

---

9-1-2007

## Paving Hessenberg Varieties by Affines

Julianna S. Tymoczko

*University of Iowa*, [jtymoczko@smith.edu](mailto:jtymoczko@smith.edu)

Follow this and additional works at: [https://scholarworks.smith.edu/mth\\_facpubs](https://scholarworks.smith.edu/mth_facpubs)



Part of the [Mathematics Commons](#)

---

### Recommended Citation

Tymoczko, Julianna S., "Paving Hessenberg Varieties by Affines" (2007). Mathematics and Statistics: Faculty Publications, Smith College, Northampton, MA.  
[https://scholarworks.smith.edu/mth\\_facpubs/109](https://scholarworks.smith.edu/mth_facpubs/109)

This Article has been accepted for inclusion in Mathematics and Statistics: Faculty Publications by an authorized administrator of Smith ScholarWorks. For more information, please contact [scholarworks@smith.edu](mailto:scholarworks@smith.edu)

## PAVING HESSENBERG VARIETIES BY AFFINES

JULIANNA S. TYMOCZKO

ABSTRACT. Regular nilpotent Hessenberg varieties form a family of subvarieties of the flag variety arising in the study of quantum cohomology, geometric representation theory, and numerical analysis. In this paper we construct a paving by affines of regular nilpotent Hessenberg varieties for all classical types, generalizing results of de Concini-Lusztig-Procesi and Kostant. This paving is in fact the intersection of a particular Bruhat decomposition with the Hessenberg variety. The nonempty cells of the paving and their dimensions are identified by combinatorial conditions on roots. We use the paving to prove these Hessenberg varieties have no odd-dimensional homology.

## 1. INTRODUCTION

This paper studies the topology of regular nilpotent Hessenberg varieties, a family of subvarieties of the flag variety introduced in [dMPS] that arise naturally in contexts as diverse as numerical analysis, number theory, and representation theory. Their geometry encodes deep algebraic and combinatorial properties, including the quantum cohomology of the flag variety [Ko]. We prove that regular nilpotent Hessenberg varieties in classical Lie types have a *paving by affines*, a cell decomposition like CW-decompositions but with weaker closure relations. This paving permits us to describe the varieties' cohomology, for instance to show that it vanishes in odd dimensions. Moreover, this paving can be realized as the intersection of the Hessenberg variety with a particular Bruhat decomposition, so the dimensions of the nonempty cells are characterized by combinatorial conditions.

Let  $G$  be a complex linear algebraic group of classical type,  $B$  a fixed Borel subgroup, and  $\mathfrak{g}$  and  $\mathfrak{b}$  their Lie algebras. A Hessenberg space  $H$  is a linear subspace of  $\mathfrak{g}$  that contains  $\mathfrak{b}$  and that is closed under Lie bracket with  $\mathfrak{b}$ , namely  $[H, \mathfrak{b}]$  is contained in  $H$ . Fix an element  $X$  in  $\mathfrak{g}$  and a Hessenberg space  $H$ . The Hessenberg variety  $\mathcal{H}(X, H)$  is the subvariety of the flag variety  $G/B$  consisting of  $gB/B$  satisfying  $g^{-1}Xg \in H$ , or equivalently  $\text{Ad } g^{-1}(X) \in H$ .

An important special case is when  $G = GL_n(\mathbb{C})$ ,  $B$  consists of the upper-triangular invertible matrices,  $\mathfrak{g}$  is the set of all  $n \times n$  matrices, and  $\mathfrak{b} \subseteq \mathfrak{g}$  the subset of all upper-triangular matrices. In this case, a Hessenberg space  $H$  is equivalent to a nondecreasing function  $h : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$  satisfying  $h(i) \geq i$  for all  $i$ , by the rule that  $H$  is a subspace of  $\mathfrak{g}$  whose matrices vanish in positions  $(i, j)$

---

2000 *Mathematics Subject Classification.* 14M15, 14F25, 14L35.

*Key words and phrases.* Hessenberg varieties, paving, Bruhat decomposition.

whenever  $i > h(j)$ . The flags in  $GL_n(\mathbb{C})/B$  can be written as nested subspaces  $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{n-1} \subseteq \mathbb{C}^n$  where each  $V_i$  is  $i$ -dimensional. The Hessenberg variety  $\mathcal{H}(X, H)$  is the collection of flags for which  $XV_i \subseteq V_{h(i)}$  for each  $i$ .

Let  $N$  be a regular nilpotent element of  $\mathfrak{g}$ , namely let  $N$  be in the dense adjoint orbit within the nilpotent elements in  $\mathfrak{g}$ . When  $G = GL_n(\mathbb{C})$ , the regular nilpotent elements are those which consist of a single Jordan block. In this paper we prove the existence of a paving by affines for regular nilpotent Hessenberg varieties. Pavings, defined formally in Section 2.1, are like CW-decompositions but with weaker closure relations. The paving in this paper can be described explicitly.

**Theorem.** *Fix a Hessenberg space  $H$  with respect to the Borel  $\mathfrak{b}$  and fix a regular nilpotent element  $N$  in  $\mathfrak{b}$ . The Bruhat decomposition  $BwB/B$  of the flag variety intersects the Hessenberg variety  $\mathcal{H}(N, H)$  in a paving by affine cells. The cell  $P_w = \mathcal{H}(N, H) \cap BwB/B$  corresponding to  $w$  is nonempty if and only if  $\text{Ad } w^{-1}(E_{\alpha_j})$  is in  $H$  for each simple root vector  $E_{\alpha_j}$ . If  $P_w$  is nonempty its dimension is given by  $\dim(\mathfrak{b} \cap \text{Ad } w(\mathfrak{b}^- \cap H)) - \text{rank}(G)$ , where  $\mathfrak{b}^-$  denotes the opposite Borel.*

Theorem 4.3 gives the complete statement of the paving result, including conditions on roots which determine when  $P_w$  is nonempty and, if so, its dimension. The arbitrary regular nilpotent Hessenberg variety  $\mathcal{H}(N, H)$  is also paved by affines (Corollary 4.5), since it is homeomorphic to an  $\mathcal{H}(N', H)$  satisfying the conditions of the Theorem. Using this paving, Corollary 4.6 proves regular nilpotent Hessenberg varieties have no odd-dimensional cohomology.

This result generalizes from type  $A_n$  some of the work of [T], where the reader may find explicit examples for  $GL_n(\mathbb{C})$ . It also extends results of [dCLP] beyond the Springer fiber, namely when  $H = \mathfrak{b}$  and  $X$  is arbitrary. (The cohomology of the Springer fiber carries a natural Weyl group action which geometrically constructs all the irreducible representations of the Weyl group; see [S], [BM], [L], and [CG], among others.) The Theorem strengthens work of B. Kostant [Ko] for the Peterson variety, i.e. when  $H$  is generated by  $\mathfrak{b}$  together with the root spaces corresponding to the negative simple roots. In [Ko], Kostant intersected the Peterson variety with a different Bruhat decomposition that paved it by affine varieties rather than the affine cells  $\mathbb{C}^k$  used here. (Kostant's paving gave an open dense subvariety of the Peterson variety with coordinate ring isomorphic to the quantum cohomology ring of the flag variety.) The topology of regular nilpotent Hessenberg varieties also gives information about algebraic invariants of ad-nilpotent ideals in a Borel subalgebra, which are closely related to Hessenberg spaces [ST]. This paper parallels the results for regular semisimple Hessenberg varieties in [dMPS] using a different approach. We understand D. Peterson has uncirculated results overlapping these [C], [BC, Theorem 3]; we infer that the methods used here are substantively different.

The second section has background information, including the definition of pavings, Bruhat decompositions, and the decomposition of the nilradical of  $\mathfrak{g}$  into subspaces called *rows*. The third section identifies the restriction and projection of the map  $\text{ad}N$  to the individual rows. The fourth uses these results to show that

the cells of this Bruhat decomposition intersect the Hessenberg variety  $\mathcal{H}(N, H)$  in an iterated tower of affine fiber bundles.

This paper was part of the author's doctoral dissertation. The author gratefully acknowledges the comments from and helpful discussions with Mark Goresky, David Kazhdan, Arun Ram, Konstanze Rietsch, and Eric Sommers, and especially those of the author's advisor, Robert MacPherson.

