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Conformal or near conformal Quantum Field Theories QFT) would benefit from a rigorous non-perturbative lattice formulation beyond the flat Euclidean space, $\mathbb{R}^d$. Although all UV complete QFT are generally acknowledged to be perturbatively renormalizable on smooth Riemann manifolds, non-perturbative realization on simplicial lattices (triangulation) encounter difficulties as the UV cut-off is removed. We review the Quantum Finite Element (QFE) method that combines classical Finite Element with new quantum counter terms designed to address this. The construction for maximally symmetric spaces ($\mathbb{S}^d$, $\mathbb{R}\times\mathbb{S}^{d-1}$ and $\mathbb{A}dS^{d+1}$) is outlined with numerical tests on $\mathbb{R}\times\mathbb{S}^2$ and a description of theoretical and algorithmic challenges for d = 3, 4 QFTs.

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1. Introduction

Lattice Field Theory (LFT) on regular, most often hypercubic, lattices has proven to be a powerful method to extract *ab initio* predictions of non-perturbative field theory – most notably for Quantum Chromodynamics (QCD). The numerical methods are generally implemented on a flat toroidal Euclidean lattice with fixed lattice spacing, a, providing the UV cut-off $\Lambda = \pi/a$. The discrete rotation and translation symmetries form a subgroup of the Poincare group sufficient to implement Wilsonian renormalization of quantum field theory (QFT) at a second order phase boundary with a minimum of relevant tuning parameters. Indeed for many perturbative renormalized field theories, QCD in particular, the lattice provides the only definition of the full non-perturbative QFT.

The goal of the Quantum Finite Elements (QFE) is to extend lattice field theory to Riemann manifolds beyond flat space. While this is a technically difficult task, such an extension is plausible based on the literature [1–3] which generalizes the renormalization of the Feynman perturbation expansion to smooth Euclidean Riemann manifolds. The challenge is to find a rigorous non-perturbative lattice framework. Beyond this theoretical goal, there are important aspects of non-perturbative quantum field theories that are best understood on curved space-time manifolds.

One target applications is to Conformal Field Theory (CFT) mapped by Weyl transform from Euclidean $\mathbb{R}^d$ to the Riemann sphere, $\mathbb{S}^d$, where the free energy gives direct access to the central charge [4]. Another CFT application is radial quantization on a cylinder [5], $\mathbb{R}\times\mathbb{S}^{d-1}$, where "time" translations are generated by the dilatation operator, giving direct access to conformal dimensions and the conformal partial wave expansion [6]. In addition small mass deformations in multi-flavor gauge theories give weakly broken CFTs to probe possible new physics beyond the standard model (BSM) with composite Higgs and/or dark matter. Finally lattices for the AdS^{d+1} manifold can provide new framework [7, 8] to study the AdS/CFT conjectures [9] as well as non-perturbative aspects of quantum gravity.

There are two steps to the QFE lattice construction. First a sequence of discrete approximations to both the base Riemann manifold and the quantum field is introduced designed to converge to the **classical** field theory in the continuum limit. This talk focuses on this first step which is largely solved by leveraging techniques from the finite element method (FEM) and discrete exterior calculus (DEC). We have extended these methods sufficient to implement the lattice action with interacting scalar, non-Abelian gauge and Dirac fermions fields. A careful refinement for a sequence of simplicial complexes (2D triangles, 2D tetrahedrons, etc.), with its Voronoi dual, appears to guarantee convergence to all solution to the continuum Euler Lagrange equation of motion (EOM).

The second and more difficult step is unique to quantum field theory. Due to the local variation in the ultraviolet cut-off (e.g lattice spacing) on a simplicial complex, the QFE prescription must include the addition of local counter terms to compensate to this local *scheme dependence* in order converge to the quantum field theory on the continuum manifold as the cut-off is removed [1]. By evaluating the UV divergent lattice perturbation diagrams we believe in principle perturbative

[1]There is possible alternative solution by replacing our fixed sequence of maximally *smooth* simplicial lattices by a quenched randomized ensemble constrained on average to the target manifold. We prefer to avoid the complexity of a double Monte Carlo sampling in both metric (base) and field (fiber) space, unless and until we wishes to explore the dynamical interaction with quantum gravity in the spirit of Regge calculus.

counter terms are sufficient for renormalizable QFT. At present our numerical tests are restricted to ϕ^4 theory on $\mathbb{S}^2$ for the 2D Ising CFT and on $\mathbb{R}\times S^2$ for the 3D Ising model. They are examples of maximally symmetric spaces so that UV divergences are uniform. The construction of counter terms is an ongoing project both for the existence of local QFE counter terms and efficient algorithms to define them. This is discussed briefly in the companion article by Evan Owen [10].

