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On the quantum moduli space of M-theory compactifications

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Abstract

We study the moduli space of M-theories compactified on G_2 manifolds which are asymptotic to a cone over quotients of $\mathbf{S}^3 \times \mathbf{S}^3$. We show that the moduli space is composed of several components, each of which interpolates smoothly among various classical limits corresponding to low energy gauge theories with a given number of massless $U(1)$ factors. Each component smoothly interpolates among supersymmetric gauge theories with different gauge groups. © 2002 Published by Elsevier Science B.V.

1. Introduction

The study of M-theory compactifications on seven-dimensional manifolds X of G_2 holonomy has been motivated by the fact that such compactifications result in unbroken supersymmetry in four dimensions. The properties of the compactification manifold X determine the particle spectrum of the corresponding four-dimensional theory. It has been shown in recent years that compactifications on singular manifolds can result in low energy physics containing interesting massless spectra. Specifically, certain singular G_2 manifolds give rise to $\mathcal{N} = 1$ supersymmetric gauge theories at low energies, as shown, for example, in [1–3]. There, X was taken to be asymptotic to a quotient of a cone on $\mathbf{S}^3 \times \mathbf{S}^3$, and the singularities of X took the form of families of *ADE* singularities giving *ADE* gauge theories at low energies.

Subsequently, the quantum moduli space of M-theories on G_2 manifolds X which are asymptotic to a cone on $\mathbf{S}^3 \times \mathbf{S}^3$ or quotients thereof has been studied in [4]. It was shown

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that the moduli space is a Riemann surface of genus zero, which interpolates smoothly between different semiclassical spacetimes.

The purpose of this paper is to generalize the construction of [4] to other quotients of $\mathbf{S}^3 \times \mathbf{S}^3$ and obtain the moduli spaces for those as well. Our quotients contain those in [4] as special cases. We propose that the moduli space for our quotients consists of several branches classified according to the number of massless $U(1)$ factors that appear in the low energy gauge theories corresponding to semiclassical points. Each branch of the moduli space interpolates smoothly between the different semiclassical points appearing on it; hence, we get smooth interpolation between supersymmetric gauge theories with different gauge groups.

This paper is organized as follows: in Section 2 we review the M-theory dynamics on the cone on $Y = \mathbf{S}^3 \times \mathbf{S}^3$ given in [4]. In Section 3, we describe quotients of this cone by discrete groups of the form $\Gamma = \Gamma_1 \times \Gamma_2 \times \Gamma_3$ where the Γ_i are *ADE* subgroups of $SU(2)$; these *ADE* groups must be chosen carefully in order to obtain known low energy gauge theories from the compactification. In Section 4, we turn to the description of the moduli space $\mathcal{N}_\Gamma$ of M-theories on these quotients, beginning with the classical moduli space and concluding with the quantum moduli space.

While this paper was being completed, we received [5] which has overlap with the case where Γ_3 is trivial (or $r = 1$ in our notation of Section 3).

2. Dynamics of M-theory on the cone over $\mathbf{S}^3 \times \mathbf{S}^3$

In this section, we review the M-theory dynamics on a manifold X of G_2 holonomy which is asymptotic at infinity to a cone over $Y = \mathbf{S}^3 \times \mathbf{S}^3$ [4]. The manifold Y can be described as a homogeneous space $Y = SU(2)^3/SU(2)$, where the equivalence relation is $(g_1, g_2, g_3) \sim (g_1 h, g_2 h, g_3 h)$, $g_i, h \in SU(2)$. Viewed this way, this manifold has $SU(2)^3$ symmetry via left action on each of the three factors in Y, as well as a “triality” symmetry S_3 permuting the three factors. Up to scaling, there is a unique metric with such symmetries given by

$$d\Omega^2 = da^2 + db^2 + dc^2, \tag{1}$$

where $a, b, c \in SU(2)$, $da^2 = -\operatorname{Tr}(a^{-1}\,da)^2$, the trace is taken in the fundamental representation of $SU(2)$, and a, b, c are related to g_1, g_2, g_3 by $a = g_2 g_3^{-1}$ and cyclic permutations thereof.