## 2. BACKGROUND AND DEFINITIONS

This section contains background and definitions needed from the literature for the rest of the paper. The first subsection recalls the necessary results about pavings. The second subsection reviews the Bruhat decomposition as well as parameterizations of each Schubert cell. The third subsection defines a partition of the positive roots into *rows*. These rows span subspaces of  $\mathfrak{b}$  which are abelian or Heisenberg, simplifying later computations.

**2.1. Pavings.** Pavings are common decompositions of algebraic varieties.

**Definition 2.1.** A paving of an algebraic variety  $X$  is an ordered partition into disjoint  $X_0, X_1, X_2, \dots$  so that each finite union  $\bigcup_{i=0}^j X_i$  is Zariski-closed in  $X$ .

The  $X_i$  are the cells of the paving. Note that pavings have weaker closure relations than CW-decompositions since the boundary of a cell is not required to be contained in cells of smaller dimension.

**Definition 2.2.** A paving by affines of  $X$  is a paving so that each  $X_i$  is homeomorphic to affine space.

The following is the main reason we use pavings [F, 19.1.11].

**Lemma 2.3.** *Let  $X = \bigcup X_i$  be a paving by a finite number of affines with each  $X_i$  homeomorphic to  $\mathbb{C}^{d_i}$ . The cohomology groups of  $X$  are given by  $H^{2k}(X) = \bigoplus_{i:d_i=k} \mathbb{Z}$ .*

**2.2. Bruhat decompositions.** Fix a Borel subgroup  $B$  and a maximal torus  $T$  in  $G$ , and let  $W$  denote the Weyl group of  $G$ , namely the quotient  $N(T)/T$  of the normalizer of  $T$ . The subgroup  $B$  determines a decomposition of the flag variety  $G/B$  into cosets  $BwB/B$  indexed by the elements  $w$  of the Weyl group  $W$ .

In fact, this is a classic paving by affines. Recall that the length of the element  $w$  is the minimal number of simple transpositions  $s_1, \dots, s_n$  required to write  $w = s_{i_1} \cdots s_{i_\ell(w)}$ . The next lemma is proven in [Ch], among others.

**Lemma 2.4.** *The cells  $BwB/B$  of the Bruhat decomposition form a paving by affines when ordered in any way subordinate to the partial order determined by the length of  $w$ .*

We use an explicit description of the affine cells of this paving. Let  $\Phi$  denote the roots of  $\mathfrak{g}$  and  $\Phi^+$  the roots corresponding to  $\mathfrak{b}$ . Recall the partial order on  $\Phi$  given by  $\alpha > \beta$  if and only if  $\alpha - \beta$  is a sum of positive roots. Write  $\mathfrak{g}_\alpha$  for the

root space corresponding to  $\alpha$ ,  $U$  for the maximal unipotent subgroup of  $B$ ,  $U^-$  for its opposite subgroup, and  $\mathfrak{n}$  for the Lie algebra of  $U$ .

**Lemma 2.5.** *Fix  $w$  in  $W$ . The following are homeomorphic:*

- (1) the Schubert cell  $BwB/B$ ;
- (2) the subgroup  $U_w = \{u \in U : w^{-1}uw \in U^-\}$ ;
- (3) the Lie subalgebra  $\mathfrak{n}_w = \bigoplus_{\alpha \in \Phi^+ : w^{-1}\alpha < 0} \mathfrak{g}_\alpha$ .

*Proof.* The subgroup  $U_w$  forms a set of coset representatives for  $BwB/B$  and is a product of root subgroups  $U_w = \prod_{w^{-1}\alpha < 0} U_\alpha$  for any fixed order of the roots [H, Theorems 28.3 and 28.4, Proposition 28.1]. Since  $\mathfrak{n}_w$  is nilpotent its image under the exponential map is  $\exp \mathfrak{n}_w = U_w$ , as in [K, page 50].  $\square$

Let  $\Phi_w = \{\alpha \in \Phi^+ : w^{-1}\alpha < 0\}$  be the set of roots indexing  $U_w$  and  $\mathfrak{n}_w$ .

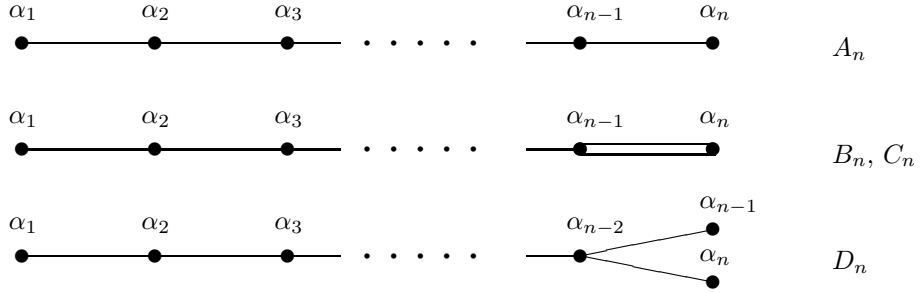
**2.3. Rows.** This subsection describes a partition of positive roots into *rows* which facilitates inductive proofs because each row generates an abelian or Heisenberg subalgebra of  $\mathfrak{g}$ . The subsection also includes a table enumerating the roots in each row of classical type. A version of this decomposition is used elsewhere, e.g. [Ste].

We are motivated by  $GL_n(\mathbb{C})$ , where the unipotent group  $U$  can be taken to be upper-triangular matrices with ones along the diagonal. In this case, the  $i^{\text{th}}$  row corresponds to the subgroup of  $U$  with nonzero entries only along the  $i^{\text{th}}$  row and the diagonal. Direct computation shows that this subgroup is abelian and that the product of the rows is  $U$ .

Write  $\alpha \geq \beta$  to indicate either  $\alpha > \beta$  or  $\alpha = \beta$ . The roots of the  $i^{\text{th}}$  row are

$$\Phi_i = \{\alpha \in \Phi^+ : \alpha \geq \alpha_i, \alpha \not\geq \alpha_j \text{ for each } j = 1, \dots, i-1\}.$$

The following labelling of the simple roots in classical types



gives the partition into rows of Table 1, which is used throughout this paper.

The rows generate subalgebras of the Lie algebra which we also call rows. We use  $\mathfrak{n}_i$  to denote the subalgebra spanned by the root spaces corresponding to the roots of  $\Phi_i$ , so

$$\mathfrak{n}_i = \bigoplus_{\alpha \in \Phi_i} \mathfrak{g}_\alpha.$$

Recall that  $\mathfrak{g}$  is an abelian Lie algebra if the derived algebra  $[\mathfrak{g}, \mathfrak{g}]$  is zero. We call  $\mathfrak{g}$  a Heisenberg Lie algebra if its lower central series  $\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0$

Type	Row $\Phi_i$
$A_n$	$\{\sum_{j=i}^k \alpha_j : i \leq k \leq n\}$
$B_n$	$\{\sum_{j=i}^k \alpha_j : i \leq k \leq n\} \cup \{\sum_{j=k}^n \alpha_j + \sum_{j=i}^n \alpha_j : i+1 \leq k \leq n\}$
$C_n$	$\{\sum_{j=i}^k \alpha_j : i \leq k \leq n\} \cup \{\sum_{j=k}^{n-1} \alpha_j + \sum_{j=i}^n \alpha_j : i \leq k \leq n-1\}$
$D_n$	$\{\sum_{j=i}^k \alpha_j : i \leq k \leq n-1\}$ $\cup \{\sum_{j=i}^{n-2} \alpha_j + \alpha_n + \sum_{j=k}^{n-1} \alpha_j : i+1 \leq k \leq n\}$

FIGURE 1. Rows in classical types

vanishes after two steps, if its derived algebra  $[\mathfrak{g}, \mathfrak{g}]$  is a one-dimensional subalgebra, and if for all  $X$  in  $\mathfrak{g}$  the map  $\text{ad}X$  surjects onto  $[\mathfrak{g}, \mathfrak{g}]$  unless  $X$  is in  $[\mathfrak{g}, \mathfrak{g}]$ . The next proposition gives a family of examples of Heisenberg algebras.

