathrm{Tr}'\log\left(\Delta_{1,i}/\Lambda^2\right), \tag{3.4}$$

where Tr' is a trace with zero modes omitted.

For a three-manifold Q with finite fundamental group, and a non-trivial irreducible representation ω_i, there are no zero modes. If ω_i is trivial, there are zero modes: they are simply the zero-form 1 and a three-form which is the covariantly constant Levi-Civita volume form. When there are zero modes, $\mathcal{T}_i$ does not simply equals $\mathcal{K}_i$; there is a correction for the zero modes. The correction is explained in Appendix B, and in the present situation is very

[5] We define the torsion as in [25] and much of the physics literature; the definition differs by a factor of -2 from that in [24]. We also note that what we call $e^{\mathcal{T}}$ is called the torsion by many authors.

simple. If V_Q is the volume of Q, then the torsion of the trivial representation (which we denote by $\mathcal{O}$) is

$$\mathcal{T}_{\mathcal{O}} = \mathcal{K}_{\mathcal{O}} - \log(V_Q \Lambda^3). \tag{3.5}$$

The Ray-Singer theorem again implies that $\mathcal{T}_{\mathcal{O}}$ is independent of the metric of Q, and hence in particular independent of Λ. In the mathematical literature, (3.5) would be written with $\Lambda = 1$. In general, for any irreducible representation ω_i, trivial or not,

$$\mathcal{T}_i = \mathcal{K}_i - \delta_{\omega_i,\mathcal{O}} \log(V_Q \Lambda^3). \tag{3.6}$$

3.2 Sum Over Massive Particles

Now we will describe the threshold correction. We let g_M be the underlying gauge coupling as deduced from M-theory in the supergravity approximation (we compute it in detail in section 4), and we let $g_a(\mu)$, $a = 1, 2, 3$ be the standard model $U(1)$, $SU(2)$, and $SU(3)$ gauge couplings measured at an energy μ, which we assume to be far below the cutoff Λ.

The tree level relations among the g_a depend on how they are embedded in G. For the usual embedding[6] of the standard model in $SU(5)$, the tree level relations are $g_a^2 = g_M^2/k_a$, with $(k_1, k_2, k_3) = (5/3, 1, 1)$.

The one-loop relation is[7]

$$\frac{16\pi^2}{g_a^2(\mu)} = \frac{16\pi^2 k_a}{g_M^2} + b_a \log(\Lambda^2/\mu^2) + \mathcal{S}_a, \tag{3.7}$$

where b_a are the one-loop beta function coefficients, and $\mathcal{S}_a$ are the one-loop threshold corrections. They are given by very similar sums over, respectively, the massless and massive states. The formula for b_a is

$$b_a = 2\,\mathrm{Str}_{\,m=0} Q_a^2 \left(\frac{1}{12} - \chi^2\right). \tag{3.8}$$

Here Str is a supertrace (bosons contribute with weight $+1$, fermions with weight -1) over massless helicity states; χ is the helicity operator; and Q_a is a generator of the a^{th} factor of the standard model group with $\mathrm{Tr}\, Q_a^2 = k_a/2$.

[6] Conventionally, $SU(2)$ and $SU(3)$ are generated by 5×5 matrices of trace-squared equal to $1/2$, and $U(1)$ is generated by the hypercharge, $\mathrm{diag}(1/2, 1/2, -1/3, -1/3, -1/3)$, whose trace-squared is $(5/3) \cdot 1/2$.

[7] The threshold corrections are called Δ_a in [6], but we will call them $\mathcal{S}_a$ to avoid confusion with the use of Δ for the Laplacian.

As explained in the introduction to this section, until section 3.5 we consider the effects of the seven-dimensional vector multiplets only. In this case, the massless particles are simply the vector multiplets of the unbroken group $SU(3) \times SU(2) \times U(1)$. So, letting $\mathcal{R}_{\mathcal{A}}$ denote the adjoint representation of the standard model, and recalling that the helicities of a vector multiplet are $1, -1, 1/2, -1/2$, we see that for this case,

$$b_a = -3\mathrm{Tr}_{\mathcal{R}_{\mathcal{A}}} Q_a^2. \tag{3.9}$$

The definition of $\mathcal{S}_a$ is rather similar to (3.8) except that the trace runs over massive states and includes a logarithmic factor depending on the mass:

$$\mathcal{S}_a = 2\,\mathrm{Str}_{\,m\neq 0} Q_a^2 \left(\frac{1}{12} - \chi^2\right) \log(\Lambda^2/m^2). \tag{3.10}$$

If we combine these formulas, we can write

$$\begin{aligned}\frac{16\pi^2}{g_a^2(\mu)} = \frac{16\pi^2 k_a}{g_M^2} &+ 2\,\mathrm{Str}_{\,m=0} Q_a^2 \left(\frac{1}{12} - \chi^2\right) \log(\Lambda^2/\mu^2) \\ &+ 2\,\mathrm{Str}_{\,m\neq 0} Q_a^2 \left(\frac{1}{12} - \chi^2\right) \log(\Lambda^2/m^2).\end{aligned} \tag{3.11}$$

We see that every helicity state, massless or massive, makes a contribution to the low energy couplings that has the same dependence on the cutoff Λ, independent of its mass.

In a unified four-dimensional GUT theory, the quantum numbers of the tree level particles are G-invariant, though the masses are not. Hence, the coefficient of $\log \Lambda$ in (3.11) is G-invariant. The precise value of Λ is therefore irrelevant in the sense that a change in Λ can be absorbed in redefining the unified coupling (which in a four-dimensional theory would usually be called g_{GUT} rather than g_M).

In our supersymmetric gauge theory on $\mathbf{R}^4 \times Q$, the sum (3.11) arises from a one-loop diagram in the seven-dimensional gauge theory; the coupling renormalization from the one-loop diagram has been expanded as a sum over Kaluza-Klein harmonics. The seven-dimensional gauge theory is unrenormalizable, and divergences in loop diagrams should be expected. The divergences, however, are proportional to gauge-invariant local operators on Q. Since we have assumed that the G-symmetry is broken only by the choice of a background flat connection, all local operators are G-invariant. Hence the divergences are G-invariant and can be, again, absorbed in a redefinition of g_M.

In practice (by using the relation to analytic torsion), we will define the sum in (3.11) with zeta-function regularization, and then it will turn out that there are no divergences at all – Λ will cancel out completely. With a different regularization, there would be divergences, but they would be G-invariant. Of course, if we really had a proper understanding of M-theory, we would use whatever regularization it gives.

3.3 Relation Of The Threshold Correction To The Torsion

The threshold correction $\mathcal{S}_a$ can be written $\mathcal{S}_a = \sum_i \mathcal{S}_{a,i}$, where $\mathcal{S}_{a,i}$ is the contribution to the sum in (3.10) coming from states that transform in the representation $\mathcal{R}_i$. We can further factor the trace of Q_a^2 – which depends only on $\mathcal{R}_i$ – from the rest of the sum:

$$\mathcal{S}_{a,i} = 2\mathrm{Tr}_{\mathcal{R}_i} Q_a^2 \ \mathrm{Str}_{\mathcal{H}_i} \left(\frac{1}{12} - \chi^2\right) \log\left(\frac{\Lambda^2}{m^2}\right). \tag{3.12}$$

Here $\mathcal{H}_i$ is defined by saying that the space $\mathcal{H}$ of one-particle massive states decomposes under the standard model as $\mathcal{H} = \oplus \mathcal{R}_i \otimes \mathcal{H}_i$.

Now let us evaluate the sum over helicity states. As we reviewed in section 2, each eigenvector of the zero-form Laplacian $\Delta_{0,i}$ contributes to $\mathcal{H}_i$ a vector multiplet, with helicities $1, -1, 1/2, -1/2$. m^2 for this multiplet is equal to the eigenvalue of $\Delta_{0,i}$. For such a multiplet, we have $\mathrm{Str}\left(\frac{1}{12} - \chi^2\right) = -3/2$. Each eigenfunction of the one-form Laplacian $\Delta_{1,i}$ similarly contributes to $\mathcal{H}_i$ a chiral multiplet, with helicities $0, 0, 1/2, -1/2$, for which $\mathrm{Str}\left(\frac{1}{12} - \chi^2\right) = 1/2$. Inserting these values in (3.12) and using (3.4) and (3.6), we find that

$$\mathcal{S}_{a,i} = 2\mathrm{Tr}_{\mathcal{R}_i} Q_a^2 \cdot \mathcal{K}_i = 2\mathrm{Tr}_{\mathcal{R}_i} Q_a^2 \cdot \left(\mathcal{T}_i + \delta_{\omega_i,\mathcal{O}} \log(V_Q \Lambda^3)\right). \tag{3.13}$$

Here $\delta_{\omega_i,\mathcal{O}}$ is 1 if ω_i is the trivial representation (which we denote as $\mathcal{O}$) and otherwise zero.[8]

Elimination Of Λ Dependence

Now we can eliminate the Λ dependence, which appears explicitly in (3.7) in the contribution from the massless particles, and arises in (3.13) because massless contributions are omitted in $\mathcal{S}_a$. The Λ dependence of $1/g_a^2(\mu)$

[8] If it should happen that ω_i is reducible for some i, then $\delta_{\omega_i,\mathcal{O}}$ should be understood as the number of copies of $\mathcal{O}$ in ω_i. In what follows, we assume for ease of exposition that the ω_i are all irreducible, as occurs for $G = SU(5)$. This restriction is inessential.

cancels only when we include all states, massive and massless, and we will now exhibit this cancellation.

