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Sub-riemannian Heat Kernels and Mean Curvature Flow of Graphs

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Abstract. We introduce a sub-Riemannian analogue of the Bence-Merriman-Osher algorithm [42] and show that it leads to weak solutions of the horizontal mean curvature flow of graphs over sub-Riemannian Carnot groups. The proof follows the nonlinear semi-group theory approach originally introduced by L. C. Evans [27] in the Euclidean setting and is based on new results on the relation between sub-Riemannian heat flows of characteristic functions of subgraphs and the horizontal mean curvature of the corresponding graphs.

1. Introduction

The study of mean curvature flow in the sub-Riemannian setting is still at a very early stage, with several key properties, such as existence, uniqueness and regularity, still unknown. The notion itself of motion by mean curvature is understood only in special cases, as in the evolution of graphs over groups.

In this paper we give two contributions to this topic. First we establish in the Carnot group setting a formula relating mean curvature of a graph and the heat flow of the characteristic function of the corresponding sub-graph (see Lemma 4). This formula seems to be new also in the Riemannian setting (see Corollary 1). Then, using such formula we prove convergence of an analogue of the Bence-Merrriman-Osher [42] algorithm in the Carnot group setting (see Theorem 2), yielding a easily implementable time-step approximation of the sub-Riemannian mean curvature flow of graphs over Carnot groups. This convergence can in turn be used to define a notion of mean curvature flow also across characteristic points.

1.1. **Carnot groups.** A Lie group $\mathbb{G}$ is called a homogeneous stratified Lie group if its Lie algebra admits a stratification

$$\mathcal{G} = V^1 \oplus \cdots \oplus V^r \text{ with } [V^i, V^j] = V^{i+j} \text{ and } [V^i, V^r] = 0.$$

We will let n denote the topological dimension of $\mathbb{G}$. Given a positive definite bilinear form g_0 on V^1 we call the pair $(\mathbb{G}, g_0)$ a Carnot group and the corresponding left invariant metric g_0 a sub-Riemannian metric. For each $X \in V^i$ let $d(X) = i$ be the degree of X and let $Q = \sum_{i=1}^r i \dim(V_i)$

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the homogeneous dimension of $\mathbb{G}$. Let us consider an orthonormal basis $\{X_1, ..., X_n\}$ of $\mathcal{G}$ fitted to the stratification, i.e. such that

$$\text{the horizontal layer } H_0 := V^1 \text{ of } \mathbb{G} \text{ is generated by } X_1, \dots, X_m, \tag{1.1}$$

while $\{X_i\}_{d(X_i)=k}$ generates V^k. These assumptions allow to define homogeneous dilations δ_λ and an homogeneous pseudo-norm $|\cdot|_0$ on $\mathcal{G}$ (and through exponential coordinates) on G. If $v \in \mathcal{G}$ is expressed in term of the element of the basis as $v = \sum_i v_i X_i$,

$$\delta_\lambda(v) := (\lambda^{d(1)} v_1, \cdots, \lambda^{d(n)} v_n), \quad |v|_0^{2r!} := \sum_{i=1}^n |v_i|^{2r!/d(i)}. \tag{1.2}$$

In this setting we call horizontal gradient of a function $u : \mathbb{G} \to \mathbb{R}$ the vector

$$\nabla_0 u = (X_1 u, \cdots X_m u), \tag{1.3}$$

and we say that u is in C^1_X if its horizontal gradient exists and it is continuous. The graph of a C^1_X function $u : \mathbb{G} \to \mathbb{R}$ can be considered a surface in $\mathbb{G} \times \mathbb{R}$. The product $\mathbb{G} \times \mathbb{R}$ can be endowed with the structure of Lie group and in particular with a Carnot group structure by setting $X_{n+1} = \partial_{n+1}$, and $d(n+1) = 1$,

$$\text{the horizontal layer } \tilde{H}_0 = H_0 \times \mathbb{R} \text{ of } \mathbb{G} \times \mathbb{R} \text{ is generated by } (X_1, \cdots, X_m, X_{n+1}). \tag{1.4}$$