**Proposition 2.6.** *In type  $A_n$ ,  $B_n$ , or  $D_n$  each row  $\mathfrak{n}_i$  is abelian. In type  $C_n$  the row  $\mathfrak{n}_i$  is Heisenberg when  $i$  is not  $n$  and is abelian when  $i = n$ .*

*Proof.* Both proofs rely on the property that

$$(2.1) \quad [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \begin{cases} 0 & \text{if } \alpha + \beta \text{ is not a root, and} \\ \mathfrak{g}_{\alpha+\beta} & \text{if } \alpha + \beta \text{ is a root.} \end{cases}$$

By inspection of Figure 1 we see that for no choice of  $\alpha, \beta$  in  $\Phi_i$  in types  $A_n, B_n$ , and  $D_n$  is the sum  $\alpha + \beta$  a root. This implies that  $\mathfrak{n}_i$  is abelian. (The argument applies to  $\Phi_n = \{\alpha_n\}$  in type  $C_n$  as well.)

In type  $C_n$  let  $\gamma_i$  denote the root  $\sum_{j=i}^{n-1} 2\alpha_j + \alpha_n$ . If  $\alpha$  is any root in  $\Phi_i$  other than  $\gamma_i$  then the difference  $\gamma_i - \alpha$  is a root in  $\Phi_i$ . This is the Heisenberg property. Since  $\gamma_i + \gamma_i$  is not a root, the subalgebra  $\mathfrak{n}_i$  is Heisenberg.  $\square$

We define  $U_i$  to be the subgroup associated to  $\mathfrak{n}_i$ . The group  $U_i$  can be characterized either as the product  $\prod_{\alpha \in \Phi_i} U_\alpha$  or as the exponential  $\exp(\mathfrak{n}_i)$ .

**Proposition 2.7.** *The unipotent group  $U$  factors as the product  $U = U_1 U_2 \cdots U_n$ .*

*Proof.*  $U = \prod_{\alpha \in \Phi^+} U_\alpha$  for any fixed ordering of the positive roots [H, Proposition 28.1]. The rows are abelian possibly up to a root subgroup, which can be ordered last.  $\square$

The intersection of a Schubert cell with a row is  $(U_i \cap U_w)wB/B$  or equivalently  $\exp(\mathfrak{n}_i \cap \mathfrak{n}_w)wB/B$ .

### 3. ADJOINT ACTIONS ON ROWS

We now begin our study of adjoint actions on regular nilpotent elements. Fix a regular nilpotent  $N$  in  $\mathfrak{n}$  and a Hessenberg space  $H$  with respect to  $\mathfrak{b}$ . For each group element  $g$ , we will choose an appropriate  $u$  in  $U$  and reduce the problem of determining if  $\text{Ad } g^{-1}(N)$  is in  $H$  to the question of whether a summand of

$\text{Ad } u^{-1}(N)$  is in a fixed subspace of  $\mathfrak{n}_i$ . This will rely on the key fact that the adjoint representation of a row is “almost” linear, made precise in Proposition 3.1.

Fix a root vector  $E_\alpha$  to generate the root space  $\mathfrak{g}_\alpha$  and define  $m_{\alpha,\beta}$  by  $[E_\alpha, E_\beta] = m_{\alpha,\beta}E_{\alpha+\beta}$ . By Equation (2.1), the coefficient  $m_{\alpha,\beta}$  is nonzero if and only if  $\alpha + \beta$  is a root. The set  $\{E_\alpha : \alpha \in \Phi^+\}$  form a basis for the Lie algebra  $\mathfrak{n}$ . We refer to the expansion  $Y = \sum y_\alpha E_\alpha$  as the basis vector expansion of  $Y$ .

Let  $\rho_i : \mathfrak{n} \rightarrow \mathfrak{n}_i$  be the vector space projection determined by this basis of root vectors. For each  $N$  in  $\mathfrak{n}$ , define the map  $\theta_i(N) : \mathfrak{n} \rightarrow \mathfrak{n}_i$  by the equation  $\theta_i(N)(X) = \rho_i \text{Ad } \exp X(N)$ .

**Proposition 3.1.** *Fix  $N$  in  $\mathfrak{n}$  and  $X$  in  $\mathfrak{n}_j$ .*

(1) *If  $i < j$  then  $\theta_i(N)(X) = \rho_i \left( N - \text{ad}(N)(X) + \frac{\text{ad}^2(X)(N)}{2} \right)$ ;*

(2) *if  $i = j$  then*

$$\theta_i(N)(X) = \begin{cases} \rho_i(N - \text{ad}(N)(X)) & \text{in types } A_n, B_n, D_n, \text{ and} \\ \rho_i \left( N - \text{ad}(N)(X) + \frac{\text{ad}^2(X)(N)}{2} \right) & \text{in type } C_n; \end{cases}$$

(3) *and if  $i > j$  then  $\theta_i(N)(X) = \rho_i N$ .*

Furthermore, when  $i = j$  in type  $C_n$  the image of  $\rho_i \text{ad}^2(X)$  lies in  $\mathfrak{g}_{\gamma_i}$ .

*Proof.* Recall that  $\text{Ad}(\exp X) = \exp(\text{ad}X) = \sum_{n=0}^{\infty} \frac{(\text{ad}X)^n}{n!}$  as in [K, Proposition 1.93]. Write  $X = \sum_{\alpha \in \Phi_j} x_\alpha E_\alpha$  and  $Y = \sum_{\alpha \in \Phi^+} y_\alpha E_\alpha$  in terms of the basis. The adjoint operator  $\text{ad}(X)(Y) = [X, Y]$  can be expanded as  $\text{ad}(X)(Y) = \sum_{\alpha+\beta \in \Phi} m_{\alpha,\beta} x_\alpha y_\beta E_{\alpha+\beta}$  for the nonzero coefficients  $m_{\alpha,\beta}$  by Equation (2.1).

The rest of the proof follows from this relation. Each element in the image of  $\text{ad}^3 X$  is a linear combination of root vectors  $E_\beta$  with  $\beta \geq 3\alpha_j$ , since  $X$  is in  $\mathfrak{n}_j$ . By Table 1 no such root  $\beta$  exists, so  $\theta_i(N)$  is a polynomial of degree at most two in the  $x_\alpha$ .

Now let  $i = j$ . In types  $A_n$ ,  $B_n$ , and  $D_n$  there is no root in  $\Phi_i$  greater than  $2\alpha_i$ , so  $\text{ad}^2 X$  vanishes. In these types  $\theta_i(N)$  is affine. In type  $C_n$  there is a unique root greater than  $2\alpha_i$ , namely the root  $\gamma_i = \sum_{k=i}^{n-1} 2\alpha_k + \alpha_n$ . The image of  $\rho_i \text{ad}^2 X$  must thus be in  $\mathfrak{g}_{\gamma_i}$ .

Finally, choose  $i > j$  and let  $c_\beta E_\beta$  be a nonvanishing summand in the expansion of  $\sum_{n=1}^{\infty} \text{ad}^n X(N)$ . Then  $\beta$  is the sum  $\beta_1 + \dots + \beta_k + \alpha$  for  $\beta_1, \dots, \beta_k$  in  $\Phi_j$ . By definition  $\beta$  is contained one of the rows  $\Phi_1, \Phi_2, \dots$ , or  $\Phi_j$ . This means that  $\theta_i(N)(X) = \rho_i N$ .  $\square$

We include the next lemma for ease of reference. It is a restatement of known results.

**Lemma 3.2.** *The element  $N = \sum_{\alpha \in \Phi^+} n_\alpha E_\alpha$  is a regular nilpotent element of  $\mathfrak{n}$  if and only if  $n_{\alpha_i}$  is nonzero for each simple root  $\alpha_i$ . The set of regular nilpotent elements of  $\mathfrak{n}$  is exactly the orbit  $\text{Ad } B(N)$  for each regular  $N$  in  $\mathfrak{n}$ .*

*Proof.* Use [CM, Lemma 4.1.4] for regular nilpotents. The lemma says in this case that  $\text{Ad } B(N)$  is exactly the set of regular nilpotents in  $\mathfrak{n}$ , and that  $\text{Ad } B(N)$

is the set  $\text{Ad } T(N) + [\mathfrak{n}, \mathfrak{n}]$  where  $T$  is the maximal torus in  $B$ . Since  $N$  can be taken to be the sum of the simple root vectors by [CM, Theorem 4.1.6], the claim follows.  $\square$

The element  $N$  in  $\mathfrak{n}$  defines a map in the endomorphism ring  $\text{End}(\mathfrak{n}_i)$  that is the linear part of  $\theta_i(N)$  when  $\theta_i(N)$  is affine.