The total Λ-dependent contribution to $\mathcal{S}_a$ is, using (3.13),

$$2\sum_i \mathrm{Tr}_{\mathcal{R}_i} Q_a^2 \delta_{\omega_i,\mathcal{O}} \log(V_Q\Lambda^3). \tag{3.14}$$

We can write the adjoint representation $\mathcal{A}$ of the low energy gauge group $SU(3)\times SU(2)\times U(1)$ as

$$\mathcal{A} = \oplus_i' \mathcal{R}_i, \tag{3.15}$$

where the sum $\oplus_i'$ runs over all $\mathcal{R}_i$ such that $\omega_i = \mathcal{O}$. The reason for this is that when we turn on a flat gauge background of a subgroup $H \subset G$ to break the gauge group G to a subgroup, the unbroken group is the subgroup that transforms trivially under H, in other words, its Lie algebra is the union of the $\mathcal{R}_i$ for which $\omega_i = \mathcal{O}$. With (3.15), we can rewrite (3.14) as

$$2\mathrm{Tr}_{\mathcal{A}} Q_a^2 \log(V_Q\Lambda^3), \tag{3.16}$$

and this is the Λ dependence of $\mathcal{S}_a$.

The Λ-dependence that is explicit in (3.7) is in the term $b_a \log(\Lambda^2/\mu^2)$. Using (3.9), we see that the Λ dependence cancels with that in (3.16), and moreover, making use of (3.13) and (3.16) as well as (3.9), we can rewrite the formula for the threshold corrections in a Λ-independent and useful form:

$$\frac{16\pi^2}{g_a^2(\mu)} = \frac{16\pi^2 k_a}{g_M^2} + b_a \log\left(\frac{1}{V_Q^{2/3}\mu^2}\right) + \mathcal{S}_a', \tag{3.17}$$

with

$$\mathcal{S}_a' = 2\sum_i \mathcal{T}_i \ \mathrm{Tr}_{\mathcal{R}_i} Q_a^2. \tag{3.18}$$

3.4 Evaluation For $SU(5)$

Now, let us evaluate this formula for $G = SU(5)$. The adjoint representation decomposes under the standard model as

$$(\mathbf{8},\mathbf{1})^0 \oplus (\mathbf{1},\mathbf{3})^0 \oplus (\mathbf{1},\mathbf{1})^0 \oplus (\mathbf{3},\mathbf{2})^{-5/6} \oplus (\overline{\mathbf{3}},\mathbf{2})^{5/6}, \tag{3.19}$$

where $SU(3)\times SU(2)$ representations have been labeled mostly by their dimension, and the superscript is the $U(1)$ charge. Thus, the representations of $H = U(1)$ that enter are the trivial representation $\mathcal{O}$, a nontrivial representation ω that corresponds to charge $-5/6$, and the dual representation $\overline{\omega}$

for charge 5/6. Since complex conjugation of the eigenfunctions exchanges ω and $\overline{\omega}$ without changing the eigenvalues of the Laplacian, we have $\mathcal{T}_\omega = \mathcal{T}_{\overline{\omega}}$. Hence, the can write the formula for $\mathcal{S}'_a$ just in terms of the two torsions $\mathcal{T}_{\mathcal{O}}$ and $\mathcal{T}_\omega$.

To make this explicit, we need to take a few traces. The traces of (Q_1^2, Q_2^2, Q_3^2) in the representation $(\mathbf{8},\mathbf{1})^0 \oplus (\mathbf{1},\mathbf{3})^0 \oplus (\mathbf{1},\mathbf{1})^0$ are $(0,2,3)$, and their traces in the representation $(\mathbf{3},\mathbf{2})^{-5/6} \oplus (\mathbf{3},\overline{\mathbf{2}})^{5/6}$ are $(25/3, 3, 2)$. So

$$\begin{aligned} \mathcal{S}'_1 &= \frac{50}{3}\mathcal{T}_\omega \\ \mathcal{S}'_2 &= 4\mathcal{T}_{\mathcal{O}} + 6\mathcal{T}_\omega \\ \mathcal{S}'_3 &= 6\mathcal{T}_{\mathcal{O}} + 4\mathcal{T}_\omega. \end{aligned} \tag{3.20}$$

Since $(b_1, b_2, b_3) = -3(0,2,3)$, we can write this as

$$\mathcal{S}'_a = -\frac{2}{3} b_a \left(\mathcal{T}_{\mathcal{O}} - \mathcal{T}_\omega\right) + 10 k_a \mathcal{T}_\omega. \tag{3.21}$$

So we get our final formula for the low energy gauge couplings in the one-loop approximation:

$$\frac{16\pi^2}{g_a^2(\mu)} = \left(\frac{16\pi^2}{g_M^2} + 10\mathcal{T}_\omega\right) k_a + b_a \log\left(\frac{\exp\left(\frac{2}{3}(\mathcal{T}_\omega - T_{\mathcal{O}})\right)}{\mu^2 V_Q^{2/3}}\right). \tag{3.22}$$

We might compare this to a naive one-loop renormalization group formula that we might write in a GUT theory. This would read

$$\frac{16\pi^2}{g_a^2(\mu)} = \left(\frac{16\pi^2}{g_{GUT}^2}\right) k_a + b_a \log(M_{GUT}^2/\mu^2). \tag{3.23}$$

We see that the two formulas agree if

$$\begin{aligned} \frac{16\pi^2}{g_{GUT}^2} &= \frac{16\pi^2}{g_M^2} + 10\mathcal{T}_\omega \\ M_{GUT}^2 &= \left(\frac{\exp(\mathcal{T}_\omega - \mathcal{T}_{\mathcal{O}})}{V_Q}\right)^{2/3}. \end{aligned} \tag{3.24}$$

The first formula tells us how the coupling g_M used in M-theory should be compared to the g_{GUT} that is inferred from low energy data. Our computation really only makes sense if the difference between g_M and g_{GUT} is much smaller than either, since otherwise higher order corrections would be important. Moreover, a different regularization (such as M-theory may supply) might have given a different answer for the shift in g_M, so this shift is unreliable. The second formula shows how the parameter V_Q of the compactification is related to M_{GUT} as inferred from low energy data.[9] This relationship is meaningful and independent of the regularization. We can write the relation as

$$V_Q = \frac{L(Q)}{M_{GUT}^3}, \quad \text{with } L(Q) = \exp(\mathcal{T}_\omega - \mathcal{T}_\mathcal{O}). \tag{3.25}$$

A noteworthy fact – though a simple consequence of $SU(5)$ group theory – is that the massive Kaluza-Klein harmonics have made no correction at all to the prediction of the theory for $\sin^2\theta_W$.

We now want to make our result completely explicit in a simple example. To allow for $SU(5)$ breaking, Q cannot be the most obvious compact three-manifold with $b_1 = 0$, which would be a sphere $\mathbf{S}^3$. We can, however, take Q to be what is arguably the next simplest choice, a lens space. We describe $\mathbf{S}^3$ by complex variables z_1, z_2 with $|z_1|^2 + |z_2|^2 = 1$, and take $\mathbf{Z}_q$ to act by

$$\gamma : z_i \to \exp(2\pi i/q) z_i \tag{3.26}$$

for some positive integer q. Then we define

$$Q = \mathbf{S}^3/\mathbf{Z}_q. \tag{3.27}$$

To break $SU(5)$ to the standard model, we assume that the action of γ is accompanied by a gauge transformation by

$$U_\gamma = \exp\left(2\pi i(w/q)\mathrm{diag}(2,2,2,-3,-3)\right) \tag{3.28}$$

with some integer w such that $5w$ is not divisible by q. The torsions in this case are

$$\mathcal{T}_\mathcal{O} = -\log q, \quad \mathcal{T}_\omega = \log\left(4\sin^2(5\pi w/q)\right), \tag{3.29}$$

giving

$$L(Q) = 4q\sin^2(5\pi w/q). \tag{3.30}$$

(See [24],[25] as well as Appendix A for some computations.) Hence the relation between M_{GUT} and V_Q is in this model

$$M_{GUT} = \left(\frac{4q\sin^2(5\pi w/q)}{V_Q}\right)^{1/3}, \quad V_Q = \frac{4q\sin^2(5\pi w/q)}{M_{GUT}^3}. \tag{3.31}$$

We actually can generalize (3.26) slightly to a transformation γ that maps $z_i \to \exp(2\pi i m_i/q) z_i$ for integers m_i that are prime to q. The quotient

[9] We define M_{GUT} as the mass parameter that appears in making a fit like (3.23) to low energy data; it is of course not necessarily the mass of any particle.

$\mathbf{S}^3/\mathbf{Z}_q$ is still called a lens space. By replacing γ by a power of itself, there is no loss of generality to take, say $m_2 = 1$; we then denote m_1 simply as m. In this more general case (see [25], pp. 168-9), $\mathcal{T}_{\mathcal{O}}$ is unchanged, and $\mathcal{T}_\omega$ becomes $\log(4|\sin(5\pi w/q)\sin(5\pi jw/q)|)$ where $jm \equiv 1$ modulo q, so $L(Q) = 4q|\sin(5\pi w/q)\sin(5\pi jw/q)|$.