The metric for a cone on Y is

$$ds^2 = dr^2 + r^2\,d\Omega^2, \tag{2}$$

where $d\Omega^2$ is the metric on Y. Such a cone can be constructed by filling in one of the three $SU(2) \sim \mathbf{S}^3$ factors of Y to a ball. We denote the manifold obtained by filling in a given g_i by X_i. The metric on a manifold X, asymptotic to X_i at infinity, can be written with a new radial variable y, which is related to r by

$$y = r - \frac{r_0^3}{4r^2} + O\big(1/r^5\big), \tag{3}$$

as

$$ds^2 = dy^2 + \frac{y^2}{36}\left(da^2 + db^2 + dc^2 - \frac{r_0^3}{2y^3}\left(f_1\,da^2 + f_2\,db^2 + f_3\,dc^2\right) + O\left(\frac{r_0^6}{y^6}\right)\right), \tag{4}$$

where r_0 is a parameter denoting the length scale of X_i, and $(f_{i-1}, f_i, f_{i+1}) = (1, -2, 1)$ (indices are understood mod 3). When $y \to \infty$ or $r \to \infty$, this becomes precisely the cone (2).

We will need to study the 3-cycles of Y in order to understand the relations between the periods of the M-theory C-field and the membrane instanton amplitudes, which we shall need in order to describe the moduli space.

The 3-cycles D_j of Y are given by projections of the jth factor of $SU(2)^3$ to Y. Hence, $D_j \cong \mathbf{S}^3$. The third Betti number of Y is two, so the three D_j satisfy the relation

$$D_1 + D_2 + D_3 = 0. \tag{5}$$

The intersection numbers of the D_i are given by

$$D_i \cdot D_j = \delta_{j,i+1} - \delta_{j,i-1}. \tag{6}$$

At X_i, where the ith factor is filled in, D_i shrinks to zero and the relation (5) reduces to $D_{i-1} + D_{i+1} = 0$ (where again the indices are understood mod 3).

At each X_i, there is a supersymmetric 3-cycle Q_i given by $g_i = 0$. It can be shown that Q_i is homologous to $\pm D_{i-1}$ and $\mp D_{i+1}$, where the sign depends on orientation.

A manifold still has G_2 holonomy up to third order in r_0/y if we take the f_j of (4) to be any linear combination of $(1, -2, 1)$ and its permutations—so we have G_2 holonomy as long as

$$f_1 + f_2 + f_3 = 0. \tag{7}$$

These f_j can be interpreted as volume defects of the cycle D_j at infinity: the volume of D_j depends linearly on a positive multiple of f_j. Furthermore, since at the classical manifold X_i, only one of the D_j vanishes, only one of the f_j (namely f_i) can be negative. So the classical moduli space may contain manifolds with the relation (7) as long as only one of the f_j is negative [4,6].

The periods of the C-field along the cycles D_j are $\alpha_j = \int_{D_j} C$. We combine them with the f_j into holomorphic observables η_j where now the C-field period is a phase:

$$\eta_j = \exp\left(\frac{2k}{3} f_{j-1} + \frac{k}{3} f_j + i\alpha_j\right), \tag{8}$$

where k is a parameter. The relation (7) means that the η_j are not independent, but instead they obey

$$\eta_1\eta_2\eta_3 = \exp\left(i \sum \alpha_j\right). \tag{9}$$

(It can be shown that due to a global anomaly in the membrane effective action, the right-hand side above is -1.)

The moduli space at the classical approximation is given by three branches $\mathcal{N}_i$, each of which contains one of the points X_i with $r_0 \to \infty$. On X_i, α_i vanishes and the parameters f_j are such that $\eta_i = 1$. So on $\mathcal{N}_i$ the functions η_j obey

$$\eta_i = 1, \quad \eta_{i-1}\eta_{i+1} = -1. \tag{10}$$

At the quantum level, there are corrections to this statement. It has been suggested in [3] that the different classical points X_i are continuously connected to one another. Hence they should appear on the same branch of the moduli space $\mathcal{N}$. We proceed now with the assumption that the only classical points are the X_i, which are the points where some of the η_j have a zero or pole. As explained in [4], since a component of $\mathcal{N}$ which contains a zero of a holomorphic function η_j must also contain its pole, and since the only points at which the η_j are singular are associated with one of the X_i, it follows indeed that all X_i are contained on a single component of $\mathcal{N}$. Furthermore, each η_j has a simple zero and simple pole in $\mathcal{N}$. The existence of such functions on $\mathcal{N}$ means that the branch containing the zero and pole has genus zero. In addition, any of the η_j can be identified as a global coordinate of $\mathcal{N}$. Choosing any η_j gives a complete description for this branch of $\mathcal{N}$.