The metric g_0 is extended to a metric $\tilde{g}_0$ in $\tilde{H}_0$ by requesting that X_{n+1} is orthogonal to $\mathcal{G}$ and has unit norm. We call horizontal normal of the graph of u the projection on the horizontal plane $\tilde{H}_0$ of the Euclidean normal and set

$$\nu_0 = \frac{(-\nabla_0 u, 1)}{\sqrt{1 + |\nabla_0 u|^2}}.$$

Let us explicitly note that such graphs do not contain any characteristic points, i.e. points where the horizontal space is contained in the tangent space. In the literature several equivalent definitions of the horizontal mean curvature h_0 have been proposed at non characteristic points of a C^2 surface M. To quote a few: h_0 can be defined in terms of the first variation of the area functional [19, 32, 13, 48, 50, 44, 11], as horizontal divergence of the horizontal unit normal or as limit of the mean curvatures h_ϵ of suitable Riemannian approximating metrics g_ϵ [11]. If the surface is not regular, the notion of curvature can be expressed in the viscosity sense (we refer to [5], [6], [52], [53], [40], [2], [41], [9] for viscosity solutions of PDE in the sub-Riemannian setting).

The curvature h_0 of the graph of a function u can be written as

$$h_0 = \frac{-1}{\sqrt{1 + |\nabla_0 u|^2}} \sum_{i,j=1}^m \left(\delta_{ij} - \frac{X_i u X_j u}{1 + |\nabla_0 u|^2} \right) X_i X_j u \tag{1.5}$$

The horizontal mean curvature flow of a graph over a Carnot group $\mathbb{G}$ is the flow $t \to \tilde{M}_t := \{(x, u(x,t)) | x \in \mathbb{G}\} \subset \mathbb{G} \times \mathbb{R}$ in which each point in the evolving manifold moves along the horizontal normal, with speed given by the horizontal mean curvature. The evolution of a family of graphs of functions $t \to u(\cdot, t)$ is then characterized by the following equation

$$\partial_t u = -Au, \quad \text{where } Au = -\sum_{i,j=1}^m \left(\delta_{ij} - \frac{X_i u X_j u}{1 + |\nabla_0 u|^2} \right) X_i X_j u, \tag{1.6}$$

Here we will consider viscosity solution of this equation:

Definition 1. A continuous function $u : \mathbb{G} \times (0,\infty) \to \mathbb{R}$ is a weak sub-solution (resp. super-solution) of equation (1.6) if for every $(x,t) \in \mathbb{G} \times \mathbb{R}$ and every smooth $\phi : \mathbb{G} \times (0,\infty) \to \mathbb{R}$ such that $u - \phi$ has a maximum (resp. a minimum) at (x,t) then

$$\partial_t \phi(x,t) \leq \ (resp.\ \geq) \ \sum_{i,j=1}^{m} \left(\delta_{ij} - \frac{X_i\phi(x,t)X_j\phi(x,t)}{1+|\nabla_0\phi(x,t)|^2} \right) X_i X_i \phi(x,t).$$

Solutions are functions which are simultaneously super-solutions and a sub-solutions.

Existence and uniqueness of viscosity solutions to this equation, attaining an assigned initial condition f has been established in [9] (see also the recent [28] as well as [25] for a probabilistic interpretation of the flow). One of the goals of this paper is to provide a discrete approximation of this motion, called diffusion driven motion by mean curvature.

1.2. **Diffusion driven motion by mean curvature.** In the Euclidean setting, the motion by mean curvature can be obtained through an algorithm introduced by Merriman, Bence, Osher [42], which relates the mean curvature flow and the heat flow.