**Definition 3.3.** For each  $N$  in  $\mathfrak{n}$  the map  $\psi_i(N)$  is defined as the restriction and projection  $\psi_i(N) = (\rho_i \circ \text{ad}N)|_{\mathfrak{n}_i}$ .

As before, let  $\{E_\alpha : \alpha \in \Phi^+\}$  be a fixed basis of root vectors for  $\mathfrak{g}$  with structure constants given by  $[E_\alpha, E_\beta] = m_{\alpha, \beta} E_{\alpha+\beta}$ . Write  $N$  in terms of this basis as  $N = \sum_{\alpha \in \Phi^+} n_\alpha E_\alpha$ .

The next lemma establishes properties of  $\psi_i(N)$  with respect to this basis, where the linear map  $\psi_i(N)$  is identified with its matrix. The entries of this matrix are indexed by pairs of roots  $(\alpha, \beta)$  in  $\Phi_i \times \Phi_i$ . For instance, the entry at position  $(\alpha, \beta)$  is the coefficient of  $E_\alpha$  in  $\psi_i(N)(E_\beta)$ .

**Lemma 3.4.** *Fix  $N$  in  $\mathfrak{n}$ .*

- (1) *The  $(\alpha, \beta)$  position of  $\psi_i(N)$  has entry  $m_{\alpha-\beta, \beta} n_{\alpha-\beta}$  if  $\alpha - \beta$  is a positive root and zero otherwise.*
- (2) *If  $X$  is in  $\mathfrak{n}_{i-j}$  for some positive  $j$  then  $\psi_i(\text{Ad } \exp X(N)) = \psi_i(N)$ .*

*Proof.* The first part follows from the construction of the basis.

The second part follows from the first once we identify the coefficients of  $E_{\alpha-\beta}$  in  $\text{Ad } \exp X(N)$ , for each pair of roots  $\alpha$  and  $\beta$  in  $\Phi_i$ . If the difference  $\alpha - \beta$  is a root then it must be in a row indexed by  $k$ , where  $k$  is at least  $i$ . The coefficient of  $E_{\alpha-\beta}$  in  $\text{Ad } \exp X(N)$  is the same as that in  $\rho_k \text{Ad } \exp X(N)$ , which is  $n_{\alpha-\beta}$  by Proposition 3.1.  $\square$

**Corollary 3.5.** *Fix a regular nilpotent  $N$  in  $\mathfrak{n}$  in types  $A_n$ ,  $B_n$ , or  $C_n$ . The map  $\psi_i(N)$  is a regular nilpotent element of  $\text{End}(\mathfrak{n}_i)$ .*

*Proof.* Order the basis  $\{E_\alpha : \alpha \in \Phi_i\}$  by the height of  $\alpha$  from highest to lowest. By Table 1 this is a total order in which each root differs by a simple root from the next.

Consider the matrix for  $\psi_i(N)$  with respect to this basis. The entries on and below the diagonal correspond to differences  $\alpha - \beta$  which are not positive. The matrix for  $\psi_i(N)$  is zero in these positions by Lemma 3.4. The  $(\alpha, \beta)$  position is immediately above the diagonal if and only if  $\alpha$  is immediately before  $\beta$  in the height order. In this case  $\alpha - \beta$  is a simple root  $\alpha_j$  and the corresponding entry of  $\psi_i(N)$  is  $m_{\alpha_j, \alpha - \alpha_j} n_{\alpha_j}$ . This is nonzero because  $N$  is regular nilpotent, by Lemma 3.2. Since  $\psi_i(N)$  is an upper-triangular matrix with nonzero entries above the diagonal, it too is regular nilpotent, using Lemma 3.2 for  $\mathfrak{gl}_n$ .  $\square$

The following lemma is similar to the previous and is necessary to handle technical difficulties in type  $D_n$ , where  $\psi_i(N)$  is not a regular nilpotent operator. Another analogue of the previous lemma for type  $D_n$  is given in Lemma 4.2.



**Lemma 3.6.** Fix  $X = \sum_{\beta \in \Phi_{i+1}} x_\beta E_\beta$  in  $\mathfrak{n}_{i+1}$  and a root  $\alpha$  in  $\Phi_i$  with  $\alpha \not\asymp 2\alpha_{i+1}$ , all in type  $D_n$ .

(1) The coefficient of  $E_\alpha$  in  $\text{Ad exp } X(N)$  is

$$n_\alpha + \sum_{\beta \in \Phi^+ : \alpha - \beta \in \Phi_{i+1}} m_{\alpha - \beta, \beta} x_{\alpha - \beta} n_\beta.$$

(2) If  $x_\beta$  is zero for each  $\beta < \alpha$  then the coefficient of  $E_\alpha$  in  $\text{Ad exp } X(N)$  is  $n_\alpha$ .

*Proof.* By Proposition 3.1 the projection

$$\rho_i \text{Ad exp } X(N) = \rho_i N + \rho_i [X, N] + \rho_i \frac{1}{2} [X, [X, N]].$$

The coefficient of  $E_\alpha$  in this expansion is

$$\begin{aligned} n_\alpha &+ \sum_{\beta \in \Phi_{i+1} : \alpha - \beta \in \Phi^+} m_{\beta, \alpha - \beta} x_\beta n_{\alpha - \beta} \\ &+ \sum_{\beta_1, \beta_2 \in \Phi_{i+1} : \alpha - \beta_1 - \beta_2 \in \Phi^+, \alpha - \beta_1 \in \Phi^+} \frac{c_{\alpha, \beta_1, \beta_2} x_{\beta_1} x_{\beta_2} n_{\alpha - \beta_1 - \beta_2}}{2} \end{aligned}$$

for nonzero  $c_{\alpha, \beta_1, \beta_2}$  determined by the  $m_{\beta, \alpha - \beta}$ . Since  $\alpha$  is not greater than  $2\alpha_{i+1}$  the difference  $\alpha - \beta_1 - \beta_2$  is not positive for any  $\beta_1, \beta_2$  in  $\Phi_{i+1}$ . Thus the projection  $\rho_i(\text{Ad exp } X(N))$  simplifies to  $\rho_i(N + [X, N])$ , expanded in Part 1. Part 2 follows immediately.  $\square$

In the next lemma, retain the assumption that the basis vectors  $\{E_\alpha : \alpha \in \Phi_i\}$  in the  $i^{\text{th}}$  row are ordered by height from highest to lowest, with an arbitrary order fixed for the two roots of same height in type  $D_n$ .

**Lemma 3.7.** Fix a regular nilpotent element  $N$  in  $\mathfrak{n}$ , a Hessenberg space  $H$ , and a Weyl group element  $w$  so that  $\text{Ad } w^{-1}(E_{\alpha_j}) \in H$  for each simple root  $\alpha_j$ . If  $\alpha$  is in  $\Phi_i$  and  $E_\alpha$  is not in  $\text{Ad } w(H)$  then the first nonzero entry in the  $\alpha$  row of  $\psi_i(N)$  is  $m_{\alpha_j, \alpha - \alpha_j} n_{\alpha_j}$  for some simple root  $\alpha_j$ . Furthermore if  $E_{\alpha - \alpha_j}$  is any basis vector whose root differs from  $\alpha$  by a simple root then  $E_{\alpha - \alpha_j}$  is in  $\mathfrak{n}_w \cap \mathfrak{n}_i$ .

*Proof.* Lemma 3.4 shows that the first entry in the  $\alpha$  row that can be nonzero occurs in the columns  $\beta$  for which  $\alpha - \beta$  is as small a positive root as possible, namely when  $\alpha - \beta$  is simple. The root  $\alpha$  cannot be  $\alpha_i$  because  $E_\alpha$  is not in  $\text{Ad } w(H)$ . At least one root  $\beta < \alpha$  in  $\Phi_i$  differs from  $\alpha$  by a simple root  $\alpha_j$ , by inspection of Table 1. The entry of  $\psi_i(N)$  is  $m_{\alpha_j, \alpha - \alpha_j} n_{\alpha_j}$ , which is nonzero because  $N$  is regular, by Lemma 3.2.