3.5 Inclusion Of Quarks, Leptons, And Higgs Bosons

So far we have solely considered the case that the normal space to Q has only the standard orbifold singularity, so that the only charged particles with masses $\leq 1/R_Q$ are the Kaluza-Klein harmonics. Now we want to introduce quarks, leptons, Higgs bosons, and possibly other charged light fields such as messengers of gauge-mediated supersymmetry breaking. We do this by assuming at points $P_i \in Q$ the existence of certain more complicated singularities of the normal bundle. These generate charged massless $SU(5)$ multiplets (which may ultimately get masses at a lower scale if a superpotential is generated or supersymmetry is spontaneously broken). If the singularities of the P_i are generic, each one contributes a new irreducible $SU(5)$ multiplet M_i of massless chiral superfields. Specific singularities that generate chiral multiplets transforming in the $\mathbf{5}$, $\mathbf{10}$, $\overline{\mathbf{10}}$, and $\overline{\mathbf{5}}$ of $SU(5)$ have been studied in [20] - [23].

It is believed that these singularities are *conical*. This is definitely true in a few cases in which the relevant G_2 metrics are conical metrics that were constructed long ago [38],[39] (and found recently [20] to generate massless chiral multiplets). Since a conical metric introduces no new length scale that is positive but smaller than the eleven-dimensional Planck length, we expect that these singularities, apart from the massless multiplets M_i, introduce no new particles of masses $\leq 1/R_Q$ that need to be considered in evaluating the threshold corrections.

We can also argue, a little less rigorously, that the singularities of the normal bundle that produces massless chiral superfields in the $\mathbf{5}$, $\mathbf{10}$, etc., have no effect on the Kaluza-Klein harmonics of the seven-dimensional vector multiplet on $\mathbf{R}^4 \times Q$. To show this, we consider the construction in [22], where the association of massless chiral multiplets with singularities was argued using duality with the heterotic string. In this argument, the existence in the G_2 description of a conical singularity that generates massless chiral superfields was related, in a heterotic string description that uses a $\mathbf{T}^3$ fibration, to the existence of a certain zero mode of the Dirac equation on a special $\mathbf{T}^3$ fiber. The existence of this zero mode generates a localized massless multiplet in the $\mathbf{5}$, $\mathbf{10}$, etc., as shown in [22], but does nothing at all to the seven-dimensional vector multiplet (which has no exceptional zero mode on the $\mathbf{T}^3$ in question).

Granted these facts, to incorporate the effects of the multiplets M_i, all we have to do is add their contributions to the starting point (3.7) or to the final result (3.22). If all of the new multiplets are massless down to the scale of supersymmetry breaking, then, for μ greater than this scale, all we have to do is add a contribution to (3.7) due to the new light fields. Let Δb_a be the contribution of the new light fields to the beta function coefficients b_a – note that since the M_i form complete $SU(5)$ multiplets, they contribute to each b_a in proportion to k_a. The contribution of the new fields to (3.7) is to add to the right hand side

$$\Delta b_a \log(\widetilde{\Lambda}^2/\mu^2), \tag{3.32}$$

which is the contribution due to renormalization group running of the M_i from their cutoff $\widetilde{\Lambda}$ down to μ. We do not exactly know what effective cutoff $\widetilde{\Lambda}$ to use for the M_i, but it is of order M_{11}. Anyway, the exact value of $\widetilde{\Lambda}$ does not matter; it can be absorbed in a small correction to g_M (this correction is no bigger than other unknown corrections due for example to possible charged particles with masses of order M_{11}). In fact, up to a small shift in g_M, it would not matter if we replace $\widetilde{\Lambda}$ by the (presumably lower) mass $\exp\left(\frac{1}{3}(\mathcal{T}_\omega - \mathcal{T}_\mathcal{O})\right)/V_Q^{1/3}$ that appears in (3.22). So if all components of the M_i are light, we can take our final answer to be simply that of (3.22), but with all b_a redefined (by the shift $b_a \to b_a + \Delta b_a$) to include the effects of the M_i. In other words, if all components of M_i are light, we simply have to take the b_a in (3.22) to be the exact β function coefficients of the low energy theory.

The assumption that all components of the M_i are light is inconsistent with the measured value of the weak mixing angle $\sin^2\theta_W$. That measured value (and the longevity of the proton) is instead compatible with the hypothesis that all components of the M_i are light except for the color triplet partners of the ordinary $SU(2)\times U(1)$ Higgs bosons; we call those triplets T and $\widetilde{T}$. Let m_T be the mass of T and $\widetilde{T}$ (we assume this mass comes from a superpotential term $T\widetilde{T}$, in which case T and $\widetilde{T}$ have equal masses), and let $\Delta b_a^{T,\widetilde{T}}$ be their contribution to the beta functions. These are not proportional to k_a since T and $\widetilde{T}$ do not form a complete $SU(5)$ multiplet! Then (3.32) should be replaced by

$$(\Delta b_a - b_a^{T,\widetilde{T}})\log(\widetilde{\Lambda}^2/\mu^2) + b_a^{T,\widetilde{T}}\log(\widetilde{\Lambda}^2/m_T^2), \tag{3.33}$$

the idea being that the T, $\widetilde{T}$ contributions run only from $\widetilde{\Lambda}$ down to m_T, while the others run down to μ. Up to a small correction to g_M, we can again replace $\widetilde{\Lambda}$ in (3.33) by

$$\widetilde{\Lambda} \to \exp\left(\frac{1}{3}(\mathcal{T}_\omega - \mathcal{T}_\mathcal{O})\right) V_Q^{-1/3}. \tag{3.34}$$

If we do this, then (3.22) is replaced by

$$\begin{aligned}\frac{16\pi^2}{g_a^2(\mu)} = \left(\frac{16\pi^2}{g_M^2} + 10\mathcal{T}_\omega + \delta\right) k_a + b_a \log\left(\frac{\exp(\frac{2}{3}(\mathcal{T}_\omega - T_\mathcal{O}))}{\mu^2 V_Q^{2/3}}\right) \\ + b_a^{T,\widetilde{T}} \log\left(\frac{\exp(\frac{2}{3}(\mathcal{T}_\omega - T_\mathcal{O}))}{m_T^2 V_Q^{2/3}}\right).\end{aligned} \tag{3.35}$$

Here b_a are the full beta functions of the low energy theory below the mass m_T, and $b_a^{T,\widetilde{T}}$ is the additional contribution to the beta functions from $T, \widetilde{T}$ between m_T and the effective GUT mass $M_{GUT} = \exp\left(\frac{1}{3}(\mathcal{T}_\omega - \mathcal{T}_\mathcal{O})\right)/V_Q^{1/3}$. Finally, δ expresses an unknown shift in the effective value of g_M; this shift is presumably unimportant within the accuracy of the computation.

Assuming that low energy threshold corrections are small, the fit to low energy measurements of gauge couplings is improved if rather than $m_T \sim M_{GUT}$ we take

$$\frac{m_T}{M_{GUT}} \sim 10^{-2}. \tag{3.36}$$

It is at least somewhat plausible in the present model that m_T/M_{GUT} would be small since the superpotential term $m_T T\widetilde{T}$ probably has to arise (like the terms that lead to quark and lepton masses) from membrane instantons. We discuss this issue further in section 6.

4 Couplings And Scales

One virtue of computing the threshold corrections is that we can make somewhat more precise the formulas for the parameters M_{GUT}, α_{GUT}, and G_N that are read off from the eleven-dimensional supergravity action.

We write the gravitational action in eleven dimensions as

$$\frac{1}{2\kappa_{11}^2}\int_{\mathbf{R}^4\times X} d^{11}x\sqrt{g}R. \tag{4.1}$$

Denoting the volume of X as V_X, this reduces in four dimensions simply to

$$\frac{V_X}{2\kappa_{11}^2}\int_{\mathbf{R}^4} d^4x\sqrt{g}R. \tag{4.2}$$

The four-dimensional Einstein-Hilbert action is

$$\frac{1}{16\pi G_N}\int_{\mathbf{R}^4} d^4x\sqrt{g}R. \tag{4.3}$$

So

$$G_N = \frac{\kappa_{11}^2}{8\pi V_X}. \tag{4.4}$$

Now let us work out the correctly normalized Yang-Mills action on $\mathbf{R}^4 \times Q$. For a system of n Type IIA D6-branes, the Yang-Mills action (see eqns. (13.3.25) and (13.3.26) of [47]) is

$$\frac{1}{4(2\pi)^4 g_s(\alpha')^{3/2}}\int d^7x\sqrt{g}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}, \tag{4.5}$$

where g_s is the string coupling constant and Tr is the trace in the fundamental representation of $U(n)$. In the GUT literature, one usually writes $F_{\mu\nu} = \sum_a F^a_{\mu\nu}Q_a$, where Q_a are generators of $U(n)$ normalized to $\mathrm{Tr}\, Q_aQ_b = \frac{1}{2}\delta^{ab}$. So (4.5) can be written

$$\frac{1}{8(2\pi)^4 g_s(\alpha')^{3/2}}\int d^7x\sqrt{g}\sum_a F^a_{\mu\nu}F^{\mu\nu\, a}. \tag{4.6}$$