The algorithm is organized in two steps[1]:

- given $\lambda \in \mathbb{R}$ and a function f, the characteristic function χ_{S_λ} of each of its $\lambda-$(sub)level sets $S_\lambda = \{\tilde{x} : \ f(\tilde{x}) > \lambda\}$ is diffused for a time t, via the subriemannian heat flow, giving rise to a smooth function $w_\lambda(\tilde{x},t)$.
- at time t one defines a new function $H(t)f$ by requiring that its $\lambda-$level sets be $\{\tilde{x} : w_\lambda(\tilde{x},t) > 1/2\}$.

This procedure leads to the definition of a two step algorithm, which given a function f allows to define a new function $H(t)f$. The mean curvature flow $t \to u(\cdot,t)$ with initial data f, can be recovered applying iteratively this two step algorithm for shorter and shorter time intervals (or equivalently through iteratively applying the operator H).

The convergence of the Euclidean version of this scheme has been proved by Evans [27] and Barles, Georgelin [3]. In particular Evans gave a proof based on nonlinear semi-group theory and on a pointwise study of heat flow of sets in terms of curvature of their boundaries. We also refer to Ishii [35], Ishii, Pires, Souganidis [36], Leoni [39] and Chambolle, Novaga [12] for extensions of this algorithm to more general setting. Let us mention that other convergent schemes for nonlinear parabolic equations have been proposed in [18], [20], [21], [22], [51] and [26].

1.3. **Diffusion driven motion by horizontal mean curvature in Carnot groups.** One of our main results is an extension of the algorithm in [42] to the degenerate parabolic setting of sub-Riemannian Lie groups. A motivation for such extension comes from problems of visual perception and modeling of the visual cortex. Geometric models for the visual cortex as a contact structure go back to work by Hoffmann [34], Petitot and Tondut [46] and Citti, Petitot and Sarti [47]. Later, in [15], Citti and Sarti endowed this contact structure with a sub-Riemannian metric, and proposed a model of perceptual completion based on two cortical mechanisms applied in sequence. This two-steps process lead to a sub-Riemannian diffusion driven motion by horizontal mean curvature in the cortical structure which is responsible for formation of subjective surfaces. In the model they propose they study the evolution of a graph, and this is one of the main reasons why in this paper we focus on diffusion driven motion of graphs.

[1] See Definition 6 for the precise definition

The diffusion driven motion by curvature problem can be formulated also in the general setting of evolving arbitrary surfaces in Carnot groups (where it could have a wealth of applications in the study of minimal surfaces and isoperimetric problems). In this wider context, however, the notion of horizontal mean curvature is not yet known, because the presence of characteristic points cannot be excluded. Although in this paper we do not study this more general setting we do provide a relation between mean curvature and heat kernels which could be used for that purpose (see Proposition 3).

In Section 2 we will begin the proof of our main theorem, by stating a preliminary technical result, which allows to express sub-Riemannian quantities as limit of Riemannian ones. In particular we express the heat kernel as limit of the corresponding Riemannian ones. This instrument follows from a generalization of the celebrated lifting method of Rothschild and Stein [49], and in the time independent case has been developed in [14]. The adaptation to the heat operator provides a powerful and general instrument for transfer results from the Riemannian to the sub-Riemannian setting. The proof of the technical result can be found in the frecent preprint [10].

In Section 3 we introduce the diffusion driven motion by mean curvature, and we restate the definition of the mean curvature operator in the setting of non linear semi-groups. In particular we recognize that the curvature operator A defined in (1.6) is non dissipative (see definition 2 below). According to a Theorem of Crandall and Liggett [17, Theorem I, page 266] this allows to give a weak definition of solution of (1.6) in terms of a non linear semi-group:

Theorem. *If A is non dissipative operator on a Banach space $\mathcal{B}$, then for all $f \in \mathcal{B}$ the limit*

$$M(t)(f) := \lim_{j\to\infty, \lambda j\to t} (I + \lambda A)^{-j} f \tag{1.7}$$

exists locally uniformly in t.