We now show that  $\alpha - \alpha_j$  is in  $\Phi_w$  for any such  $\alpha_j$ . By hypothesis  $E_{w^{-1}\alpha_j}$  is in  $H$  but  $E_{w^{-1}(\alpha - \alpha_j) + w^{-1}\alpha_j}$  is not. Since  $H$  is closed under bracket with  $\mathfrak{b}$ , the root space  $E_{w^{-1}(\alpha - \alpha_j)}$  is in the opposite Borel  $\mathfrak{b}^-$  and so  $w^{-1}(\alpha - \alpha_j)$  is negative.  $\square$

## 4. ITERATED TOWERS OF AFFINE FIBER BUNDLES

We are now ready to prove the main lemmata of the paper. They construct affine spaces which are pieces of the intersection of a Schubert cell with  $\mathcal{H}(N, H)$ . The main theorem then uses these affine spaces to show that each Schubert cell in  $\mathcal{H}(N, H)$  has the structure of an iterated tower of affine fiber bundles.

Each Hessenberg space  $H$  is the direct sum of root spaces [dMPS, Lemma 1]. We define  $\Phi_H$  to be the set of roots such that  $H = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi_H} \mathfrak{g}_\alpha$ . Using this correspondance, define  $H^c = \bigoplus_{\alpha \in \Phi_H^c} \mathfrak{g}_\alpha$  to be the complementary sum of root spaces.  $H^c$  is an ad-nilpotent ideal inside  $\mathfrak{b}^-$ , as discussed in [ST, Section 10].

**Lemma 4.1.** *Fix  $N$  in  $\mathfrak{n}$ ,  $w$  in  $W$ , and a Hessenberg space  $H$  so that each simple root vector  $E_{\alpha_j}$  is in  $\text{Ad } w(H)$ . In types  $A_n$ ,  $B_n$ , and  $C_n$ , the set*

$$\mathcal{X}_i(N) = \{X \in \mathfrak{n}_i \cap \mathfrak{n}_w : \rho_i \text{Ad } \exp X(N) \in \rho_i \text{Ad } w(H)\}$$

*is homeomorphic to an affine space of dimension  $|\Phi_w \cap \Phi_i \cap w\Phi_H|$ .*

*Proof.* The image  $\text{Ad } w(H)$  is the direct sum of root spaces and the Cartan subalgebra, so  $\rho_i \text{Ad } w(H) = \text{Ad } w(H) \cap \mathfrak{n}_i$ . By definition of  $\theta_i(N)$ , the preimage  $\theta_i(N)^{-1}(\text{Ad } w(H) \cap \mathfrak{n}_i) = \{X : \rho_i \text{Ad } \exp X(N) \in \text{Ad } w(H) \cap \mathfrak{n}_i\}$ . It follows that  $\mathcal{X}_i(N) = \mathfrak{n}_i \cap \mathfrak{n}_w \cap \theta_i(N)^{-1}(\text{Ad } w(H) \cap \mathfrak{n}_i)$ .

In types  $A_n$  and  $B_n$ , the map  $\theta_i(N)$  is affine by Proposition 3.1 and so the preimage of the linear subspace  $\text{Ad } w(H) \cap \mathfrak{n}_i$  of  $\mathfrak{n}_i$  is affine. The intersection  $\mathcal{X}_i(N)$  of this affine preimage with the linear subspace  $\mathfrak{n}_i \cap \mathfrak{n}_w$  is also affine.

In type  $C_n$ , recall that  $\gamma_i = \sum_{j=i}^{n-1} 2\alpha_j + \alpha_n$  and consider the commutative diagram

$$\begin{array}{ccc} \mathfrak{n}_w \cap \mathfrak{n}_i & \xrightarrow{\theta_i(N)} & \mathfrak{n}_i \\ \downarrow & & \downarrow \\ (\mathfrak{n}_w \cap \mathfrak{n}_i) / \mathfrak{g}_{\gamma_i - \alpha_i} & \xrightarrow{\tilde{\theta}_i(N)} & \mathfrak{n}_i / \mathfrak{g}_{\gamma_i} \end{array}$$

whose vertical arrows are vector space quotients by the root space. Define the map  $\tilde{\theta}_i(N)$  so the diagram commutes. It is well-defined because the image  $\theta_i(N)(\mathfrak{g}_{\gamma_i - \alpha_i})$  is the coset  $\rho_i N + \mathfrak{g}_{\gamma_i} = \theta_i(N)(0) + \mathfrak{g}_{\gamma_i}$ . It is affine since the image of  $\text{ad}^2 \mathfrak{n}_i$  is in  $\mathfrak{g}_{\gamma_i} \oplus \bigoplus_{j < i} \mathfrak{n}_j$ , so the preimage  $\tilde{\theta}_i(N)^{-1}(\text{Ad } w(H))$  is affine.

The element  $[X]$  in the preimage  $\tilde{\theta}_i(N)^{-1}(\text{Ad } w(H))$  pulls back to the coset  $X + \mathfrak{g}_{\gamma_i - \alpha_i}$  in  $\mathfrak{n}_w \cap \mathfrak{n}_i$ . The image  $\theta_i(N)(X + \mathfrak{g}_{\gamma_i - \alpha_i})$  lies in  $\text{Ad } w(H) + \mathfrak{g}_{\gamma_i}$  by commutativity of the diagram. Moreover, the restriction of  $\theta_i(N)$  to  $X + \mathfrak{g}_{\gamma_i - \alpha_i}$  is affine because the image  $\text{ad}^2 \mathfrak{g}_{\gamma_i - \alpha_i}$  is zero. (The image  $\theta_i(N)(X + \mathfrak{g}_{\gamma_i - \alpha_i})$  intersects  $\text{Ad } w(H)$  in exactly one point if  $\mathfrak{g}_{\gamma_i} \not\subseteq \text{Ad } w(H)$  and otherwise is contained in  $\text{Ad } w(H)$ .) The fiber over  $[X]$  intersects  $\mathcal{X}_i(N)$  in the affine space given by the preimage of  $\theta_i(N)(X + \mathfrak{g}_{\gamma_i - \alpha_i}) \cap \text{Ad } w(H)$  under  $\theta_i(N)$ . The linear map  $\psi_i(N)$  thus determines the dimension of  $\mathcal{X}_i(N)$ .

We now show that the dimension of each of these fibers in types  $A_n$ ,  $B_n$ , and  $C_n$  is constant if  $N$  is regular. It suffices to study the linear part of the affine

operators and to prove that the map  $\mathfrak{n}_w \cap \mathfrak{n}_i \xrightarrow{\psi_i(N)} \mathfrak{n}_i \rightarrow \text{Ad } w(H^c) \cap \mathfrak{n}_i$  is full rank, since then the translation in  $\theta_i(N)$  does not affect the dimension of the kernel. Since  $\mathfrak{n}_i = (\text{Ad } w(H^c) \oplus \text{Ad } w(H)) \cap \mathfrak{n}_i$  this kernel is precisely the preimage of  $\text{Ad } w(H) \cap \mathfrak{n}_i$ .

The restricted matrix for  $\psi_i(N) : \mathfrak{n}_w \cap \mathfrak{n}_i \rightarrow \text{Ad } w(H^c) \cap \mathfrak{n}_i$  consists of the rows  $\alpha$  in  $\Phi_i \cap w\Phi_H^c$  and the columns  $\beta$  in  $\Phi_w \cap \Phi_i$ . Row  $\alpha$  of the full matrix for  $\psi_i(N)$  has its first nonzero entry  $m_{\alpha_j, \alpha - \alpha_j} n_{\alpha_j}$  in position  $(\alpha, \alpha - \alpha_j)$  for some simple root  $\alpha_j$ , by Lemma 3.7. The columns indexed by  $\alpha - \alpha_j$  are distinct in types  $A_n$ ,  $B_n$ , and  $C_n$  because the  $i^{\text{th}}$  row is totally ordered by height in these types. Finally, the root  $\alpha - \alpha_j$  is in  $\Phi_w \cap \Phi_i$  by Lemma 3.7, so the rank of the matrix induced by  $\psi_i(N)$  on  $\mathfrak{n}_w \cap \mathfrak{n}_i \rightarrow \text{Ad } w(H^c) \cap \mathfrak{n}_i$  is  $|\Phi_i \cap w\Phi_H^c|$ . Note that  $\Phi_i \cap w\Phi_H^c$  is a set of positive roots and  $w^{-1}(\Phi_i \cap w\Phi_H^c)$  is a set of negative roots, since  $H$  contains  $\mathfrak{b}$ . This means that  $\Phi_i \cap w\Phi_H^c$  is contained in  $\Phi_w \cap \Phi_i$ , so the kernel of this matrix has dimension  $|\Phi_w \cap \Phi_i| - |\Phi_i \cap w\Phi_H^c| = |\Phi_w \cap \Phi_i \cap w\Phi_H|$ .  $\square$