2. Classical Field Theory on Riemann Manifold

At the classical level we introduce a sequence of d-dimensional simplicial Delaunay complex $\mathbb{C}^d$ and the circumcenter dual Voronoi complex $\mathbb{C}^*$. Here we give a rough description of this elegant and crucial Finite Element Method (FEM) referring for details to our earlier QFE publications [11–13]. It starts with introducing a piece-wise linear interpolation of the metric $g_{\mu\nu}(x)$ field identifying each site i with a co-ordinate x^μ_i and edge lengths $l_{ij} = |x_i - x_i| + O(a^2)$. This guarantees the topology of the complex conforms the Riemann manifold upon refinement approaching the continuum metric in zero lattice spacing a limit to $O(a^2)$. Unlike Regge calculus (RC) and random lattice methods, we do not average over an ensemble of near by complexes.

A d-dimensional simplicial complex $\mathbb{C}_d$ is built out of a sequence elementary n-dimensional cells $\sigma_n(i_0, i_1, \cdots i_n)$ (sites, edges, triangle, tetrahedrons etc.) for $n = 0, 1, .., d$ with volume $|\sigma_n|$ and associated $n-d$ dimensional dual polytopes σ^*_n. Orthogonality to the dual lattice $\mathbb{C}^*_d$ at circumcenters implies the volume of the hybrid cells

$$|\sigma_n \wedge \sigma^*_n| = \frac{n!(d-n)!}{d!}|\sigma_n||\sigma^*_n| \ . \tag{1}$$

is a proper tessellation. It is crucial to realize this simplicial complex and its dual preserve the algebraic structure of the continuum differential geometry under the banner of Category Theory. On this lattice scaffolding, we next introduce scalar (on sites), vector (on links) tensor (on triangles) and Dirac fields which for example obey discrete analogues of stokes theorem relating exterior derivatives to boundary forms etc.

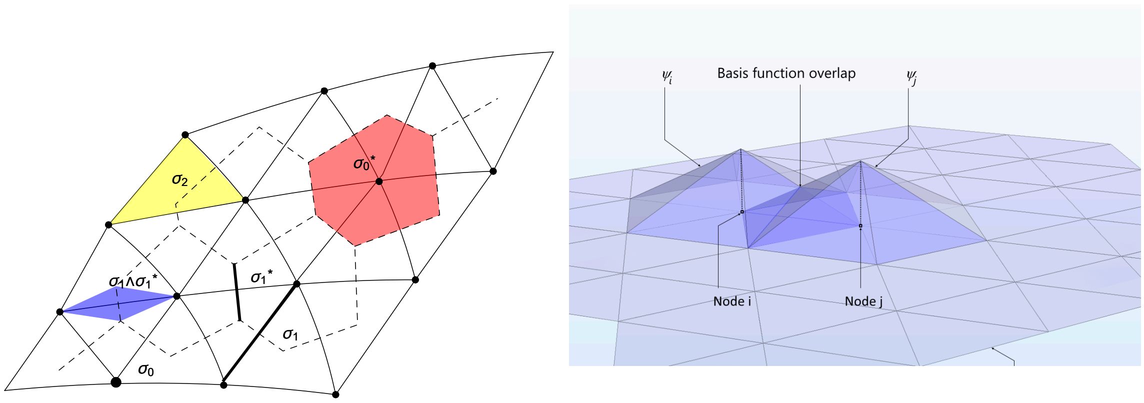

Figure 1: On the left a 2D simplicial complex on the base manifold and on the right the linear finite element basis.