The main result of section 3 is the proof that this notion of weak semi-group solution coincides with the viscosity one, given in Definition 1:

Proposition 1. *For every continuous and periodic function f, the weak semi-group solution $M(t)(f)$ is the viscosity solution u of motion by curvature, with initial datum f.*

Note that a non linear semi-group provides us with a discrete approximation of this motion, but the approximation is non-linear, while we were looking for a linear discrete approximation.

In Section 4 we give a definition of the evolution operator H analogous to the one defined in [27]. However we are able to provide a substantial simplification, since we work only with graphs (and not with more general level sets), and we evolve only the subgraph of the given initial datum. Calling χ_S the characteristic function of its subgraph, we evolve it with the sub-Riemannian heat flow in $\mathbb{G} \times \mathbb{R}$:

$$\partial_t w = \sum_{d(i)=1} X_i^2 w \quad w(\tilde{x}, 0) = \chi_S(\tilde{x}). \tag{1.8}$$

The new function $H(t)f$ is defined requiring that its sub-graph is $\{\tilde{x} : w(\tilde{x}, t) < 1/2\}$.

Note that the right-hand side of the heat equation (1.8) is a sum of squares of $m + 1$ vector fields in $\mathbb{R}^{n+1}$, and consequently it is only degenerate parabolic. However, in view of the work of Hörmander [33], Rothschild and Stein [49] (see also Jerison and Sanchez-Calle [37]) such operator is hypoelliptic and admits a fundamental solution Γ with good estimates (see Proposition 5 and

(2.6)). We conclude Section 4 with Theorem 3, which contains many properties of the operator H and in particular

Proposition 2. *For every continuous periodic function f and g*

$$||H(t)f - H(t)g|| \leq ||f - g|| \text{ for all } t \geq 0. \tag{1.9}$$

Section 5 contains the main step and the geometric core of the proof: we will prove that the operator H locally approximates motion by horizontal mean curvature. The evolution of the characteristic function of a given set S is expressed through the heat kernel Γ as

$$w(\tilde{x}, t) = \int \Gamma(\tilde{y}^{-1}\tilde{x}, t)\chi_S(\tilde{y})d\tilde{y}.$$

Following Evans [27], we need to perform a point-wise asymptotic expansion of w for t near 0, assuming that the boundary of S is sufficiently smooth. In the Euclidean case the expression of Γ is explicit, and the proof in [27, Theorem 4.1] is a direct computation. In contrast, in our sub-Riemannian setting the heat kernel does not have in general explicit expression and in order to extend Evans' result we are obliged to use a deeper method, based on geometrical properties of the space. Our approach uses in full strength the properties of the sub-Riemannian heat kernel and its Riemannian approximation. In this extension, the results stated in Section 2 play a crucial role. Proceeding in this fashion we then establish some new formula, emphasizing a link between the heat kernel and the mean curvature, which are of independent interest.

These new formula are closely related to the notion of heat content developed by De Giorgi [23], [24] in the Euclidean case, and recently extended to the sub-Riemannian case in [7]. In these works the integral of the function w is considered under low regularity on the set S, and it is proved that its derivatives tend to the perimeter. Higher terms expansions of the integral of w in the Euclidean case have been established in [4] where it has been proved that next term in the expansion depends on the mean curvature.

Here we establish instead a pointwise estimate of the function w involving the curvature of ∂S

Proposition 3. *If S is the subgraph of a smooth function f and $\tilde{x} \in \partial S$, one has*

$$w(\tilde{x}, t) = \frac{1}{2} - \sqrt{t}h_0(\tilde{x}) \int_\Pi \Gamma(\tilde{z}, 1)d\sigma_0(\tilde{z}) + O(t), \text{ as } t \to 0.$$

where Π is the intrinsic tangent plane to S at the point $\tilde{x}$, as defined in the statement of Lemma 4.

Note that this expression can be considered as a weak definition of horizontal mean curvature, which does not rely on differentiability of the surface or on other geometric considerations. The same argument provided in the proof of Proposition 3 yields an analogue result for arbitrary hypersurfaces of a Riemannian manifold (see Corollary 1).