3. Quotients and low energy gauge groups

Here, we begin our study of manifolds which are asymptotic to a cone over quotients of Y. We shall consider a discrete group action of $\Gamma = \Gamma_1 \times \Gamma_2 \times \Gamma_3$ on Y where the Γ_i will be chosen from *ADE* subgroups of $SU(2)$ in such a way that the low energy physics is known.

We begin with the simplest case where $\Gamma = \mathbf{Z}_p \times \mathbf{Z}_q \times \mathbf{Z}_r$. Each $\mathbf{Z}_n$ is embedded in $SU(2)$ via

$$\beta^k = \begin{pmatrix} e^{2\pi ik/n} & 0 \\ 0 & e^{-2\pi ik/n} \end{pmatrix}, \tag{11}$$

where β is the generator of $\mathbf{Z}_n$ and $k = 0, 1, \ldots, n-1$. The action of Γ on $Y = SU(2)^3/SU(2)$ is given by

$$(\gamma, \delta, \epsilon) \in \mathbf{Z}_p \times \mathbf{Z}_q \times \mathbf{Z}_r : (g_1, g_2, g_3) \mapsto (\gamma g_1, \delta g_2, \epsilon g_3), \tag{12}$$

and we denote the resulting quotient space by Y_Γ.

The spaces $X_{i,\Gamma}$, obtained by filling in the ith $SU(2)$ factor of Y_Γ, are quotients of $\mathbf{R}^4 \times \mathbf{S}^3$ where the $\mathbf{R}^4$ corresponds to the filled-in factor. Choosing $i = 1$ and gauging g_2 away using the right diagonal $SU(2)$ action, the identification corresponding to $(\gamma^k, \delta^l, \epsilon^m) \in \Gamma$ is

$$(g_1, 1, g_3) \sim \left(\gamma^k g_1 \delta^{-l}, 1, \epsilon^m g_3 \delta^{-l}\right), \tag{13}$$

where $g_1 \in \mathbf{R}^4$ and $g_3 \in SU(2) \sim \mathbf{S}^3$. The set $(0, 1, g_3)$ with g_3 varying in $SU(2)$ is a fixed point of the action of the $\mathbf{Z}_p$ subgroup of Γ, and this singularity is identical to the standard A_{p-1} singularity of codimension four of the form $\mathbf{R}^4/\mathbf{Z}_p$ or $\mathbf{C}^2/\mathbf{Z}_p$, which gives an $SU(p)$ gauge theory at low energies.

Depending on the values of the integers q and r, there may be additional, unfamiliar singularities for which we do not know the low energy physics. Namely, there may be values of g_3 which are fixed under a nontrivial subgroup of $\mathbf{Z}_q \times \mathbf{Z}_r$, i.e., where the following holds

$$\epsilon^m g_3 \delta^{-l} = g_3. \tag{14}$$

This is the same as looking for elements g_3 of $SU(2)$ which diagonalize δ^l:

$$\epsilon^m = g_3 \delta^l g_3^{-1}. \tag{15}$$

Choosing the orders q and r of δ and ϵ to be relatively prime, $(q,r)=1$, ensures that there are no solutions of this equation (since then the orders of the left- and right-hand sides of (15) are relatively prime). Similarly, we choose $(p,q)=(p,r)=1$, and so there are no singularities at $X_{i,\Gamma}$ other than the *ADE* singularities whose low energy physics is known: an A_{p-1} singularity on $\mathbf{S}^3/(\mathbf{Z}_q \times \mathbf{Z}_r)$ at $X_{1,\Gamma}$, an A_{q-1} singularity on $\mathbf{S}^3/(\mathbf{Z}_r \times \mathbf{Z}_p)$ at $X_{2,\Gamma}$, and an A_{r-1} singularity on $\mathbf{S}^3/(\mathbf{Z}_p \times \mathbf{Z}_q)$ at $X_{3,\Gamma}$, with the discrete group action on $\mathbf{S}^3$ given by the appropriate cyclic permutation of the action on g_3 in (13).

Now consider also the non-abelian *ADE* groups. Again, we would like to choose Γ such that we will only get singularities whose physics at low energies we understand—namely, *ADE* singularities. For this purpose we review the relevant properties of the DE groups. For information about these groups, see [7].