Going to M-theory, the relation between κ_{11}, g_s, and α' is (eqn. (14.4.5) of [47])

$$\kappa_{11}^2 = \frac{1}{2}(2\pi)^8 g_s^3(\alpha')^{9/2}. \tag{4.7}$$

Combining these, we see that the Yang-Mills action in seven dimensions is

$$\frac{1}{4g_7^2}\int d^7x\sqrt{g}\sum_a F^a_{\mu\nu}F^{\mu\nu\, a} = \frac{1}{8(2\pi)^{4/3}2^{1/3}\kappa_{11}^{2/3}}\int d^7x\sqrt{g}\sum_a F^a_{\mu\nu}F^{\mu\nu\, a}. \tag{4.8}$$

The conventional action in four dimensions is

$$\frac{1}{4g_{GUT}^2}\int d^4x\,\sqrt{g}\,F^a_{\mu\nu}F^{\mu\nu\, a}, \tag{4.9}$$

with $\alpha_{GUT} = g_{GUT}^2/4\pi$. So after reducing (4.8) to four dimensions on $\mathbf{R}^4 \times Q$ and getting a factor of V_Q, the volume of Q, from the integral over Q, we identify

$$\alpha_{GUT} = \frac{(4\pi)^{1/3}\kappa_{11}^{2/3}}{V_Q}. \tag{4.10}$$

Combining (4.4) and (4.10), we have

$$G_N = \frac{\alpha_{GUT}^3 V_Q^{2/3}}{32\pi^2 a}, \tag{4.11}$$

with $a = V_X/V_Q^{7/3}$. Using (3.25), we can write this as

$$G_N = \frac{\alpha_{GUT}^3 L(Q)^{2/3}}{32\pi^2 a M_{GUT}^2}. \tag{4.12}$$

Here M_{GUT} is the unification scale as inferred from low energy data (but if there are extra light particles not presently known, they must be included in the extrapolation). For the simplest lens space, $L(Q) = 4q\sin^2(5\pi w/q)$, as noted in (3.29). As we explained in discussing eqn. (2.2), we consider it plausible that for many phenomenologically interesting manifolds of G_2 holonomy, there is an upper bound on a that is of order 1. If this is so (and of course it could be proved in principle, and the upper bound on a computed, for a given G_2 manifold X), then (4.12) is a lower bound on G_N that depends on the values of the GUT parameters, the readily computed constant $L(Q)$, as well as the more problematic bound on a.

Within the uncertainties, such a bound may well be saturated in nature. If we use the often-quoted values $M_{GUT} = 2.2\times 10^{16}$ GeV and $\alpha_{GUT} \sim 1/25$, then, with $G_N = 6.7\times 10^{-39}\,\mathrm{GeV}^{-2}$, we need approximately $L(Q)^{2/3}/a = 15$. We recall, however, that these values correspond to a minimal three family plus Higgs boson spectrum below the GUT scale, and that the doublet-triplet splitting mechanism [19] that we are assuming in the present paper really leads to extra light fields in vector-like $SU(5)$ multiplets. If we use the values $M_{GUT} \sim 8\times 10^{16}$ GeV, $\alpha_{GUT} \sim .2$, which are typical values found in [9] for certain models that have TeV-scale vector-like fields transforming as $\overline{\mathbf{5}}\oplus\mathbf{10}$ plus their conjugates, then we get $L(Q)^{2/3}/a = 1.7$. Raising the value of α_{GUT} to .2 or .3 will also significantly reduce the membrane action and so alleviate the problems with quark and lepton masses that we consider in section 6.

Alternatively, (4.4) and (4.11) can be combined to give

$$\kappa_{11}^2 = \frac{\alpha_{GUT}^3 L(Q)^3}{4\pi M_{GUT}^9}. \tag{4.13}$$

This formula is attractive because it is expresses the fundamental eleven-dimensional coupling κ_{11} in terms of quantities – α_{GUT} and M_{GUT} – about whose values we have at least some idea from experiment, and another quantity – $L(Q)$ – that is readily calculable in a given model.

The eleven-dimensional Planck mass M_{11} has been defined ([47], p. 199) by $2\kappa_{11}^2 = (2\pi)^8 M_{11}^{-9}$. So we can express (4.13) as a formula for M_{11}:

$$M_{11} = \frac{2\pi M_{GUT}}{\alpha_{GUT}^{1/3} L(Q)^{1/3}}. \tag{4.14}$$

One important result here is that M_{GUT} is parametrically smaller than M_{11} – by a factor of $\alpha_{GUT}^{1/3}$. This factor of $\alpha_{GUT}^{1/3}$ is the reason that it makes sense to use perturbation theory – as we have done in computing threshold corrections in section 3.[10]

Regrettably, the precise factors in the definition of M_{11} have been chosen for convenience. We really do not know if the characteristic mass scale at which eleven-dimensional supergravity breaks down and quantum effects become large is M_{11}, or $2\pi M_{11}$, or for that matter $M_{11}/2\pi$. [11] This uncertainty will unfortunately be important in section 5.

5 Proton Decay

In this section, we will analyze the gauge boson contribution to proton decay in the present class of models.

First, we recall how the analysis goes in four-dimensional GUT's. We will express the analysis in a way that is convenient for the generalization to $\mathbf{R}^4 \times Q$. The gauge boson contribution to proton decay comes from the matrix element of an operator product

$$g_{GUT}^2 \int d^4x\, J^\mu(x)\widetilde{J}_\mu(0)D(x,0), \tag{5.1}$$

where J and $\widetilde{J}$ are the currents in emission and absorption of the color triplet gauge bosons. We have used translation invariance to place one current at the origin, and $D(x,0)$ is the propagator of the heavy gauge bosons that transform as $(\mathbf{3},\mathbf{2})^{-5/6}$ of $SU(3)\times SU(2)\times U(1)$. Because the proton is so large compared to the range of x that contributes appreciably in the integral, we can replace $J^\mu(x)$ by $J^\mu(0)$, and then use

$$\int d^4x\, D(x,0) = \frac{1}{M^2} \tag{5.2}$$

(with M the mass of the heavy gauge bosons) to reduce (5.1) to

$$\frac{g_{GUT}^2 J_\mu \widetilde{J}^\mu(0)}{M^2}. \tag{5.3}$$

[10]It also means that the radius of Q and presumably of X is of order $\alpha_{GUT}^{-1/3}$ in eleven-dimensional Planck units, so that membrane instanton actions (which scale as length cubed) are of order $1/\alpha_{GUT}$. This fact will be troublesome in section 6.

[11]Just to get a feel for what the characteristic mass scale might be, note that for $(q,w)=(2,1)$, we have $M_{11}=2.0\times10^{17}$GeV for the often-quoted values of M_{GUT} and α_{GUT}, and $M_{11}=4.3\times10^{17}$GeV for the values in [9] .

(5.2) is a direct consequence of the equation for the propagator, which is

$$\left(\Delta + M^2\right) D(x,0) = \delta^4(x), \tag{5.4}$$

with $\Delta = -\eta^{\mu\nu}\partial_\mu\partial_\nu$ the Laplacian. Of course, in deriving (5.2), we should be careful in defining the operator product $J_\mu \widetilde{J}^\mu$; doing so leads to some renormalization group corrections to the above tree level derivation. These can be treated the same way in four dimensions and in the G_2-based models, and hence need not concern us here.

In $\mathbf{R}^4 \times Q$, the idea is similar, except that the currents are localized at specific points on Q, which we will call P_1 and P_2. The gauge boson propagator is a function $D(x, P; y, P')$ with $x, y \in \mathbf{R}^4$, and $P, P' \in Q$; the equation it obeys is

$$\left(\Delta_{\mathbf{R}^4} + \Delta_Q\right) D(x, P; y, P') = \delta^4(x-y)\delta(P, P'). \tag{5.5}$$

Here $\Delta_{\mathbf{R}^4}$ is the Laplacian on $\mathbf{R}^4$, acting on the x variable, and similarly Δ_Q is the Laplacian on Q, acting on P. Now we set P and P' to be two of the special points P_1 and P_2 on Q (with enhanced singularities in the normal directions) at which chiral matter fields are supported. The analog of (5.1) is

$$g_7^2 \int d^4x J_\mu(x, P_1)\widetilde{J}^\mu(0, P_2) D(x, P_1; 0, P_2), \tag{5.6}$$

where we have used translation invariance to set $y = 0$. Again, because the proton is so large compared to the range of x that contributes significantly to the integral, we can set x to 0 in $J_\mu(x, P_1)$, giving us

$$g_7^2 J_\mu(0; P_1)\widetilde{J}^\mu(0; P_2) \int d^4x\, D(x, P_1; 0, P_2). \tag{5.7}$$

Now it follows from (A.15) that the function

$$F(P, P') = \int d^4x\, D(x, P; 0, P') \tag{5.8}$$

obeys

$$\Delta_Q F(P, P') = \delta(P, P'). \tag{5.9}$$

In other words, F is the Green's function of the scalar Laplacian on Q (for scalar fields valued in the $(\mathbf{3}, \mathbf{2})^{-5/6}$ representation). In particular, F is bounded for P away from P', and for $P \to P'$,

$$F(P, P') \to \frac{1}{4\pi|P - P'|}, \tag{5.10}$$

with $|P - P'|$ denoting the distance between these two points. The proton decay interaction is

$$g_7^2 J_\mu(0;P_1)\widetilde{J}^\mu(0;P_2)\,F(P_1,P_2). \tag{5.11}$$

More exactly, this is the contribution for fermions living at the points P_1, P_2. It must be summed over possible P_i.