Remark 1. See the proof of [1, Theorem 3.3] and [27, Theorem 4.1] for similar results in the Euclidean setting. In both papers the explicit form of the Euclidean heat kernel plays a key role.

From the latter we infer:

Proposition 4. *For every continuous periodic function f one has*

$$t^{-1}(I - H(t))(f)(\tilde{x}) \to -Af(\tilde{x}),$$

uniformly as $t \to 0$.

This approximation result allows to apply a general theorem of Brezis and Pazy [8], which ensures the following:

Theorem 1. *If A is an $m-$dissipative operator $\{H(t)\}_{t\geq 0}$ satisfying* (1.9) *and*

$$(I+\lambda A)^{-1} f = \lim_{t\to 0^+} \Big(I + \lambda t^{-1}(I - H(t))\Big)^{-1} f, \tag{1.10}$$

for every f and $\lambda > 0$, then for every $f \in \bar{D}(A)$ and $t \geq 0$ then one has

$$M(t) f = \lim_{j\to\infty} H\Big(\frac{t}{j}\Big)^j f, \text{ uniformly for } t \text{ in compact sets} \tag{1.11}$$

On the other side, since we have already proved the relation between the weak semi-group and the viscosity solutions of motion by curvature, we can finally deduce our main approximation result for the curvature flow:

Theorem 2. *If f is a continuous and periodic function, and u is the unique viscosity solution to* (1.6)*, with initial datum f, then*

$$u(\tilde{x}, t) = \lim_{j\to\infty} H\Big(\frac{t}{j}\Big)^j f, \text{ uniformly for } \tilde{x} \in \mathbb{G} \text{ and } t \text{ in compact sets .}$$

2. The sub-Riemannian structure as limit of its Riemannian approximation

In this section we recall that the sub-Riemannian structure $(\mathbb{G}, g_0)$ in a Carnot group can be interpreted as a degenerate limit (in the Gromov-Hausdorff sense) of Riemannian spaces $(\mathbb{G}, g_\epsilon)$ as $\epsilon \to 0$, [31]. In [14], Citti and Manfredini studied the relation between the sub-Riemannian Laplace operator and its Riemannian approximation. In particular they established bounds, uniform in the approximating parameter ϵ, for the fundamental solution of the $g_\epsilon-$Laplace operators. Here we recall a recent extension of these bounds to the degenerate parabolic case by Manfredini and the first two named authors [10].

We will work in the Carnot groups $\mathbb{G}$ and $\tilde{\mathbb{G}} = \mathbb{G} \times \mathbb{R}$, whose Lie algebras we denote by $\mathcal{G}$ and by $\tilde{\mathcal{G}}$. On $\tilde{\mathbb{G}}$ one can define a semi-norm $|\tilde{x}|_0 = |(x, x_{n+1})|_0 := |x|_0 + |x_{n+1}|$ where $|x|_0$ is as in (1.2), a distance $\tilde{d}_0(\tilde{x}, \tilde{y}) = |\tilde{y}^{-1}\tilde{x}|_0$, and dilations $\delta_\lambda(\tilde{x}) = (\delta_\lambda(x), \lambda x_{n+1})$ analogous to the same objects defined $\mathbb{G}$.

For each $\epsilon > 0$ set

$$X_1^\epsilon = X_1, \cdots, X_m^\epsilon = X_m, X_{m+1}^\epsilon = \epsilon X_{m+1}, \cdots X_n^\epsilon = \epsilon X_n, \text{ and } X_{n+1}^\epsilon = X_{n+1}$$