As in the abelian case, we let $\Gamma = \Gamma_1 \times \Gamma_2 \times \Gamma_3$ act on Y by

$$(\gamma,\delta,\epsilon) \in \Gamma_1 \times \Gamma_2 \times \Gamma_3 : (g_1, g_2, g_3) \mapsto (\gamma g_1, \delta g_2, \epsilon g_3), \tag{16}$$

from which Eqs. (13) and (14) follow in the same way as before.

The binary dihedral groups $\mathbf{D}_q$ have order $4q-8$ and are generated in $SU(2)$ by two elements:

$$\mathbf{D}_q = \left\langle \begin{pmatrix} e^{\frac{\pi i}{q-2}} & 0 \\ 0 & e^{-\frac{\pi i}{q-2}} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle. \tag{17}$$

Since all $\mathbf{D}_q$ groups share an element of order 4, we cannot choose more than one of the Γ_i to be a dihedral group, since otherwise we would get solutions to (14). Hence we let $\Gamma = \mathbf{Z}_p \times \mathbf{D}_q \times \mathbf{Z}_r$ with $(p,r) = (p, 2(q-2)) = (r, 2(q-2)) = 1$.

We turn to the E series. A singularity $\mathbf{R}^4/G$ which gives at low energies $\mathbf{E}_6$, $\mathbf{E}_7$, or $\mathbf{E}_8$ gauge groups corresponds to G being the tetrahedral group $\mathbf{T}_{24}$, the octahedral group $\mathbf{O}_{48}$, or the icosahedral group $\mathbf{I}_{120}$. The orders of these groups are 24, 48, and 120, respectively, and each of them has elements of orders 3 and 4, so we cannot have more than one E group appearing in Γ. The group $\mathbf{I}_{120}$ also has elements of order 5. Hence, in addition to $(p,r)=1$, for $\Gamma = \mathbf{Z}_p \times \mathbf{E}_6 \times \mathbf{Z}_r$ or $\Gamma = \mathbf{Z}_p \times \mathbf{E}_7 \times \mathbf{Z}_r$, we need also $(p, 2\cdot 3) = (r, 2\cdot 3) = 1$, and for $\Gamma = \mathbf{Z}_p \times \mathbf{E}_8 \times \mathbf{Z}_r$, we need $(p, 2\cdot 3\cdot 5) = (r, 2\cdot 3\cdot 5) = 1$.

Therefore, our group Γ is always chosen to be of the form $\Gamma = \mathbf{Z}_p \times \Gamma_2 \times \mathbf{Z}_r$ where Γ_2 is an A, D, or E group, and p, r, and Γ_2 satisfy the conditions noted above, which can be summarized by

$$(p,N) = (r,N) = (p,r) = 1,$$

Table 1
Low energy gauge theories at $X_{i,\Gamma}$

Γ_2	$X_{1,\Gamma}$		$X_{2,\Gamma}$		$X_{3,\Gamma}$	
$\mathbf{Z}_q$	$SU(p)$	$\mathbf{S}^3/(\mathbf{Z}_q \times \mathbf{Z}_r)$	$SU(q)$	$\mathbf{S}^3/(\mathbf{Z}_r \times \mathbf{Z}_p)$	$SU(r)$	$\mathbf{S}^3/(\mathbf{Z}_p \times \mathbf{Z}_q)$
$\mathbf{D}_q$	$SU(p)$	$\mathbf{S}^3/(\mathbf{D}_q \times \mathbf{Z}_r)$	$SO(2q)$	$\mathbf{S}^3/(\mathbf{Z}_r \times \mathbf{Z}_p)$	$SU(r)$	$\mathbf{S}^3/(\mathbf{Z}_p \times \mathbf{D}_q)$
$\mathbf{T}_{24}$	$SU(p)$	$\mathbf{S}^3/(\mathbf{T}_{24} \times \mathbf{Z}_r)$	$\mathbf{E}_6$	$\mathbf{S}^3/(\mathbf{Z}_r \times \mathbf{Z}_p)$	$SU(r)$	$\mathbf{S}^3/(\mathbf{Z}_p \times \mathbf{T}_{24})$
$\mathbf{O}_{48}$	$SU(p)$	$\mathbf{S}^3/(\mathbf{O}_{48} \times \mathbf{Z}_r)$	$\mathbf{E}_7$	$\mathbf{S}^3/(\mathbf{Z}_r \times \mathbf{Z}_p)$	$SU(r)$	$\mathbf{S}^3/(\mathbf{Z}_p \times \mathbf{O}_{48})$
$\mathbf{I}_{120}$	$SU(p)$	$\mathbf{S}^3/(\mathbf{I}_{120} \times \mathbf{Z}_r)$	$\mathbf{E}_8$	$\mathbf{S}^3/(\mathbf{Z}_r \times \mathbf{Z}_p)$	$SU(r)$	$\mathbf{S}^3/(\mathbf{Z}_p \times \mathbf{I}_{120})$