Given (5.10), if it is possible to have P_1 very close to P_2, this will give the dominant contribution. But how close will the P_i be? The smallest that the denominator in (5.10) will get is if $P_1 = P_2$, in other words if the currents J_μ and $\widetilde{J}^\mu$ in the proton decay process act on the same $\mathbf{10}$ or $\overline{\mathbf{5}}$ of $SU(5)$, living at some point $P = P_1 = P_2$ on Q. If this is the case, then the result (5.7) is infinite. M-theory will cut off this infinity, but we do not know exactly how. The best we can say is that the cutoff will occur at a distance of order the eleven-dimensional Planck length, as this is the only scale that is relevant in studying the conical singularity at P. If we naively say that setting $P_1 = P_2$ means replacing $1/|P_1 - P_2|$ by $1/R_{11} = M_{11}$ (with R_{11} the eleven-dimensional Planck length; M_{11} was evaluated in (4.14)), then we would replace $F(P,P)$ by $M_{11}/4\pi$. Unfortunately, as noted at the end of section 4, we have no idea whether M_{11} or some multiple of it is the natural cutoff in M-theory. This is an important uncertainty, since (for example) 4π is a relatively large number and the proton decay rate is proportional to the square of the amplitude! All we can say is that the effective value of $F(P,P)$, though uncalculable with the present understanding of M-theory, is model-independent; it does not depend on the details of X or Q, but is a universal property of M-theory with the conical singularity P. The effective interaction is

$$\sum_P C\frac{M_{11}g_7^2}{4\pi}J_\mu\widetilde{J}^\mu(P), \tag{5.12}$$

where C is a constant that in principle depends only on M-theory and not the specific model. So C might conceivably be computed in the future (if better methods are discovered) without knowing how to pick the right model. We have made explicit the fact that the interaction is summed over all possibilities for $P = P_1 = P_2$. Subleading (and model-dependent) contributions with $P_1 \neq P_2$ have not been written.

Let us compare this to the situation in four-dimensional GUT's. The currents J and $\widetilde{J}$ receive contributions from particles in the $\mathbf{10}$ and $\overline{\mathbf{5}}$ of $SU(5)$. Thus we can expand

$$J_\mu\widetilde{J}^\mu = J_\mu^{\mathbf{10}}\widetilde{J}^{\mu\,\mathbf{10}} + J_\mu^{\overline{\mathbf{5}}}\widetilde{J}^{\mu\,\overline{\mathbf{5}}} + J_\mu^{\mathbf{10}}\widetilde{J}^{\mu\,\overline{\mathbf{5}}} + J_\mu^{\overline{\mathbf{5}}}\widetilde{J}^{\mu\,\mathbf{10}}. \tag{5.13}$$

Among these terms, the $\mathbf{10}\cdot\mathbf{10}$ operator product contributes to $p \to \pi^0 e_L^+$, $\overline{\mathbf{5}}\cdot\overline{\mathbf{5}}$ does not contribute to proton decay, and the cross terms contribute to

$p \to \pi^0 e_R^+$ and $p \to \pi^+ \overline{\nu}_R$. Assuming that the points supporting **10**'s are distinct from the points supporting $\overline{\mathbf{5}}$'s, the above mechanism, in comparison to four-dimensional GUT's, enhances the decay $p \to \pi^0 e_L^+$ relative to the others.

Using the formulas in section 4, we can evaluate the product $g_7^2 M_{11}$. Reading off g_7^2 from (4.8) and M_{11} from (4.14), we find that the effective interaction is

$$\sum_{P_i} C J_\mu \widetilde{J}^\mu(P_i) \cdot \frac{2\pi L(Q)^{2/3} \alpha_{GUT}^{2/3}}{M_{GUT}^2}. \tag{5.14}$$

The equivalent formula in four-dimensional GUT's is

$$J_\mu^T \widetilde{J}^{T\,\mu} \frac{g_{GUT}^2}{M^2} = J_\mu^T \widetilde{J}^{T\,\mu} \frac{4\pi\alpha_{GUT}}{M^2}, \tag{5.15}$$

where the superscript T refers to the total current for all fermion multiplets. Moreover, M is the mass of the color triplet gauge bosons, and so may not coincide with M_{GUT}, which is the unification scale as deduced from the low energy gauge couplings; in some simple four-dimensional models, the ratio M/M_{GUT} is computable. The above formulas show that, in principle, the decay amplitude for $p \to \pi^0 e_L^+$ in the G_2-based theory is enhanced as $\alpha_{GUT} \to 0$ by a factor of $\alpha_{GUT}^{-1/3}$ relative to the corresponding GUT amplitude. The enhancement means that, in some sense, in the models considered here, proton decay is not purely a gauge theory phenomenon but a reflection of M-theory.

In practice, in nature, $\alpha_{GUT}^{-1/3}$ is not such a big number. Whether the effect we have described is really an enhancement of $p \to \pi^0 e_L^+$ or a suppression of the other decays depends largely on the unknown M-theory constant C. The factor $L(Q)^{2/3}$ can also be significant numerically. For example, for the simplest lens space, with the minimal choice $w = 1$, $q = 2$, we get $L(Q) = 4q \sin^2(5\pi w/q) = 8$, whence $L(Q)^{2/3} = 4$.

Even if C and the other factors in (5.14) were all known, the proton lifetime would also depend on how the light quarks and leptons are distributed among the different P_i, or equivalently, how they are distributed among the different **10**'s of $SU(5)$. The same remark applies in four-dimensional GUT's: the proton decay rate can be reduced by mixing of quarks and leptons among themselves as well as with other multiplets, including multiplets that have GUT-scale masses.

The arrangement of the different quarks and leptons among the P_i will also affect the flavor structure of proton decay. What we have referred to as $p \to \pi^0 e_L^+$ will also contain an admixture of other modes with π^0 replaced by

K^0, and/or e_L^+ replaced by μ_L^+. As in four-dimensional GUT's, these relative decay rates are model-dependent.

6 Difficulties With The Model

Finally, in this concluding section, we will explain some of the difficulties in making a realistic model of physics based on the class of models that we have been exploring.

One key cluster of issues is common to all known string and M-theory approaches to phenomenology. These center around the need for a good mechanism of supersymmetry breaking that solves problems such as the SUSY flavor problem, the smallness of the cosmological constant, and the high degree of stability of the vacuum we live in. We will not say anything here about these general issues, and instead focus on issues that are more or less special to the particular G_2 framework.

In the models described in [19] and further explored here, $SU(5)$ multiplets M_i containing Higgs bosons, quarks and leptons, and possibly messengers of gauge-mediated supersymmetry breaking are supported at points P_i on Q. As long as the P_i are distinct, superpotential interactions that ultimately lead to quark and lepton masses come from membrane instanton corrections and hence are exponentially small as $\alpha_{GUT} \to 0$. (For a study of superpotentials from membrane instantons on smooth G_2 manifolds, see [46].)

In fact, the membrane action scales as $(\text{length})^3$, and hence, since we saw in section four that $R_Q \sim {\alpha_{GUT}}^{-1/3}$, the membrane action scales as $1/\alpha_{GUT}$ if all lengths in X are scaled the same way. We would need to know details about X in order to compute the coefficients of $1/\alpha_{GUT}$ for various membrane instantons, so it is hard to be specific here. On the plus side of the ledger, small changes in membrane instanton actions could produce a wide range of quark and lepton masses, such as is seen in nature.

Quarks and leptons of the first two generations – and especially the first – have very small masses, but probably not as small as one might get from a membrane instanton with an action of order $1/\alpha_{GUT}$ if we take the usual value $\alpha_{GUT} \sim 1/25$. So it is probably necessary to assume that, because of extra $SU(5)$ multiplets that survive below the GUT scale, α_{GUT} is significantly larger than the usual estimate. For a proposal with extra light vector-like matter leading to $\alpha_{GUT} \sim .2 - .3$, see [9] . (We recall that at least some extra light vector-like matter – possibly serving as messengers of gauge-mediated supersymmetry breaking – is needed in the doublet-triplet splitting framework of [19] because of certain anomalies. See also [11] .)

The top quark mass is so large that it is not plausible to interpret it as being subject to any exponential suppression at all. So we might want to assume that the point P_1 that supports a Higgs **5** of $SU(5)$ coincides with the point P_2 that supports one of the **10**'s. In a case with such multiple singularities, one obtains a superpotential coupling (hopefully $\mathbf{5}\cdot\mathbf{10}\cdot\mathbf{10}$ in this example) that is "of order one," with no exponential suppression at all. (This is shown in an example in [20].) This might give us a large top quark mass. However, if we want to split Higgs doublets and triplets using the mechanism described in [19], we really must take the Higgs $\overline{\mathbf{5}}$ to be supported at a point P_3 that is distinct from P_1 and P_2 (in fact, P_3 and P_1 lie on different components of the fixed point set of the global symmetry that leads to doublet-triplet splitting). So the bottom quark mass presumably has at least some exponential suppression involving a membrane instanton action. Since $m_b/m_t \sim 1/40$, which is not all that small, and m_b can be suppressed by large $\tan\beta$ in supersymmetric models as well as by a large membrane instanton action, in practice we will have to suppose that $\tan\beta$ is not too large and than a certain membrane instanton has rather small action.