The next lemma uses a similar approach for type  $D_n$ , where there are technical difficulties because  $\psi_i(N)$  is not regular nilpotent. Define a partition of  $\Phi_i$ :

$$\begin{aligned} \Phi_i^0 &= \{\alpha \in \Phi_i : \alpha \leq \sum_{j=i}^{n-2} \alpha_j\} \\ \Phi_i^1 &= \{\sum_{j=i}^{n-1} \alpha_j, \alpha_n + \sum_{j=i}^{n-2} \alpha_j\} \\ \Phi_i^2 &= \{\alpha \in \Phi_i : \alpha \geq \sum_{j=i}^n \alpha_j\}. \end{aligned}$$

The superscript indicates how many of the simple roots  $\{\alpha_{n-1}, \alpha_n\}$  are summands of the roots in part  $\Phi_i^j$ . Write  $\mathfrak{n}_i^j$  for the subspace  $\bigoplus_{\alpha \in \Phi_i^j} \mathfrak{g}_\alpha$ , as well as  $\rho_i^j$  for the projection  $\mathfrak{n} \rightarrow \mathfrak{n}_i^j$  and  $\theta_i^j$  for the composition  $\rho_i^j \circ \theta_i$ . In type  $D_n$ , normalize the basis  $\{E_\alpha\}$  so that

$$\begin{aligned} m_{\sum_{j=i}^{n-2} \alpha_j, \alpha_{n-1}} &= m_{\sum_{j=i}^{n-2} \alpha_j, \alpha_n} = \\ m_{\alpha_i, \sum_{j=i+1}^{n-1} \alpha_j} &= m_{\alpha_i, \alpha_n + \sum_{j=i+1}^{n-2} \alpha_j} = \\ m_{\sum_{j=i+1}^{n-1} \alpha_j, \alpha_n} &= m_{\alpha_n + \sum_{j=i+1}^{n-2} \alpha_j, \alpha_{n-1}} = 1 \end{aligned}$$

for all  $i$  simultaneously. This is possible by, for instance, [Sa, page 54].

This lemma proves that for each  $X$  in  $\mathfrak{n}_i^0$  and  $Y$  in  $\mathfrak{n}_{i+1}^1 \oplus \mathfrak{n}_{i+1}^2$ , the map

$$\begin{aligned} \theta_D : \mathfrak{n}_i^0 \oplus \mathfrak{n}_{i+1}^1 \oplus \mathfrak{n}_{i+1}^2 &\rightarrow \mathfrak{n}_{i+1}^2 \oplus \mathfrak{n}_i^1 \oplus \mathfrak{n}_i^0 \\ (X, Y) &\mapsto \begin{pmatrix} \theta_{i+1}^2(N)(Y) \\ \rho_i^1 \text{Ad } \exp X (\text{Ad } \exp Y(N)) \\ \theta_i^0(\text{Ad } \exp Y(N))(X) \end{pmatrix} \end{aligned}$$

is affine and surjects onto the subspace of  $\text{Ad } w(H)$  in its image. The main step is to write the linear part of  $\theta_D$  as a matrix whose first column and last row are zero, and to show that the remaining minor is block diagonal, each of whose diagonal blocks is invertible when  $N$  is regular.

**Lemma 4.2.** *The set*

$$\mathcal{X}_i(N) = \left\{ \begin{array}{l} (X, Y) \in \mathfrak{n}_i^0 \oplus \mathfrak{n}_{i+1}^1 \oplus \mathfrak{n}_{i+1}^2 : \\ \theta_D(X, Y) \in \rho_{i+1}^2 \text{Ad } w(H) \oplus \rho_i^1 \text{Ad } w(H) \oplus \rho_i^0 \text{Ad } w(H) \end{array} \right\}$$

is affine of dimension

$$|\Phi_w \cap (\Phi_i^0 \cup \Phi_{i+1}^1 \cup \Phi_{i+1}^2)| - |(\Phi_i^0 \cup \Phi_i^1 \cup \Phi_{i+1}^2) \cap w\Phi_H^c|.$$

*Proof.* We prove that  $\theta_i^0(\text{Ad exp } Y(N)) = \theta_i^0(N)$  for each  $Y$  in  $\mathfrak{n}_{i+1}^1 \oplus \mathfrak{n}_{i+1}^2$  by showing that  $\text{Ad exp } Y(N)$  differs from  $N$  only in root spaces which do not affect the map  $\theta_i^0$ . Indeed, each root in  $\Phi_{i+1}^1 \cup \Phi_{i+1}^2$  is greater than at least one of  $\alpha_{n-1}$  or  $\alpha_n$ , while no root from  $\Phi_i^0$  is greater than either  $\alpha_{n-1}$  or  $\alpha_n$ . For each  $Y$  in  $\mathfrak{n}_{i+1}^1 \oplus \mathfrak{n}_{i+1}^2$ , the coefficient of  $E_\beta$  in the root vector expansion of  $\text{Ad exp } Y(N)$  agrees with that of  $N$  for all  $\beta \leq \sum_{j=i}^{n-2} \alpha_j$ , by Lemma 3.6.2. These are the only basis vectors that affect either the translation or, by Lemma 3.4, the linear part of the affine operators, so  $\theta_i^0(N) = \theta_i^0(\text{Ad exp } Y(N))$ .

The maps  $\theta_i^0(N)$  and  $\theta_{i+1}^2(N)$  are affine operators on  $\mathfrak{n}_i^0$  and  $\mathfrak{n}_{i+1}^1 \oplus \mathfrak{n}_{i+1}^2$  respectively by Proposition 3.1. We write  $\rho_i^1 \text{Ad exp } X(\text{Ad exp } Y(N))$  explicitly to see that it, too, is affine in the  $X_\alpha$  and  $Y_\alpha$ . The coefficient of  $E_{\sum_{j=i}^{n-1} \alpha_j}$  in  $\theta_i(\text{Ad exp } Y(N))(X)$  is

$$n_{\sum_{j=i}^{n-1} \alpha_j} + m_{\sum_{j=i+1}^{n-1} \alpha_j, \alpha_i} n_{\alpha_i} Y_{\sum_{j=i+1}^{n-1} \alpha_j} + \sum_{k=i+1}^{n-1} m_{\sum_{j=i}^{k-1} \alpha_j, \sum_{j=k}^{n-1} \alpha_j} n_{\sum_{j=k}^{n-1} \alpha_j} X_{\sum_{j=i}^{k-1} \alpha_j}$$

by Proposition 3.1 and Lemma 3.6.1. The coefficient of  $E_{\sum_{j=i}^{n-2} \alpha_j + \alpha_n}$  is obtained from this formula by exchanging  $\alpha_{n-1}$  and  $\alpha_n$ . Both coefficients are affine functions in the  $X_\alpha$  and  $Y_\alpha$  and so the map  $\theta_D$  is affine. Since  $\mathcal{X}_i(N)$  is the preimage of the linear space  $\text{Ad } w(H) \cap (\mathfrak{n}_i^0 \oplus \mathfrak{n}_i^1 \oplus \mathfrak{n}_{i+1}^2)$  under this affine map, it is affine itself.

Write the linear part of  $\theta_D$  with respect to the basis of root vectors. Order the columns  $\Phi_{i+1}^2$  from highest root to lowest root, follow with the columns  $\Phi_{i+1}^1$  ordered as in the definition, and then with the columns  $\Phi_i^0$  ordered from highest to lowest. Similarly, order the rows  $\Phi_{i+1}^2$ , then  $\Phi_i^1$ , then  $\Phi_i^0$ , within each set ordering by height or by definition.

The first column corresponds to the root  $\alpha_{i+1} + \sum_{j=i+2}^{n-2} 2\alpha_j + \alpha_{n-1} + \alpha_n$  and the last row to  $\alpha_i$ . There is no root in  $\Phi_{i+1}^2, \Phi_{i+1}^1, \Phi_i^1$ , or  $\Phi_i^0$  which is greater than the former or less than the latter, so this column and row are identically zero.