where N is the order of the group Γ_2. At $X_{1,\Gamma}$ we have an A_{p-1} singularity on $\mathbf{S}^3/(\Gamma_2 \times \mathbf{Z}_r)$, at $X_{3,\Gamma}$ we have an A_{r-1} singularity on $\mathbf{S}^3/(\mathbf{Z}_p \times \Gamma_2)$, and at $X_{2,\Gamma}$ we have an A, D, or E singularity on $\mathbf{S}^3/(\mathbf{Z}_r \times \mathbf{Z}_p)$, where here the discrete group action is given by the appropriate cyclic permutation of the action on g_3 in (13).

The low energy gauge theories obtained from compactifying M-theory on $\mathbf{R}^4 \times X_{i,\Gamma}$ are listed in Table 1. Each entry contains the gauge group and the compact 3-manifold which is the locus of the *ADE* singularity.

As we shall see below, for the cases where Γ_2 is a D or E group, there are additional semiclassical points where the low energy gauge group is different from those listed above.

We note that for the case with $r = 1$, $X_{3,\Gamma}$ is smooth and its low energy theory has no gauge symmetry. If also $p = 1$, $X_{1,\Gamma}$ is smooth as well (this is the case studied in [4]).

4. The curve of M theories on the quotient

4.1. Classical geometry

The 3-cycles D'_i of Y_Γ are the projections of the ith factor of $SU(2)^3$ to Y_Γ. Hence, for $\Gamma = \mathbf{Z}_p \times \Gamma_2 \times \mathbf{Z}_r$ we have

$$D'_1 = \mathbf{S}^3/\mathbf{Z}_p, \tag{18}$$

$$D'_2 = \mathbf{S}^3/\Gamma_2, \tag{19}$$

$$D'_3 = \mathbf{S}^3/\mathbf{Z}_r. \tag{20}$$

Using the relation (5) in Y and the fact that $D_1 \in Y$ projects to a p-fold cover of $D'_1 \in Y_\Gamma$, as well as cyclic permutations of this fact, we find

$$pD'_1 + ND'_2 + rD'_3 = 0, \tag{21}$$

where N is the order of the group Γ_2. To study the intersection numbers of the D'_i we note that $D'_1 \in Y_\Gamma$ lifts to $NrD_1 \in Y$, and similar statements are true for the other D'_i. Counting the intersection numbers in Y and then dividing by pNr (since there are pNr points in Y which project to one point in Y_Γ), we get

$$D'_1 \cdot D'_2 = r, \qquad D'_2 \cdot D'_3 = p, \qquad D'_3 \cdot D'_1 = N. \tag{22}$$

Here we see that the D'_i generate the third homology group of Y_Γ: since $(r, N) = 1$, we can find integers m, n such that

$$D'_1 \cdot (mD'_2 + nD'_3) = mr - nN = 1, \tag{23}$$

and similarly for the other cycles.

We define the periods of the M-theory C-field at infinity by

$$\alpha'_j = \int_{D'_j} C \bmod 2\pi. \tag{24}$$

Note that these are related to the α_j of Y by

$$\alpha_1 = p\alpha'_1, \qquad \alpha_2 = N\alpha'_2, \qquad \alpha_3 = r\alpha'_3. \tag{25}$$

4.2. Classical moduli space

We define our holomorphic observables to be the following functions of the periods α'_j and of the volumes f_j:

$$\eta_1 = \exp\left(\frac{2k}{3p} f_3 + \frac{k}{3p} f_1 + i\alpha'_1\right),$$
$$\eta_2 = \exp\left(\frac{2k}{3N} f_1 + \frac{k}{3N} f_2 + i\alpha'_2\right),$$
$$\eta_3 = \exp\left(\frac{2k}{3r} f_2 + \frac{k}{3r} f_3 + i\alpha'_3\right).$$

These functions are adopted from (8), where we substitute the expressions in (25) for the periods and then take the largest possible root that still leaves the η_i invariant under $\alpha'_j \mapsto \alpha'_j + 2\pi$.