Light neutrino masses come from dimension five superpotential couplings (of the form $\int d^2\theta H^2 L^2$, with H and L being Higgs boson and lepton doublets), which again arise from membrane instantons, or possibly by integrating out a heavy singlet ("right-handed neutrino"). So it is necessary to arrange that the membrane instanton generating the H^2L^2 coupling has rather small action, or that the heavy singlet is sufficiently light and sufficiently strongly coupled to H and L. (The singlet mass and couplings to HL both come from membrane instantons.)

Since the $SU(5)$ relation $m_b = m_\tau$, which is converted by renormalization group running to the weak scale to something more like $m_b = 3m_\tau$ [48], is fairly successful, we presumably want to preserve this relation by avoiding significant mixing of the third generation with other multiplets. Since analogous relations for the first two genreations are not successful, we probably do want mixing of the first two generations with other multiplets that have large masses (the use of such mixing to modify fermion mass relations was recalled in [19]). It is consistent to have significant mixing of the first two generations with other multiplets while avoiding such mixing for the third generation, because the observed values of the CKM quark mixing angles suggest that (except for neutrinos) the mixing of the third generation with the first two is very tiny. We do not have a good mechanism that would split off the third generation in this way.

Finally, we come to the question of whether the Higgs triplet mass m_T can really be as light as suggested in (3.36). In four-dimensional GUT's this might lead to trouble because of proton decay interactions mediated by the Higgs triplets. In the present context, the doublet-triplet splitting mechanism of [19] implies that, because of an exotic discrete symmetry, the Higgs triplet exchange does not generate dimension five operators. We also have to worry about dimension six operators mediated by the Higgs triplets; these are unavoidable, and give proton decay amplitudes proportional to $\lambda_1\lambda_2/m_T^2$, where the λ_i are Yukawa couplings of the Higgs triplet to quarks and leptons. The λ_i would plausibly[12] be of order $m_{q,l}/m_W$, where $m_{q,l}$ is a first or second generation quark and lepton mass and m_W is the W boson mass. Since the ratios $m_{q,l}/m_W$ range from roughly 10^{-4} to 10^{-2}, with values closer to 10^{-4} for the quarks that are part of the initial state in a proton decay process, this is plausibly enough suppression so that we can accept a Higgs triplet mass as low as $10^{-2}\ M_{GUT}$. In fact, such a Higgs triplet might not dominate proton decay, leaving the mechanism explored in section 5 as the dominant mechanism.

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[12] This estimate for the Yukawa couplings is certainly valid if we ignore mixing of the Higgs bosons and the first two generations with other multiplets.

A On The Ray-Singer Torsion In The Trivial Representation

In this appendix, we will calculate $\mathcal{K}_{\mathcal{O}}(\mathbf{S}^3)$ for the trivial representation $\mathcal{O}$ directly by summing over the non-zero eigenvalues of the Laplacian and using zeta function regularization. We did not find this calculation in the literature, and wanted to fill this gap; for more examples of the use of zeta function regularization in studying Ray-Singer torsions, see [49] .

Then we will calculate $\mathcal{K}_{\mathcal{O}}$ for the lens space $\mathbf{S}^3/\mathbf{Z}_q$ by using the relation

$$\mathcal{K}_{\mathcal{O}}(\mathbf{S}^3) = \mathcal{K}_{\mathcal{O}}(\mathbf{S}^3/\mathbf{Z}_q) + \sum_{\omega \neq \mathcal{O}} \mathcal{K}_{\omega}(\mathbf{S}^3/\mathbf{Z}_q) \tag{A.1}$$

where the sum is over non-trivial representations of $\mathbf{Z}_q$. For such non-trivial representations, $\mathcal{K}_{\omega}(\mathbf{S}^3/\mathbf{Z}_q)$ has been computed by evaluating the zeta function [25]. The relation (A.1) holds simply because each eigenform on $\mathbf{S}^3$ is in some representation of the $\mathbf{Z}_q$ action on $\mathbf{S}^3$, so we can separate the sum over eigenforms on $\mathbf{S}^3$ into a sum over the eigenforms living in the trivial representation of $\mathbf{Z}_q$ and those living in the non-trivial representations.

We recall the definition[13] of $\mathcal{K}$:

$$\mathcal{K}(\mathbf{S}^3) = \frac{3}{2} \log \det \Delta_0(\mathbf{S}^3) - \frac{1}{2} \log \det \Delta_1(\mathbf{S}^3). \tag{A.2}$$

So we need the eigenvalues of the Laplacian and their multiplicities for 0–forms and 1–forms on $\mathbf{S}^3$. We shall calculate everything first for a sphere of radius 1, and include the dependence on the radius later.

The 0–forms have eigenvalues $\lambda_{0,n} = n(n+2)$ with multiplicities $y_{0,n} = (n+1)^2$. There are two types of 1–forms: closed ones, which have the same eigenvalues and multiplicities as the 0–forms, and co-closed 1–forms, which have eigenvalues $\lambda_{1,n} = (n+1)^2$ and multiplicities $y_{1,n} = 2n(n+2)$ (see (3.19) of [49]).

The logarithm of the determinant of the Laplacian is defined by analytic continuation using zeta functions and comparing to the known analytic continuation of the Riemann zeta function. We begin by writing the zeta function of the Laplacian on 0–forms or closed 1–forms, and the zeta function

[13] In discussing $\mathcal{K}(\mathbf{S}^3)$, we omit the subscript $\mathcal{O}$ as the fundamental group of $\mathbf{S}^3$ is trivial.

of the Laplacian on co-closed 1–forms:

$$\zeta_0(s) = \sum \frac{y_{0,n}}{\lambda_{0,n}^s} = \sum_{n=1}^{\infty} \frac{(n+1)^2}{(n(n+2))^s}, \tag{A.3}$$

$$\zeta_1(s) = \sum \frac{y_{1,n}}{\lambda_{1,n}^s} = \sum_{n=1}^{\infty} \frac{2n(n+2)}{(n+1)^{2s}}. \tag{A.4}$$

These converge for sufficiently large $\mathrm{Re}(s)$. We want to analytically continue them to $s=0$, after which we define

$$-\zeta_0'(0) = \log\det\Delta_0' = \log\det\Delta_1^{closed}, \quad -\zeta_1'(0) = \log\det\Delta_1^{co-closed}. \tag{A.5}$$

In terms of these zeta functions, we have

$$\frac{1}{2}\Big[3\log\det\Delta_0' - (\log\det{\Delta_1'}^{closed} + \log\det{\Delta_1'}^{co-closed})\Big] = \frac{1}{2}\zeta_1'(0) - \zeta_0'(0). \tag{A.6}$$

We wish to rewrite the sums $\zeta_0(s)$ and $\zeta_1(s)$ in terms of the well known Riemann zeta function $\zeta(s)$ so that they can be continued to $s=0$. This is straightforward for $\zeta_1(s)$:

$$\zeta_1(s) = 2\sum_{n=1}^{\infty}\Big(\frac{(n+1)^2}{(n+1)^{2s}} - \frac{1}{(n+1)^{2s}}\Big). \tag{A.7}$$

So we can write $\zeta_1(s)$ in terms of the Riemann zeta function $\zeta(s)$:

$$\zeta_1(s) = 2((\zeta(2s-2)-1)-(\zeta(2s)-1)) = 2(\zeta(2s-2)-\zeta(2s)). \tag{A.8}$$

Since the analytic continuation of $\zeta(s)$ is well-known, this solves the problem of analytically continuing $\zeta_1(s)$. As for $\zeta_0(s)$, we rewrite it as follows:

$$\begin{aligned}\zeta_0(s) =& \sum_{n=1}^{\infty}(n+1)^2\Big[\Big(1-\frac{1}{(2s+1)}\Big)\frac{1}{(n+1)^{2s}} + \frac{1}{2(2s+1)}\Big(\frac{1}{n^{2s}}+\frac{1}{(n+2)^{2s}}\Big)\Big] \\ &+ \sum_{n=1}^{\infty}(n+1)^2\Big[\frac{1}{(n(n+2))^s} - \Big(1-\frac{1}{(2s+1)}\Big)\frac{1}{(n+1)^{2s}} \\ &\qquad - \frac{1}{2(2s+1)}\Big(\frac{1}{n^{2s}}+\frac{1}{(n+2)^{2s}}\Big)\Big].\end{aligned} \tag{A.9}$$