The simplest example is a single scalar field illustrated by ϕ^4 theory. The FEM rule on 2D triangulated planes is well known. For example classical FEM action for ϕ^4 theory on $\mathbb{R} \times \mathbb{S}^2$ has the FEM action in physical units relative to the radius, R, of the sphere,

$$S_{FEM} = \frac{a_t}{2}\Big[\sum_{y\in\langle x,y\rangle} \frac{l^*_{xy}}{l_{xy}}(\phi_{t,x} - \phi_{t,y})^2 + \frac{\sqrt{g_x}}{4R^2}\phi^2_{t,x} + \sqrt{g_x}\Big[\frac{(\phi_{t,x} - \phi_{t+1,x})^2}{a_t^2} + m_0^2\phi^2_{t,x} + \lambda_0\phi^4_{t,x}\Big]\Big] ,$$

with the Einstein summation convention for $x = 1, 2, \cdots, N$ for sites on each sphere and $t = 1, 2, \cdots, L_t$ along the length of the cylinder. Relative to Eq.1 on the 2d sphere we have used the notation: $\sqrt{g_x} = |\sigma_0^*(x)|$, $l_{xy} = |\sigma_1(xy)|$ and $l^*_{xy} = |\sigma_1^*(xy)|$. Note that the weights for FEM kinetic term is the ratio of the length of the dual link connecting circumcenter on the dual lattice and the triangle length: $\langle x, y\rangle$ links l^*_{xy}/l_{xy}. In 2d this is equivalent to piecewise linear finite elements, illustrated in Fig.1. A some what informal summary of these lattice fields are

$$\begin{array}{lll}
J = 0 & S_{scalar} = \dfrac{1}{2}\displaystyle\sum_{\langle i,j\rangle}\frac{V^*_{ij}}{l^2_{ij}}(\phi_i - \phi_j)^2 \quad , & V^*_{ij} = |\sigma_1^*(ij)\wedge\sigma_1(ij)| \\
J = 1/2 & S_{Wilson} = \dfrac{1}{2}\displaystyle\sum_{\langle i,j\rangle}\frac{V^*_{ij}}{l_{ij}}(\bar\psi_i\hat e^{j(i)}_a\gamma^a\Omega_{ij}\psi_j - \bar\psi_j\Omega_{ji}\hat e^{i(j)}_a\gamma^a\psi_j) + \text{Wilson Term} & \\
J = 1 & S_{gauge} = \dfrac{1}{2g^2N_c}\displaystyle\sum_{\triangle_{ijk}}\frac{V^*_{ijk}}{A^2_{ijk}}Tr[2 - U_{\triangle_{ijk}} - U^\dagger_{\triangle_{ijk}}] & \\
\text{FFdual} & S_{F\tilde F} = \dfrac{i\theta}{N_c}\displaystyle\sum_{\langle nijkl\rangle}\frac{V^*_{nijkl}}{A_{nij}A_{nkl}}\epsilon^{ijkl}Tr[U_{\triangle_{nij}}U_{\triangle_{nkl}}] &
\end{array} \tag{2}$$

where we define $l_{ij} = |\sigma_1(ij)|$ and $A_{ijk} = |\sigma_2(ijk)|$ on $\mathbb{C}$, and the hybrid volumes, $V^*_{ij} = |\sigma_1(ij)\wedge\sigma_1^*(ij)|$, $V^*_{ijk} = |\sigma_2(ijk)\wedge\sigma_2^*(ijk)|$ and $V^*_{nijkl} = |\sigma_4(nijkl)|$. Together these components allows the construction at the classical level field theories with scalars, Fermions and gauge theories. For example as we note in Ref. [12], the scalar kinetic term is a DEC representation of the Beltrami Laplace operator: $*d*d\phi \to \sqrt{g}\partial_\mu g^{\mu\nu}\sqrt{\partial}_\nu\phi(x)$. The transform between lattice and dual lattice is the simplicial Hodge star operator $*$. However we note for $d > 2$ this is **not** in fact given by piece-wise liner elements but follows the elegant DEC method, which in flat space was formulated in a classic paper by by Christ, Friedberg and Lee[14] for scalars and non-Abelian gauge fields.