The periods of the C-field are interpreted as the phases of the holomorphic observables. Due to (7), we have

$$\eta_1^p \eta_2^N \eta_3^r = \exp\left(i \sum_j \alpha'_j\right). \tag{26}$$

The η_j have zeros or poles at the semiclassical points $X_{i,\Gamma}$ with large r_0 in which the f_j diverge. As in Section 2, classically at the point $X_{1,\Gamma}$, $\eta_1 = 1$ and $\alpha'_1 = 0$. Hence, at this point

$$\eta_2^N \eta_3^r = \exp\bigl(i(\alpha'_2 + \alpha'_3)\bigr), \tag{27}$$

so when η_2 has a pole, η_3 has a zero and vice versa. In fact, the order of the zeros or poles of η_2 must be a multiple of r, and similarly the order of the zeros or poles of η_3 must be a multiple of N for this equation to hold. In the classical approximation, there are three branches $\mathcal{N}_i$ of the moduli space, on which we have $\eta_i = 1$ and $\eta_{i\pm1}$ obeying the relation (27) for $i = 1$ or cyclic permutations of it for $i = 2, 3$.

4.3. Quantum curve via membrane instantons

To study the quantum curve, we study the singularities, i.e., the zeros and poles of the holomorphic observables η_j, which correspond to the classical points $X_{i,\Gamma}$ with $r_0 \to \infty$.

We shall use a relation between the η_j and the amplitude for membrane instantons which wrap on supersymmetric cycles Q in X. Using chiral symmetry breaking of the low energy gauge theories, we find a clear relation between the local parameter on the moduli space and our observables, and hence can describe the moduli space.

A supersymmetric cycle in $X_{i,\Gamma}$ is given by the 3-manifolds Q_i given by $g_i = 0$:

$$Q_1 = \mathbf{S}^3/(\Gamma_2 \times \mathbf{Z}_r), \tag{28}$$

$$Q_2 = \mathbf{S}^3/(\mathbf{Z}_r \times \mathbf{Z}_p), \tag{29}$$

$$Q_3 = \mathbf{S}^3/(\mathbf{Z}_p \times \Gamma_2). \tag{30}$$

At $X_{1,\Gamma}$, Q_1 is homologous (up to orientation) to the D'_j as follows:

$$rQ_1 \sim D'_2, \tag{31}$$

$$NQ_1 \sim D'_3, \tag{32}$$

and cyclic permutations of that give the relations at $X_{2,\Gamma}$ to be $pQ_2 \sim D'_3$ and $rQ_2 \sim D'_1$, and at $X_{3,\Gamma}$ we have $NQ_3 \sim D'_1$ and $pQ_3 \sim D'_2$.

We now study the zeros and poles of the η_j. To understand the orders of the zeros and poles, we must compare the η_j to the true local parameter on $\mathcal{N}_\Gamma$ around each $X_{i,\Gamma}$ with large r_0.

One would expect at first that the membrane instanton amplitude u itself, given near $X_{i,\Gamma}$ by

$$u = \exp\left(-TV(Q_i) + i\int_{Q_i} C\right), \tag{33}$$

where T is the membrane tension and $V(Q_i)$ is the volume of Q_i, would be a good local parameter near $X_{i,\Gamma}$. However, at low energies we have a supersymmetric A, D, or E gauge theory in four dimensions, and due to chiral symmetry breaking, we expect the good local parameter—the gluino condensate—to be $u^{1/h}$ where h is the dual Coxeter number of the gauge group.

We now compare phases of the η_j to the phase of u. Let $P_{i,\Gamma}$ correspond to the manifolds $X_{i,\Gamma}$ with large r_0. For the case where $\Gamma_2 = \mathbf{Z}_q$, at $P_{1,\Gamma}$ Eq. (31) implies that the phase $\int_{D'_2} C$ of η_2 is related to the phase $\int_{Q_1} C$ of u by $\int_{D'_2} C \sim r\int_{Q_1} C$. Since the good local parameter is actually $u^{1/p}$ due to chiral symmetry breaking of the $SU(p)$ gauge theory at $P_{1,\Gamma}$, the true order of the zero of η_2 at $P_{1,\Gamma}$ is pr. The same calculation for the other η_j and $P_{i,\Gamma}$ gives the orders of zeros and poles shown in Table 2.