Now examine the minor obtained by omitting the first column and last row. Form blocks by partitioning the columns into three sets  $\Phi_{i+1}^2$ ,  $\Phi_{i+1}^1 \cup \{\sum_{j=i}^{n-2} \alpha_j\}$ , and  $\Phi_i^0 \setminus \{\sum_{j=i}^{n-2} \alpha_j\}$ , and the rows into the sets  $\Phi_{i+1}^2 \setminus \{\sum_{j=i+1}^n \alpha_j\}$ ,  $\Phi_{i+1}^1 \cup \{\sum_{j=i+1}^n \alpha_j\}$ , and  $\Phi_i^0$ . The matrix is block diagonal because each block to the left of the block  $B$  is indexed by roots not less than the roots in  $B$ . In each diagonal block, the first nonzero entry in the row for  $\alpha$  is  $m_{\alpha - \alpha_j, \alpha_j} n_{\alpha_j}$  and is located in a column  $\alpha - \alpha_j$  where  $\alpha - \alpha_j$  differs from  $\alpha$  by a simple root. The root  $\alpha - \alpha_j$  is unique if  $\alpha > \sum_{j=i+1}^n \alpha_j$  in  $\Phi_{i+1}^2$  or if  $\alpha$  is in  $\Phi_i^0$  because those root subsets are totally ordered by height, by inspection of Table 1. Thus, the first and third diagonal blocks are upper-triangular with nonzero entries along the diagonal.

Using the previous explicit calculations of  $\rho_i^1 \text{Ad exp } X(\text{Ad exp } Y(N))$  and computing the coefficient of  $E_{\sum_{j=i+1}^n \alpha_j}$  in  $\theta_{i+1}(N)(Y)$  shows that the second diagonal block is

$$\begin{pmatrix} m_{\sum_{j=i+1}^{n-1} \alpha_j, \alpha_n} n_{\alpha_n} & m_{\alpha_n + \sum_{j=i+1}^{n-2} \alpha_j, \alpha_{n-1}} n_{\alpha_{n-1}} & 0 \\ m_{\sum_{j=i+1}^{n-1} \alpha_j, \alpha_i} n_{\alpha_i} & 0 & m_{\sum_{j=i}^{n-2} \alpha_j, \alpha_{n-1}} n_{\alpha_{n-1}} \\ 0 & m_{\alpha_n + \sum_{j=i+1}^{n-2} \alpha_j, \alpha_i} n_{\alpha_i} & m_{\sum_{j=i}^{n-2} \alpha_j, \alpha_n} n_{\alpha_n} \end{pmatrix} = \begin{pmatrix} n_{\alpha_n} & n_{\alpha_{n-1}} & 0 \\ -n_{\alpha_i} & 0 & n_{\alpha_{n-1}} \\ 0 & -n_{\alpha_i} & n_{\alpha_n} \end{pmatrix}$$

by the basis normalization. This is invertible since the  $n_{\alpha_j}$  are nonzero.

This confirms that the upper minor of  $\theta_D$  is full rank independent of  $X$  and  $Y$ . We now prove that the projection of  $\theta_D$  to  $\text{Ad } w(H^c) \cap (\mathfrak{n}_i^0 \oplus \mathfrak{n}_i^1 \oplus \mathfrak{n}_{i+1}^2)$  is full rank on  $\mathfrak{n}_w \cap (\mathfrak{n}_i^0 \oplus \mathfrak{n}_{i+1}^1 \oplus \mathfrak{n}_{i+1}^2)$ . For each row  $\alpha$  that is in  $w\Phi_H^c$ , all of the nonzero columns  $\alpha - \alpha_j$  for the row are in  $\Phi_w$  by Lemma 3.7. This shows that the entries used to determine the rank of the full matrix are also in the matrix restricted and projected to  $\mathfrak{n}_w \cap (\mathfrak{n}_i^0 \oplus \mathfrak{n}_{i+1}^1 \oplus \mathfrak{n}_{i+1}^2) \rightarrow \text{Ad } w(H^c) \cap (\mathfrak{n}_i^0 \oplus \mathfrak{n}_i^1 \oplus \mathfrak{n}_{i+1}^2)$ . Consequently, this restriction is full rank and so the dimension of its kernel is  $|\Phi_w \cap (\Phi_i^0 \cup \Phi_{i+1}^1 \cup \Phi_{i+1}^2)| - |(\Phi_i^0 \cup \Phi_i^1 \cup \Phi_{i+1}^2) \cap w\Phi_H^c|$ .  $\square$

The main theorem studies the intersection of the Hessenberg variety  $\mathcal{H}(N, H)$  for  $N$  in  $\mathfrak{b}$  with the Bruhat decomposition of  $G/B$  given by the Borel subgroup corresponding to  $\mathfrak{b}$ . The previous lemmata will show that each nonempty cell in this decomposition has the structure of an iterated affine fiber bundle and so is homeomorphic to an affine cell.

**Theorem 4.3.** *Let  $N$  be a regular nilpotent element in  $\mathfrak{b}$ ,  $H$  a Hessenberg space with respect to  $\mathfrak{b}$ , and  $\mathcal{H}(N, H)$  the corresponding Hessenberg variety. Let  $P_w = \mathcal{H}(N, H) \cap BwB/B$  be the intersection of the Schubert cell corresponding to  $w$  with the Hessenberg variety. The  $\{P_w\}$  form a paving by affines of  $\mathcal{H}(N, H)$  when ordered subordinate to the partial order determined by the length of  $w$ . The cell  $P_w$  is nonempty if and only if  $w^{-1}\alpha_i \in \Phi_H$  for each simple root  $\alpha_i$ . If  $P_w$  is nonempty its dimension is  $|\Phi_w \cap w\Phi_H|$ .*

*Proof.* The  $\{P_w\}$  form a paving under any order that respects the length partial order because the Bruhat decomposition of  $G/B$  is a paving and  $\mathcal{H}(N, H)$  is closed in  $G/B$ .

Consider the set of Lie algebra elements  $\text{Ad } u(N)$  for  $u \in U_w$ . If  $u$  is in  $U$  then  $\text{Ad } u(N)$  is regular nilpotent and the coefficient of each simple root vector is nonzero, by Lemma 3.2. Thus, the element  $\text{Ad } u(N)$  can only be in  $\text{Ad } w(H)$  if each  $w^{-1}\alpha_i$  is in  $\Phi_H$ .

We now prove that the condition is sufficient for  $P_w$  to be nonempty and compute the dimension of the affine cell. Write  $U = U_1 \cdots U_n$ , factor  $u = u_1 \cdots u_n$  accordingly, and let  $X_i$  be an element of  $\mathfrak{n}_i$  with  $\exp X_i = u_i$ . Then  $\text{Ad } u(N)$  is

$$\text{Ad } u(N) = \text{Ad exp } X_1(\text{Ad exp } X_2(\cdots \text{Ad exp } X_n(N) \cdots)).$$

(In type  $D_n$ , further decompose  $X_i$  into  $X_i^0 \in \mathfrak{n}_i^0$  and  $X_i^1 \in \mathfrak{n}_i^1 \oplus \mathfrak{n}_i^2$ , and factor

$$\text{Ad } u(N) = \text{Ad } \exp X_1^1 (\text{Ad } \exp X_1^0 (\text{Ad } \exp X_2^1 (\cdots \text{Ad } \exp X_n^0(N) \cdots)).)$$

Define  $\mathcal{Z}_i$  to be the set

$$\{u_i u_{i+1} \cdots u_n : u_j \in U_w \cap U_j \ \forall j, \rho_j \text{Ad } (u_i \cdots u_n)(N) \in \rho_j \text{Ad } w(H) \ \forall j \geq i\}.$$

The set  $\mathcal{Z}_1$  is homeomorphic to the cell  $P_w$  via the map that sends  $u_1 \cdots u_n$  to the flag corresponding to  $(u_1 \cdots u_n)^{-1}w$ . There is a natural map  $\mathcal{Z}_i \rightarrow \mathcal{Z}_{i+1}$  given by  $u_i u_{i+1} \cdots u_n \mapsto u_{i+1} \cdots u_n$ . The fiber over the point  $u'$  is the set

$$\{\exp X_i : X_i \in \mathfrak{n}_w \cap \mathfrak{n}_i, \rho_i \text{Ad } \exp X_i (\text{Ad } u'(N)) \in \rho_i \text{Ad } w(H)\},$$

namely  $\exp \mathcal{X}_i(\text{Ad } u'(N))$  of Lemma 4.1. The exponential map is a homeomorphism on  $\mathfrak{n}_i$  and so by Lemma 4.1 the map  $\mathcal{Z}_i \rightarrow \mathcal{Z}_{i+1}$  is an affine fiber bundle of rank  $|\Phi_w \cap \Phi_i \cap w\Phi_H|$  for each  $i$ . The space  $P_w \cong \mathcal{Z}_1$  is thus an iterated tower of affine fiber bundles and is itself homeomorphic to an affine space of dimension  $|\Phi_w \cap w\Phi_H|$ .