On curved space the Fermion field is more subtle. The Kahler Dirac fermion (or it gauged form called staggered fermions) can be implemented by DEC but the Wilson lattice Dirac field required a novel solution given in Ref. [11]. In continuum the Dirac field action is

$$S = \frac{1}{2}\int d^dx\sqrt{g}\bar\psi(x)[\mathbf{e}^\mu(x)(\partial_\mu - \frac{i}{4}\omega_\mu(x))]\psi(x) \tag{3}$$

where the verbine $\mathbf{e}^\mu(x) = e^\mu_a(x)\gamma^a$ relates to a local co-ordinate on the tangent plane at x to the spin direction $\omega_\mu(x) \equiv \omega^{ab}_\mu\sigma_{ab}$ for the covariant derivative between tangent planes. The key problem is the proper introduction of lattice verbein on tangent planes at each site and lattice spin connection. In Eq. 2 for S_{Wilson} , $e^{j(i)}$ is the lattice verbein at site i on link connecting to j and Ω_{ij} is the lattice spin connection for the Riemann curvature. They satisfy a simplicial Tetrad identity, $e^{j(i)}_a\gamma^a\Omega_{ij} + \Omega_{ji}\hat e^{i(j)}_a\gamma^a = 0$, which guarantees gauge invariance for co-ordinate rotation on each tangent plane – perfect analogue of the Wilson Lattice gauge links U_{ij}. Indeed coupling to gauge links is accomplished by the substitution: $\Omega_{ij} \to \Omega_{ij} U_{ij}$. The addition of the Wilson term and the application this Wilson kernel to domain wall fermions is straight forward.

3. Test Case of 3D ϕ^4 theory

Recently QFE has been applied [13] to the 3D critical ϕ^4 CFT on $\mathbb{R} \times \mathbb{S}^2$ and tested against accurate results form the conformal bootstrap [15]. One must add to the classical FEM action above in Eq.2 a counter term defined earlier in Ref. [12] for the linear divergent one loop term and the logarithmically divergent two loop term.

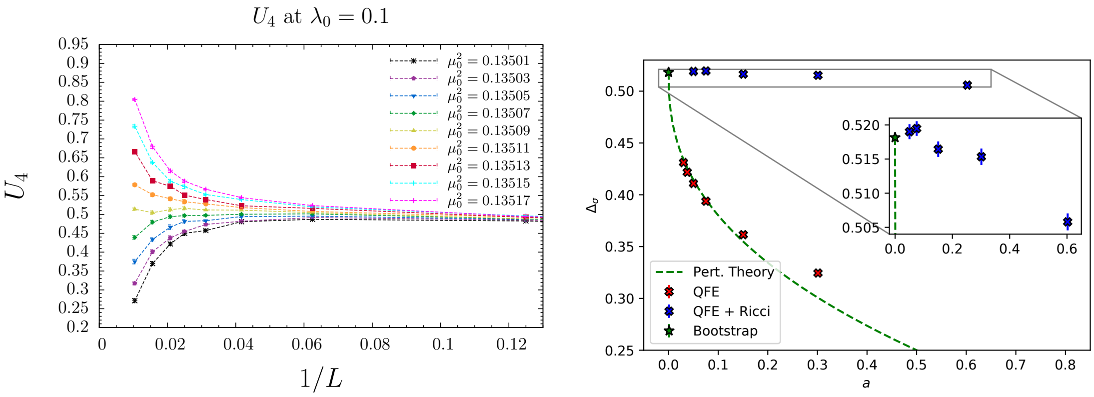

Figure 2: On the left, the Binder cumulant tuning to the critical surface in mass $\mu_0^2 = -m_0^2$ at fixed $\lambda_0 = 0.1$. On the right, the Monte Carlo data from Ref. [13] for 3D ϕ^4 on $\mathbb{R} \times \mathbb{S}^2$. The plot shows the importance of including the Ricci term to rapidly converge to the continuum bootstrap [15] value for $\Delta_\sigma = 0.5181$ at zero lattice spacing ($a = 0$).