Table 2
Behavior of η_i for $\Gamma_2 = \mathbf{Z}_q$

$\Gamma_2 = \mathbf{Z}_q$	$P_{1,\Gamma}$	$P_{2,\Gamma}$	$P_{3,\Gamma}$
η_1	1	∞^{qr}	0^{qr}
η_2	0^{pr}	1	∞^{pr}
η_3	∞^{pq}	0^{pq}	1

Table 3
Behavior of η_i for $\Gamma_2 = \mathbf{D}_q$

$\Gamma_2 = \mathbf{D}_q$	$P_{1,\Gamma}$	$P_{2,\Gamma}$	$P_{2',\Gamma}$	$P_{3,\Gamma}$
η_1	1	∞^{rh}	$\infty^{2rh'}$	0^{rN}
η_2	0^{rp}	1	-1	∞^{rp}
η_3	∞^{Np}	0^{hp}	$0^{2h'p}$	1

Table 4
Behavior of η_i for $\Gamma_2 = \mathbf{E}_a$

$\Gamma_2 = \mathbf{E}_a$	$P_{1,\Gamma}$	$P_{2,\Gamma}$	$P_{\mu t,\Gamma}$	$P_{3,\Gamma}$
η_1	1	∞^{rh}	∞^{rth_t}	0^{rN}
η_2	0^{rp}	1	$e^{2\pi i\mu/t}$	∞^{rp}
η_3	∞^{Np}	0^{hp}	0^{pth_t}	1

The cases where Γ_2 is a D or E group give similar tables, except that in these cases we get extra semiclassical points in the same way as in [4]: for the case $\Gamma_2 = \mathbf{D}_q$, we have Table 3, where $h = 2q - 2$, $h' = q - 3$, and $h + 2h' = N$. The low energy gauge theory at $P_{2',\Gamma}$ has gauge group $Sp(q - 4)$.

For Γ_2 in the E series, we have Table 4, where t, h_t, μ are given for each $\mathbf{E}_a$ as follows: let k_i be the Dynkin indices of $\mathbf{E}_a$, and let t be the positive integers which divide some of the k_i; μ runs over positive integers less than t that are prime to t, unless $t = 1$ in which case $\mu = 0$; h_t is the dual Coxeter number of the associated group K_t whose Dynkin indices are k_i/t where here the k_i run through the indices of $\mathbf{E}_a$ that divide t. The t and h_t obey the relation $\sum th_t = N$. The low energy gauge group is given by the *ADE* group corresponding to K_t.

From the relation $\sum th_t = N$ and Tables 2–4, we see that for each η_j, the total number of zeros and poles is equal. Since the total number of zeros is the same as the total number of poles for each of the η_j, it seems reasonable to assume that we have found all the zeros and poles, and hence all the semiclassical limits in our moduli space. It would seem, therefore, that we can now proceed to describe the moduli space completely, by writing our functions η_i explicitly and identifying the points $P_{i,\Gamma}$ with values of a good coordinate on the moduli space. However, as we shall see, we run into a few puzzles.

The first question we ask is: what can be said about the genus of $\mathcal{N}_\Gamma$? For the cases $p = r = 1$, which are the cases considered in [4], the function η_2 has a simple zero and a simple pole, and hence can be identified with a global coordinate on the moduli space, which can then be claimed to have genus zero. If $p, r > 1$, this is not so: none of our η_j have just a simple zero and pole, so we cannot identify the moduli space with any of the η_j, and we do not know the genus.

However, the simplest result would be that the curve has genus zero, and we proceed with this assumption. Hence, we assign the curve a global coordinate z, write the η_j as holomorphic functions of z, and see how well we can describe the curve.

For the case $\Gamma_2 = \mathbf{Z}_q$, this turns out to be straightforward; we may fix $P_{1,\Gamma}$ at $z = 0$, $P_{2,\Gamma}$ at $z = 1$, and $P_{3,\Gamma}$ at $z = \infty$, and then write our functions:

$$\eta_1 = \frac{1}{(1-z)^{qr}}, \tag{34}$$

$$\eta_2 = z^{pr}, \tag{35}$$

$$\eta_3 = \frac{(1-z)^{pq}}{z^{pq}}. \tag{36}$$

This description is unique up to possible overall factors which are related to an anomaly in the membrane effective action, analogous to the one described in Section 5 of [4].