and $\nabla_\epsilon u = (X_i^\epsilon)_{i=1,\cdots,n}$. We extend both g_0 and $\tilde{g}_0$ to Riemannian metrics g_ϵ and $\tilde{g}_\epsilon$ on $\mathcal{G}$ and $\tilde{\mathcal{G}}$ by requesting that the vectors X_i^ϵ are an orthonormal family. The degree of these new vector fields is $d_\epsilon(X_i^\epsilon) = 1$ for $\epsilon > 0$ and $i = 1, ..., n+1$. The Carnot-Caratheodory distances (see [45]) associated to these vector fields can be proved to be equivalent to the pseudo-distances $d_\epsilon(x, y) := |y^{-1}x|_\epsilon$, and $\tilde{d}_\epsilon(\tilde{x}, \tilde{y}) := |\tilde{y}^{-1}\tilde{x}|_\epsilon$, associated to the non-homogeneous norms. For every $v \in \mathcal{G}$ and $v_{n+1} \in \mathbb{R}$,

$$|v|_\epsilon := \sum_{i=1}^m |v_i| + \sum_{i=m+1}^n \min\Big(\frac{|v_i|}{\epsilon}, |v_i|^{1/d(i)}\Big) \text{ and } |(v, v_{n+1})|_\epsilon := |v|_\epsilon + |v_{n+1}|. \tag{2.1}$$

It is then clear that for every $\epsilon > 0$ one has $d_1(x, y) \le d_\epsilon(x, y) \le d_0(x, y)$. Analogous considerations hold for $\tilde{d}_0$ and $\tilde{d}_\epsilon$. It can be shown (see [31] and references therein) that $(\tilde{\mathbb{G}}, \tilde{d}_\epsilon) \to (\tilde{\mathbb{G}}, \tilde{d}_0)$ in the Gromov-Hausdorff topology.

2.1. **Riemannian mean curvature of a graph.** As it is well known, the mean curvature of the graph of a regular function u_ϵ in the metric g_ϵ can be expressed in the form:

$$h_\epsilon = \frac{1}{\sqrt{1 + |\nabla_\epsilon u_\epsilon|^2}} \sum_{i,j=1}^n a_{ij}^\epsilon(\nabla_\epsilon u_\epsilon) X_i^\epsilon X_j^\epsilon u_\epsilon \text{ with } a_{ij}^\epsilon(\xi) = \left(\delta_{ij} - \frac{\xi_i \xi_j}{1 + |\xi|^2}\right)$$

From this expression it immediately follows that $h_\epsilon \to h_0$ as $\epsilon \to 0$.

Also note that classical results establish existence, uniqueness and regularity for solutions to the elliptic equation

$$u_\epsilon - \lambda \sum_{i,j=1}^n a_{ij}^\epsilon(\nabla_\epsilon u_\epsilon) X_i^\epsilon X_j^\epsilon u_\epsilon = f \tag{2.2}$$

for any regular domain of $\mathcal{G}$, which can be identified with $\mathbb{R}^n$ (see [30]).

2.2. **Riemannian and sub-Riemannian heat kernels estimates.** For $\epsilon \ge 0$, we denote by $\Delta_\epsilon = \sum_{d_\epsilon(i)=1} X_i^\epsilon$ the Laplace operator in $\tilde{G}$ defined in terms of the metric $\tilde{g}_\epsilon$. The subelliptic heat operator is defined in (1.8) as a sum of squares of vector fields

$$L = (\partial_t - \Delta_0)u$$

while its Riemannian counterpart is

$$L_\epsilon = (\partial_t - \Delta_\epsilon)u = (\partial_t - \sum_{d(i)=1} X_i^2 - \epsilon^2 \sum_{d(i)>1} X_i^2)u$$

Both operators have fundamental solutions, which we call Γ (or occasionally Γ_0) and Γ_ϵ respectively. Note that the fundamental solution is defined on $(\mathbb{G} \times \mathbb{R} \times \mathbb{R}^+)^2$. However due to the Lie group invariance, we can fix the fundamental solution with pole in $(0, 0)$, and the value in any other point will be obtained by translation invariance, i.e. for $\epsilon \ge 0$

$$\Gamma_\epsilon((\tilde{y}, t), (\tilde{x}, \tau)) = \Gamma_\epsilon((\tilde{x}^{-1}\tilde{y}, t - \tau), (0, 0)). \tag{2.3}$$