Surprisingly it was discovered that rapid convergence to the continuum required the inclusion of the Ricci curvature term as illustrated in Fig. 2 . While this term is technically irrelevant on the critical surface, it is very important for high precision convergence to the continuum value ($\Delta_\sigma = 0.5181$) given by the bootstrap [15]. Also as discussed in the presentation by Evan Owen [10], we have varied the bare coupling and found that the counter terms may have significant non-perturbative corrections. While these corrections might be useful to improve the convergence to the continuum, we note that they are not required. The proper limit to the continuum field theory is to hold the dimensional renormalized coupling fixed: $\lambda_R = O(\lambda_0/a)$ approaching the critical surface at $m_R = O(m_0/a)$. Consequently in the continuum limit, $\lambda_0 \to 0$ and perturbative counter terms are sufficient.

4. Future direction and Challenges.

While the basic premise of QFE lattice theory seems to be working, there is much more to do to verify this and develop methods for relativistic QFT and bulk $\mathbb{A}dS^{d+1}$ to boundary $\mathbb{R}\times\mathbb{S}^d$ CFTs for d = 3 and 4. We are pursuing this step by step. Our next target is to introduce the simplicial geometry on $\mathbb{S}^3$ to repeat the 3D Ising model on Riemann sphere [16] and to study 4d gauge theories on $\mathbb{R}\times\mathbb{S}^3$.

There are two barriers to overcome to go to 4d. First the perturbative construction of counter terms, while likely feasible in principle, does require considerable analytical and computational effort, particularly when including fermions. Next, and equally challenging, is to construct sufficiently efficient parallel Monte Carlo codes. However, it should be noted that the design of simplicial lattices for QFE these applications has intentionally focused on highly regular tessellations of a few maximally symmetric manifolds that at the local level share the regularity found in hypercubic discretization of $\mathbb{R}^d$. For example as illustrated in Fig.3 for $\mathbb{S}^2$ starting from the **icosahedron** the refinement data structure on the surface is embedded in square and cubic lattice respectively. A similar strategy is being pursued starting for $\mathbb{S}^3$ refining the remarkably similar **hexacosichoron** composed of 600 hundred regular tetrahedrons the unit $\mathbb{S}^3$. Both have few domains (20 and 600 respectively) friendly to typical data parallel high performance lattice codes. continuum.

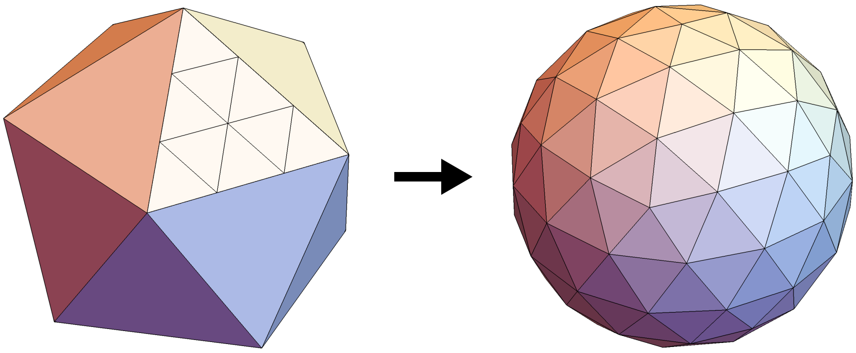

Figure 3: On the left, the Icosahedron is refine with 20 faces into a triangular graph allowing for simple data parallel domains in application to $\mathbb{S}^2$ and $\mathbb{R} \times \mathbb{S}^2$. On the right, the construction of the metric for simplicial manifold of the sphere, $\mathbb{S}^2$, is given by projecting equilateral triangular refinements of an icosahedron onto the surface of a unit sphere.

In principle the full array of sophisticated multiscale solvers and Hybrid Monte Carlo (HMC) algorithms should sit in a higher level opaque software layer with little explicit reference to the new simplicial data structures of the base lattice. The hope is that this will enable the bulk of the data parallel software components of Exascale QCD to be re-factored for the QFE applications to 4D Gauge theories of interest to the BSM studies. Not an easy task but one that appears feasible.

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