For $\Gamma_2 = \mathbf{D}_q$, we run into a puzzle. Once we fix the first three points, we have to find at what value z_4 the fourth point $P_{2',\Gamma}$ sits: our functions in this case are

$$\eta_1 = \frac{z_4^{2rh'}}{(1-z)^{rh}(z_4-z)^{2rh'}}, \tag{37}$$

$$\eta_2 = z^{rp}, \tag{38}$$

$$\eta_3 = \frac{(1-z)^{ph}(z_4-z)^{2h'p}}{z^{Np}}, \tag{39}$$

again up to overall factors. The forms of η_1 and η_3 do not constrain z_4, but to satisfy $\eta_2(z_4) = -1$, we need $z_4^{pr} = -1$ for which there are pr solutions. A similar situation arises for Γ_2 in the E series, where there are pr choices for each point beyond the first three.

The pr solutions, however, should correspond to the same point in the moduli space of M-theories, since they correspond to the same theory. Hence, it seems that we have a redundancy in our description of the moduli space; we should impose a symmetry on $\mathcal{N}_\Gamma$ which identifies the different values of z_4.

There is another, more serious puzzle which shows up, also involving possible extra classical points on $\mathcal{N}_\Gamma$: from Table 1, we see that our low energy gauge theory is compactified on a manifold which is not simply connected, but rather is of the form $\mathbf{S}^3/H$ for some discrete group H. Hence its fundamental group is equal to H. Therefore, it is possible to construct theories which have gauge fields with nontrivial Wilson loops which break the gauge symmetry. Where in $\mathcal{N}_\Gamma$ do these theories lie?

For the case $\Gamma_2 = \mathbf{Z}_q$, the point $P_{1,\Gamma}$ can have Wilson loops which are conjugacy classes of elements of $SU(p)$ of order qr. One can show that, when p, q, r are relatively prime, the number of inequivalent such elements is

$$\frac{1}{p}\binom{p+qr-1}{qr-1} = \frac{(p+qr-1)!}{p!(qr)!} \tag{40}$$

with cyclic permutations for $P_{2,\Gamma}$ and $P_{3,\Gamma}$. Furthermore, for Wilson loops that break $SU(p)$ in a way that leaves $s-1$ factors of $U(1)$, i.e.,

$$SU(p) \longrightarrow \prod_{i=1}^{s} SU(n_1) \times U(1)^{s-1},$$

where $\sum n_i = p$, the number of inequivalent Wilson loops is

$$\frac{s}{pqr}\binom{p}{s}\binom{qr}{s}. \tag{41}$$

Each set of theories with a given number $s-1$ of $U(1)$ factors should lie on a separate component $\mathcal{N}_{s,\Gamma}$ of the moduli space, since smooth interpolation means that the number of massless modes—which corresponds to $U(1)$ fields—is constant on each component. For $s>1$, we know that the theories on $\mathcal{N}_{s,\Gamma}$ do not have a mass gap due to the massless $U(1)$ field. On the other hand, the theories corresponding to the points $P_{i,\Gamma}$ with no nontrivial Wilson loops are believed to have a mass gap. Hence we claim that $\mathcal{N}_{1,\Gamma}$ contains theories with a mass gap.

For the case $r=1$, we obtain no singularity at $X_{3,\Gamma}$. Hence, for that case the mass gap of the theory at $X_{3,\Gamma}$ means that all of $\mathcal{N}_{1,\Gamma}$ has a mass gap.

Continuing with the case where $r=1$, we note a manifest symmetry between p and q in the expression (41) for the number of possible Wilson loops at each level s. At first sight, this could support the assertion that these points lie on their own branch of the moduli space, which will interpolate smoothly among them and contain no other singular points. However, chiral symmetry breaking means that the number of vacua at each classical point is given by $\prod n_i$ which is clearly not symmetric between p and q, and spoils the counting of the orders of zeros and poles.

Going back to general r and looking at $\mathcal{N}_{1,\Gamma}$ only, we see that we have smooth interpolation among theories with different gauge groups: $SU(p)$, $SU(q)$, and $SU(r)$ when $\Gamma_2=\mathbf{Z}_q$; $SU(p)$, $SO(2q)$, $Sp(q-4)$, and $SU(r)$ when $\Gamma_2=\mathbf{D}_q$; and analogously for Γ_2 in the E series, where we interpolate between $SU(p)$, K_t, and $SU(r)$, with K_t as described before. Similarly, the other branches $\mathcal{N}_{s,\Gamma}$ smoothly interpolate among theories with these gauge groups broken by Wilson lines.

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