For simplicity we will drop the $(0, 0)$ term in the argument of Γ and consider the fundamental solution as a function defined on $\mathbb{R}^{n+1} \times \mathbb{R}^+$. Let us recall some well known properties of Γ, (see references in [37])

Proposition 5. *(Subelliptic Gaussian estimates $\epsilon = 0$) For all $\tilde{x} \in \mathbb{G} \times \mathbb{R}$ and $t > 0$ we have*

(1) *(Rescaling property)*

$$\Gamma(\tilde{x}, t) = \frac{1}{t^{(Q+1)/2}} \Gamma(\delta_t(\tilde{x}), 1)$$

(2) *There exist constants C_1, C_2 depending on $\mathbb{G}$ and g_0, such that, for all $\tilde{x} \in \mathbb{G}\times\mathbb{R}$ and $t>0$, we have*

$$t^{-(Q+1)/2}\, e^{-C_1 \frac{\tilde{d}_0^2(\tilde{x},0)}{t}} \le \Gamma(\tilde{x},t) \le t^{-(Q+1)/2}\, e^{-C_2 \frac{\tilde{d}_0^2(\tilde{x},0)}{t}} \tag{2.4}$$

and

$$|X^I \partial_t^\alpha \Gamma(\tilde{x},t)| \le C_1 t^{-(Q+1)/2-\frac{d(I)}{2}-\alpha} e^{-C_2^{-1}\tilde{d}_0^2(\tilde{x},0)/t},$$

where we have let $I=(i_1,...,i_l)\in\{1,...,m\}^l$, $X^I = X_{i_1}...X_{i_l}$, $d(I)=\sum_{h=1}^l d(i_h)$ and $Q=\sum_{i=1}^r i\dim(V^i)$ is the homogeneous dimension of $\mathbb{G}$.

(3) *For every $t>0$,*

$$\int_{\mathbb{G}\times\mathbb{R}} \Gamma(\tilde{x},t)\, d\tilde{x} \equiv 1. \tag{2.5}$$

This result implies in particular that for every fixed $\epsilon \ge 0$ Gaussian estimates of the heat kernel Γ^ϵ hold with constants C_1, C_2 depending on ϵ. In this paper we will need to use of uniform estimates of the heat kernels Γ^ϵ in terms of the limit kernel Γ (see [10]):

Proposition 6. *There exist constants $C_1, C_2>0$ depending on $\mathbb{G}, g_0$, such that for every $\epsilon>0$, $\tilde{x}\in\tilde{\mathbb{G}}$ and $t>0$ one has*

$$C_1^{-1}|B_\epsilon(0,\sqrt{t})|^{-1}\exp(-C_2\tilde{d}_\epsilon(\tilde{x},0)^2/t) \le \Gamma^\epsilon(\tilde{x},t) \le C_1|B_\epsilon(0,\sqrt{t})|^{-1}\exp(-C_2^{-1}\tilde{d}_\epsilon(\tilde{x},0)^2/t). \tag{2.6}$$

For any $l,\alpha\in\mathbb{N}$ and $l-$tuple $I=(i_1,...,i_l)\in\{1,...,n\}^l$, there exists $C_3=C_3(l,k,\mathbb{G},g_0)>0$ such that for every $\epsilon\ge 0$, $\tilde{x}\in\tilde{\mathbb{G}}$ and $t>0$ one has

$$|(X^\epsilon)^I\partial_t^\alpha\Gamma^\epsilon(\tilde{x},t)| \le C_3|B_\epsilon(0,\sqrt{t})|^{-1}t^{-\frac{l}{2}-\alpha}\exp(-C_2^{-1}\tilde{d}_\epsilon(\tilde{x},0)^2/t), \tag{2.7}$$

where we have let $(X^\epsilon)^I = X^\epsilon_{i_1}...X^\epsilon_{i_l}$.

Moreover, as $\epsilon\to 0$ one has

$$X^I_\epsilon\partial_t^\alpha\Gamma^\epsilon \to X^I\partial_t^\alpha\Gamma$$

uniformly on compact sets and in a dominated way on all $\mathbb{G}$. .