The sum on the second line converges absolutely for $\mathrm{Re}(s) > -1/2$: rewriting it with $u = \frac{1}{n+1}$, it becomes

$$\sum_{n=1}^{\infty} u^{2s-2}\Big[\frac{1}{(1-u^2)^s} - \Big(1-\frac{1}{(2s+1)}\Big) - \frac{1}{2(2s+1)}\Big(\frac{1}{(1-u)^{2s}}+\frac{1}{(1+u)^{2s}}\Big)\Big], \tag{A.10}$$

and one can see that the leading order term for large n (small u) is $u^{2s-2}u^4 = u^{2s+2}$. Hence it is bounded by $1/(n+1)^{2s+2}$ which converges absolutely for $\mathrm{Re}(2s+2) > 1$ or $\mathrm{Re}(s) > -1/2$. So for $\mathrm{Re}(s) > -1/2$, we can do the sum term by term. At $s = 0$, each term in the sum vanishes, and furthermore the derivative of each term with respect to s at $s = 0$ vanishes. Therefore, for small s we can write

$$\zeta_0(s) = \sum_{n=1}^{\infty}(n+1)^2\Big[\Big(1 - \frac{1}{(2s+1)}\Big)\frac{1}{(n+1)^{2s}} + \frac{1}{2(2s+1)}\Big(\frac{1}{n^{2s}} + \frac{1}{(n+2)^{2s}}\Big)\Big], \tag{A.11}$$

which becomes, by analytic continuation,

$$\zeta_0(s) = \zeta(2s-2) + \frac{1}{(2s+1)}\zeta(2s) - \left(1 - \frac{1}{(2s+1)}\right) - \frac{1}{(2s+1)2^{2s+1}}. \tag{A.12}$$

Using known values of the Riemann zeta function

$$\zeta(-2) = 0\,, \qquad \zeta(0) = -1/2\,, \qquad \zeta'(0) = -\frac{1}{2}\log 2\pi, \tag{A.13}$$

we have at $s = 0$ the following values of the zeta functions of our Laplacians and their derivatives:

$$\begin{aligned}
\zeta_0(0) =& \zeta(-2) + \zeta(0) - \frac{1}{2} = -1,\\
\zeta_1(0) =& 2(\zeta(-2) - \zeta(0)) = 1,\\
\zeta_0'(0) =& 2\zeta'(-2) - \log\pi,\\
\zeta_1'(0) =& 4(\zeta'(-2) - \zeta'(0)) = 4\zeta'(-2) + 2\log(2\pi).
\end{aligned} \tag{A.14}$$

(The value of $\zeta_0'(0)$ can also be obtained from Proposition 3.1 of [50], or from [51].)

Now we include the dependence on the radius. The quantity that comes into the physical calculation is actually

$$-\mathrm{Tr}\log\left(\frac{m_{k,n}^2}{\Lambda^2}\right) = -\sum_n y_n \log\frac{\lambda_{k,n}}{\Lambda^2 R^2}\ , \tag{A.15}$$

where Λ is the cutoff and R is the radius of $\mathbf{S}^3$. So we should replace ζ_0 and ζ_1 as defined in equations (A.3) and (A.4) by

$$\begin{aligned}
\eta_0(s) =& (R\Lambda)^{2s}\zeta_0(s),\\
\eta_1(s) =& (R\Lambda)^{2s}\zeta_1(s).
\end{aligned} \tag{A.16}$$

The derivatives at $s = 0$ are

$$\begin{aligned}
\eta_0'(0) =& -\log R^2\Lambda^2 + \zeta_0'(0),\\
\eta_1'(0) =& \log R^2\Lambda^2 + \zeta_1'(0).
\end{aligned} \tag{A.17}$$

We now have

$$\begin{aligned}\mathcal{K}(\mathbf{S}^3,1) =&\frac{1}{2}\eta_1'(0)-\eta_0'(0)\\ =&\frac{3}{2}\log R^2\Lambda^2+\log(2\pi^2).\end{aligned} \tag{A.18}$$

Finally, we can use equation (A.1) to extend this result to a lens space. By evaluating zeta functions, Ray showed[14] [24] that for non-trivial ω,

$$\mathcal{T}_\omega(\mathbf{S}^3/\mathbf{Z}_q)=\log|\omega-1||\omega^{-j}-1|. \tag{A.19}$$

Here ω is any non-trivial q^{th} root of unity, and j is prime to q, so

$$\sum_{\omega\neq\mathcal{O}}\mathcal{K}_\omega(\mathbf{S}^3/\mathbf{Z}_q)=\sum_{\omega\neq\mathcal{O}}\mathcal{T}_\omega(\mathbf{S}^3/\mathbf{Z}_q)=\sum_{\omega\neq\mathcal{O}}\log|\omega-1||\omega^{-j}-1|=2\log q\ . \tag{A.20}$$

Therefore,

$$\begin{aligned}\mathcal{K}_{\mathcal{O}}(\mathbf{S}^3/\mathbf{Z}_q) =&\frac{3}{2}\log R^2\Lambda^2+\log 2\pi^2-2\log q\\ =&\log\left(\frac{2\pi^2}{q^2}\right)+\frac{3}{2}\log R^2\Lambda^2.\end{aligned} \tag{A.21}$$

In terms of the volume $V=2\pi^2R^3/q$ of the lens space, this becomes

$$\begin{aligned}\mathcal{K}_{\mathcal{O}}(\mathbf{S}^3/\mathbf{Z}_q) =&\log\left(\frac{2\pi^2}{q^2}\right)+\log\frac{q}{2\pi^2}V\Lambda^3\\ =&\log\frac{V\Lambda^3}{q}.\end{aligned} \tag{A.22}$$

The torsion of the lens space for the trivial representation is $\mathcal{T}_{\mathcal{O}}(\mathbf{S}^3/\mathbf{Z}_q)=\log(1/q)$. To compute this, we use the fact that, as described in [24], the lens space has a cell decomposition in which the chain group C_k is isomorphic to $\mathbf{Z}$ for $k=0,\ldots,3$. The only non-trivial boundary operator is $\partial_{2\to1}:C_2\to C_1$, which equals multiplication by q. To compute $\mathcal{T}_{\mathcal{O}}(\mathbf{S}^3/\mathbf{Z}_q)$, relative to a basis of the integral homology, we should first remove subgroups of the chain groups that generate the homology. In this case, we do this by dropping C_0 and C_3. Then the Reidemeister torsion of the lens space for the trivial representation is defined as an alternating sum of logarithms of the boundary maps; in the present case, this reduces to $-\log\partial_{2\to1}=\log(1/q)$. The Reidemeister torsion equals the Ray-Singer torsion by the conjecture of Ray and Singer, which was later proved by Cheeger, so $\mathcal{T}_{\mathcal{O}}=\log(1/q)$. In [24], (A.19) was obtained by a similar computation.

[14]We express this result using the normalization of the torsion that is used in [25] and in the present paper. Also, we include the integer m described in the last paragraph of section 3.4, with $jm\equiv 1 \bmod q$.

So (A.22) implies that

$$\mathcal{K}_{\mathcal{O}} = \mathcal{T}_{\mathcal{O}} + \log(V\Lambda^3) \tag{A.23}$$

for lens spaces. In the next appendix, we show that this relation is actually true for all three-manifolds with $b_1 = 0$. This relation was used in section 3 in evaluating the threshold corrections.

B The Volume Correction

Here we show the relation (A.23) for every three-manifold Q. In the derivation, we set $\Lambda = 1$, though from a physical point of view it is natural to include Λ.

In defining Reidemeister torsion for a representation that has non-trivial real cohomology, one has to pick a basis for the cohomology. In the case of the trivial representation $\mathcal{O}$, we can pick a basis of integral classes that generate the integral cohomology mod torsion; the Reidemeister torsion is independent of the choice of such a basis. When we speak of $\mathcal{T}_{\mathcal{O}}(Q)$, we mean the torsion defined relative to such a basis, which we call a topological basis.

In Ray-Singer torsion, instead, the natural basis would be a basis of zero modes α_i of the Laplacian that are orthonormal in the $\mathbf{L}^2$ sense, that is, $\int_Q \overline{\alpha}_i \wedge *\alpha_j = \delta_{ij}$. We call such a basis an $\mathbf{L}^2$ basis. The theorem relating Ray-Singer and Reidemeister torsion asserts that

$$\mathcal{T}_{\mathcal{O}}(Q) = \mathcal{K}_{\mathcal{O}}(Q) + \mathcal{A}, \tag{B.1}$$

where $\mathcal{A}$ is defined as follows. For each $k = 0, \ldots, 3$, we let A^k be an invertible map from a topological basis of the k^{th} cohomology to an $\mathbf{L}^2$ basis. Then the quantities $|\det(A^k)|$ are independent of the choices of bases and maps, and the general definition of $\mathcal{A}$ is

$$\mathcal{A} = \sum_{k=0}^{3} (-1)^k \log|\det(A^k)|. \tag{B.2}$$

For a three-manifold Q of $b_1(Q) = 0$, everything simplifies drastically. The nonzero Betti numbers are $b_0 = b_3 = 1$. For a topological basis, we can pick the zero-form 1 and the three-form $\epsilon_{ijk}dx^i dx^j dx^k/V_Q$, where the volume is $\int \epsilon_{ijk}dx^i dx^j dx^k = V_Q$. For an $\mathbf{L}^2$ basis, we pick $1/\sqrt{V_Q}$ and $\epsilon_{ijk}dx^i dx^j dx^k/\sqrt{V_Q}$. So $\det A^0 = 1/\sqrt{V_Q}$, $\det A^3 = \sqrt{V_Q}$, and we arrive at (A.23).

References

[1] H. Georgi, H. Quinn, and S. Weinberg, "Hierarchy Of Interactions In Unified Gauge Theories," Phys. Rev. Lett. **33** (1974) 451.

[2] H. Georgi and S. Glashow, "Unity Of All Elementary Particle Forces," Phys. Rev. Lett. **32** (1974) 438.