(In type  $D_n$  we use the sets

$$\mathcal{Z}_i = \left\{ \begin{array}{l} u_j^0 \in U_w \cap U_j^0 \ \forall j, u_j^1 \in U_w \cap (U_j^1 U_j^2) \ \forall j; \\ u_i^0 u_{i+1}^1 u_{i+1}^0 \cdots u_n^1 : \rho_i^k \text{Ad } (u_i^0 \cdots u_n^1)(N) \in \rho_i^k \text{Ad } w(H) \text{ for } k = 0, 1; \\ \rho_j \text{Ad } (u_i^0 \cdots u_n^1)(N) \in \rho_j \text{Ad } w(H) \ \forall j > i \end{array} \right\}.$$

As before, the set  $\mathcal{Z}_0$  is homeomorphic to the cell  $P_w$ . The map  $\mathcal{Z}_i \rightarrow \mathcal{Z}_{i+1}$  given by  $(u_i^0 u_{i+1}^1)u' \mapsto u'$  is an affine fiber bundle with fiber  $\exp \mathcal{X}_i(\text{Ad } u'(N))$  from Lemma 4.2. This displays  $\mathcal{Z}_0$  as an iterated tower of affine fiber bundles, and so  $P_w$  is homeomorphic to affine space. The dimension of each fiber is

$$|\Phi_w \cap (\Phi_i^0 \cup \Phi_{i+1}^1 \cup \Phi_{i+1}^2)| - |(\Phi_i^0 \cup \Phi_i^1 \cup \Phi_{i+1}^2) \cap w\Phi_H^c|.$$

Summing over  $i$  gives  $|\Phi_w| - |\Phi^+ \cap w\Phi_H^c|$ . Since each root in  $\Phi^+ \cap w\Phi_H^c$  is in  $\Phi_w$ , the total dimension is  $|\Phi_w \cap w\Phi_H|$ .  $\square$

The statement of this theorem is more concise when  $N$  is the sum of simple root vectors.

**Corollary 4.4.** *Let  $N$  be the sum of simple root vectors  $N = \sum_{\alpha_i} E_{\alpha_i}$  in  $\mathfrak{b}$ ,  $H$  a Hessenberg space with respect to  $\mathfrak{b}$ , and  $\mathcal{H}(N, H)$  the corresponding Hessenberg variety. Let  $P_w = \mathcal{H}(N, H) \cap BwB/B$  be the intersection of the Schubert cell corresponding to  $w$  with the Hessenberg variety. The  $\{P_w\}$  form a paving by affines of  $\mathcal{H}(N, H)$  when ordered subordinate to the partial order determined by the length of  $w$ . The cell  $P_w$  is nonempty if and only if  $\text{Ad } w^{-1}(N) \in H$ . If  $P_w$  is nonempty its dimension is  $|\Phi_w \cap w\Phi_H|$ .*

*Proof.* The sum of simple root vectors is regular in all classical types by Lemma 3.2. Since  $\text{Ad } w^{-1}(N) = \sum_{\alpha_i} E_{w^{-1}\alpha_i}$  and since  $H$  is a sum of root spaces, the condition  $\text{Ad } w^{-1}(N) \in H$  is equivalent to  $w^{-1}\alpha_i \in \Phi_H$  for each simple  $\alpha_i$ .  $\square$

While the criterion for nonemptiness is more complicated, this theorem also proves that all regular nilpotent Hessenberg varieties are paved by affine cells.

**Corollary 4.5.** *Fix  $\mathfrak{g}$  of classical type, let  $N$  be a regular nilpotent in  $\mathfrak{g}$ , and let  $H$  be a Hessenberg space with respect to  $\mathfrak{b}$ . The Hessenberg variety  $\mathcal{H}(N, H)$  is paved by affines.*

*Proof.* Choose an element  $\text{Ad } g^{-1}(N)$  in the regular nilpotent orbit which is also in  $\mathfrak{n}$ . The variety  $\mathcal{H}(\text{Ad } g^{-1}(N), H)$  is paved by affines  $\{P_w\}$  by Theorem 4.3. Note that  $\mathcal{H}(\text{Ad } g^{-1}(N), H) = g^{-1}\mathcal{H}(N, H)$  and that translation is a homeomorphism in  $G/B$ . This means  $\mathcal{H}(N, H)$  is paved by the affine cells  $gP_w$ .  $\square$

The existence of a paving by affines shows the following.

**Corollary 4.6.** *In classical types, for any Hessenberg space  $H$  with respect to  $\mathfrak{b}$ , the regular nilpotent Hessenberg variety  $\mathcal{H}(N, H)$  has no odd-dimensional cohomology.*

*Proof.* The existence of a paving by complex affine cells means that the odd-dimensional cohomology of  $\mathcal{H}(N, H)$  vanishes by Lemma 2.3.  $\square$

We remark that since  $\mathcal{H}(N, H)$  has no odd-dimensional cohomology, it is equivariantly formal with respect to any algebraic torus action [GKM, p.26].

#### REFERENCES

- [BC] M. Brion and J. Carrell, *The equivariant cohomology ring of regular varieties*, Mich. Math. J. **52** (2004), 189–203.
- [BM] W. Borho and R. MacPherson, *Partial resolutions of nilpotent varieties*, Asterisque 101–102, Soc. Math. France, Paris, 1983.
- [C] J. Carrell, personal communication.
- [Ch] C. Chevalley, *Sur les décompositions cellulaires des espaces  $G/B$* , Proc. Symp. Pure Math. **56** 1994, Part I, 1–25.
- [CG] N. Chriss and V. Ginzburg, *Representation Theory and Complex Geometry*, Birkhäuser, Boston, 1997.
- [CM] D. Collingwood and W. McGovern, *Nilpotent Orbits in Semisimple Lie Algebras*, Van Nostrand Reinhold, New York, 1993.
- [dCLP] C. de Concini, G. Lusztig, and C. Procesi, *Homology of the zero-set of a nilpotent vector field on a flag manifold*, J. Amer. Math. Soc. **1** 1988, 15–34.
- [dMPS] F. de Mari, C. Procesi, and M. A. Shayman, *Hessenberg varieties*, Trans. Amer. Math. Soc. **332** 1992, 529–534.
- [F] W. Fulton, *Intersection Theory*, Springer-Verlag, Berlin, 1984.
- [GKM] M. Goresky, R. Kottwitz, and R. MacPherson, *Equivariant cohomology, Koszul duality, and the localization theorem*, Inv. Math. **131** 1998, 25–83.
- [H] J. Humphreys, *Linear Algebraic Groups*, Grad. Texts in Math. 21, Springer-Verlag, New York, 1964.
- [K] A. Knapp, *Lie Groups Beyond an Introduction*, Birkhäuser, Boston, 1996.
- [Ko] B. Kostant, *Flag Manifold Quantum Cohomology, the Toda Lattice, and the Representation with Highest Weight  $\rho$* , Selecta Math. (N. S.) **2** 1996, 43–91.
- [L] G. Lusztig, *Intersection cohomology complexes on a reductive group*, Invent. Math. **75** 1984, 205–272.

- [ST] E. Sommers and J. Tymoczko, *Exponents for B-stable ideals*, Trans. Amer. Math. Soc. **358** (2006), 3493–3509.
- [Sa] H. Samelson, *Notes on Lie algebras*, Van Nostrand Reinhold Math. Studies **23**, Van Nostrand Reinhold, New York, 1969.
- [S] T. Springer, *Trigonometric sums, Green functions of finite groups and representations of Weyl groups*, Invent. Math. **36** 1976, 173–207.
- [Ste] J. Stembridge, *Explicit matrices for irreducible representations of Weyl groups*, Represent. Theory **8** 2004, 267–289.
- [T] J. Tymoczko, *Linear conditions imposed on flag varieties*, Amer. J. Math. **128** (2006), 1587–1604.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOWA, IOWA CITY, IA 52242  
*E-mail address:* [tymoczko@math.uiowa.edu](mailto:tymoczko@math.uiowa.edu)