[3] S. Dimopoulos, S. Raby, and F. Wilczek, "Supersymmetry And The Scale Of Unification," Phys. Rev. **D24** (1981) 1681.

[4] D. J. Gross, J. A. Harvey, E. Martinec, and R. Rohm, "The Heterotic String," Phys. Rev. Lett. **54** (1985) 502.

[5] P. Ginsparg, "Gauge And Gravitational Couplings In Four-Dimensional String Theories," Phys. Lett. **B197** (1987) 139.

[6] V. Kaplunovsky, "One Loop Threshold Effects in String Unification," Nucl. Phys. **B307** (1988) 145, erratum Ibid **B382** (1992) 436; for a revised version see hep-th/9205070.

[7] S. Weinberg, "Effective Gauge Theories," Phys. Lett. **B91** (1980) 51.

[8] B. A. Ovrut and H. J. Schnitzer, "Low-Energy And Threshold Calculations Using Effective Field Theory," Nucl. Phys. B **184**, 109 (1981).

[9] K. S. Babu and J. C. Pati, "The Problems Of Unification Mismatch and Low α_3: A Solution With Light Vector-Like Matter," Phys. Lett. **B384** (1995) 140, hep-ph/9606215.

[10] J. A. Bagger, K. T. Matchev, D. M. Pierce, and R.-J. Zhang, "Gauge And Yukawa Coupling Unification In Models With Gauge-Mediated Supersymmetry Breaking," Phys. Rev. Lett. **78** (1997) 1002.

[11] D. Ghilencea, M. Lanzagorta and G. G. Ross, "Unification predictions," Nucl. Phys. B **511**, 3 (1998) [arXiv:hep-ph/9707401].

[12] P. Horava and E. Witten, "Heterotic And Type I String Dynamics From Eleven-Dimensions," Nucl Phys. **B460** (1996) 506, hep-th/9510209.

[13] E. Witten, "Strong Coupling Expansion Of Calabi-Yau Compactification," Nucl. Phys. **B471** (1996) 135.

[14] G. Curio and A. Krause, "Four-flux and warped heterotic M-theory compactifications," Nucl. Phys. B **602**, 172 (2001) [arXiv:hep-th/0012152]; G. Curio and A. Krause, "G-fluxes and non-perturbative stabilisation of heterotic M-theory," Nucl. Phys. B **643**, 131 (2002) [arXiv:hep-th/0108220].

[15] M. Cvetic, G. Shiu, and A. M. Uranga, "Three Family Supersymmetric Standard-Like Models From Intersecting Brane Worlds," Phys. Rev. Lett. **87** (2001), hep-th/0107143, "Chiral Type II Orientifold Construction As M-Theory On G_2 Holonomy Spaces," hep-th/0111179.

[16] M. Cvetic, P. Langacker, and G. Shiu, "Phenomenology Of A Three Family Standard Like String Model," Phys. Rev. **D66** (2002) 066004, "A Three Family Standard-Like Orientifold Model: Yukawa Couplings And Hierarchy," Nucl. Phys. **B642** (200) 139, hep-th/0206115.

[17] R. Blumenhagen and V. Braun, "Superconformal Field Theories For Compact G_2 Manifolds," JHEP **0112** (2001) 006, hep-th/0110232.

[18] R. Blumenhagen, V. Braun, B. Kors and D. Lust, "The Standard Model on the Quintic," hep-th/0210083.

[19] E. Witten, "Deconstruction, G_2 Holonomy, And Doublet-Triplet Splitting," hep-ph/0201018.

[20] M. F. Atiyah and E. Witten, "M Theory Dynamics On A Manifold Of G_2 Holonomy," hep-th/0107177.

[21] E. Witten, "Anomaly Cancellation On G_2 Manifolds," hep-th/0108165.

[22] B. Acharya and E. Witten, "Chiral Fermions From Manifolds Of G_2 Holonomy," hep-th/0109152.

[23] P. Berglund and A. Brandhuber, "Matter From G_2 Manifolds," Nucl. Phys. **B641** (2002) 351, hep-th/0205184.

[24] D. B. Ray, "Reidemeister Torsion And The Laplacian On Lens Spaces," Adv. Math. **4** (1970) 109.

[25] D. B. Ray and I. M. Singer, "R-Torsion And The Laplacian On Riemannian Manifolds," Adv. Math. **7** (1971) 145, "Analytic Torsion," in *Partial Differential Equations*, Proc. Symp.Pure Math. Vol. XXIII (Univ. California, Berkeley, Cal. 1971), p. 167 (American Math. Society, Providence RI, 1973).

[26] E. Witten, "Symmetry Breaking Patterns In Superstring Models," Nucl. Phys. **B258** (1985) 75.

[27] Y. Kawamura, "Triplet Doublet Splitting, Proton Stability, And Extra Dimensions," Prog. Theor. Phys. **105** (2001) 999, hep-ph/0012125.

[28] L. J. Hall and Y. Nomura, "Gauge Unification In Higher Dimensions," Phys. Rev. **D64** (2001).

[29] G. Altarelli and F. Feruglio, "$SU(5)$ Grand Unification In Extra Dimensions And Proton Decay," Phys. Lett. **B511** (2001) 257.

[30] M. Kakizaki and M. Yamaguchi, "Splitting Triplet And Doublet In Extra Dimensions," hep-ph/0104103.

[31] L. J. Hall, H. Murayama, and Y. Nomura, "Wilson Lines And Symmetry Breaking On Orbifolds," hep-th/0107245.

[32] S. Raby, "Proton Decay," hep-ph/0211024; H. Dermisek, A. Mafi, and S. Raby, "SUSY GUT's Under Siege: Proton Decay," hep-ph/0007213.

[33] B. Bajc, P. F. Perez and G. Senjanovic, "Proton decay in minimal supersymmetric SU(5)," Phys. Rev. D **66**, 075005 (2002) [arXiv:hep-ph/0204311]; B. Bajc, P. F. Perez and G. Senjanovic, "Minimal supersymmetric SU(5) theory and proton decay: Where do we stand?," arXiv:hep-ph/0210374.

[34] K. S. Ganezer, "The Search for Nucleon Decay at Super-Kamiokande," Int. J. Mod. Phys. **A16** (2001) 855.

[35] C. K. Jung, "Feasibility Of A Next Generation Underground Water Cerenkov Detector: UNO," hep-ex/0005046.

[36] A. Strominger, S.-T. Yau, and E. Zaslow, "Mirror Symmetry Is T Duality," Nucl. Phys. **B479** (1996) 243, hep-th/9606040.

[37] B. Acharya, "M Theory, Joyce Manifolds, and Super Yang-Mills," hep-th/9812205.

[38] R. Bryant and S. Salamon, "On The Construction Of Some Complete Metrics With Exceptional Holonomy," Duke Math. J. **58** (1989) 829.

[39] G. W. Gibbons, D. N. Page, and C. N. Pope, "Einstein Metrics On $\mathbf{S}^3$, $\mathbf{R}^3$, and $\mathbf{R}^4$ Bundles," Commun. Math. Phys. **127** (1990) 529.

[40] M. F. Atiyah and J. Maldacena and C. Vafa, "An M Theory Flop As A Large N Duality," J. Math. Phys. **42** (2001) 3209, hep-th/0011256.

[41] B. Acharya, "On Realising $\mathcal{N} = 1$ Super Yang-Mills In M Theory," hep-th/0011089, "Confining Strings From G_2-Holonomy Spacetimes," hep-th/0101206.

[42] T. Friedmann, "On The Quantum Moduli Space Of M Theory Compactifications," Nucl. Phys. **B635** (2002) 384, hep-th/0203256.

[43] A. E. Faraggi and M. Thormeier, "String inspired Z' model with stable proton and light neutrino masses," Nucl. Phys. B **624**, 163 (2002) [arXiv:hep-ph/0109162]; A. E. Faraggi, "Proton stability in superstring derived models," Nucl. Phys. B **428**, 111 (1994) [arXiv:hep-ph/9403312].

[44] A. Schwarz, "The Partition Function Of A Degenerate Functional," Commun. Math. Phys. **67** (1979) 1.

[45] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, "Kodaira-Spencer Theory of Gravity and Exact Results For Quantum String Amplitudes," Commun. Math. Phys. **B165** (1994) 311, hep-th/9309140.

[46] J. A. Harvey and G. Moore, "Superpotentials and Membrane Instantons," hep-th/9907026.

[47] J. Polchinski, *String Theory*, vol. 2 (Cambridge University Press, 1998).

[48] A. J. Buras, J. Ellis, M. K. Gaillard, and D. Nanopoulos, "Aspects Of The Grand Unification Of Strong, Weak, And Electromagnetic Interactions," Nucl. Phys. **B135** (1979) 66.

[49] C. Nash and D. J. O'Connor, "Determinants Of Laplacians, The Ray-Singer Torsion On Lens Spaces, and The Riemann Zeta Function," J. Math. Phys. **36** (1995) 1462-1505.

[50] I. Vardi, "Determinants Of Laplacians And Multiple Gamma Functions," SIAM J. Math. Anal. **19** (1988) 493.

[51] J. Choi and H. M. Srivastava, "An Application Of The Theory Of The Doublet Gamma Function," Kyushu J. Math. **53** (1999) 209