The proof of this result can be found in [10] and is directly inspired to the arguments introduced in [14], where the time independent case is studied, and a general procedure (reminiscent of the lifting technique in [49]) is introduced for lifting the operator to a higher dimensional space.

3. Nonlinear semigroups and horizontal mean curvature flow of graphs

In this section we will restate the definition of mean curvature operator and viscosity solution of mean curvature flow in the setting of non linear semi-group theory.

Definition 2. Let $\mathcal{B}$ be a Banach space, and A a $\mathcal{B}-$valued nonlinear operator with domain $D(A)\subset\mathcal{B}$. We say that $-A$ is $m-$dissipative if

- $R(I+\lambda A)=\mathcal{B}$ for every $\lambda>0$,
- its resolvent $J_\lambda=(I+\lambda A)^{-1}$ is a single-valued contraction

The curvature flow will be defined for regular functions, periodic in the following sense:

Definition 3. Let $\mathbb{G}$ be a Carnot group with a set of generators $\{e_1, ..., e_n\}$ closed under Lie brackets, and denote by $Q \subset \mathbb{G}$ a fundamental domain, i.e. an open bounded set such that $e_i Q \cap Q = \emptyset$ for all $i = 1, ..., n$ and $\mathbb{G} = \bigcup_\alpha e_1^{\alpha_1} ... e_n^{\alpha_n} \bar{Q}$, where the union is taken among all $n-$tuples of integers $\alpha = (\alpha_1, ..., \alpha_n)$. A function $u : \mathbb{G} \to \mathbb{R}$ is periodic with respect to Q and $e_1, ..., e_n$ if $u(e_i x) = u(x)$ for all $i = 1, ..., n$.

From now on $\mathcal{B}$ will denote the space of continuous periodic functions in a Carnot group $\mathbb{G}$, and $(\mathcal{B}, ||\cdot||)$ will be the Banach space obtained by endowing $\mathcal{B}$ with the sup norm $||\cdot||$.

Definition 4. We say that $u \in \mathcal{B}$ belongs to the domain of A if there exists $f \in \mathcal{B}$ and $\lambda > 0$ such that u is a weak solution (in the sense of Definition 5 below) of

$$u - \lambda \sum_{i,j=1}^{m} a_{ij}(\nabla_0 u) X_i X_j u = f, \text{ where } a_{ij}(\nabla_0 u) := \left(\delta_{ij} - \frac{X_i u X_j u}{1 + |\nabla_0 u|^2}\right) \tag{3.1}$$

in $\mathbb{G}$. In this case we will write

$$(I + \lambda A)u = f.$$

Definition 5. A continuous function $u : \mathbb{G} \to \mathbb{R}$ is a weak sub-solution (resp. a super-solution) of

$$u - \lambda \sum_{i,j=1}^{m} \left(\delta_{ij} - \frac{X_i u X_j u}{1 + |\nabla_0 u|^2}\right) X_i X_j u = f, \tag{3.2}$$

in $\mathbb{G}$ if for every $x \in \mathbb{G}$ and smooth $\phi : \mathbb{G} \to \mathbb{R}$ such that $u - \phi$ has a maximum (resp. a minimum) at x one must have

$$u(x) - \lambda \sum_{i,j=1}^{m} \left(\delta_{ij} - \frac{X_i \phi(x) X_j \phi(x)}{1 + |\nabla_0 \phi(x)|}\right) X_i X_j \phi(x) \le (resp. \ \ge) \ f(x).$$

Solutions are functions which are simultaneously super-solutions and a sub-solutions.

Proposition 7. *The operator $-A$ introduced in Definition 4 is $m-$dissipative*

Proof. We first prove that $||u - v|| \le ||J_\lambda u - J_\lambda v||$. For $\lambda > 0$ let $f = J_\lambda u = u + \lambda A u$ and $w = J_\lambda v = v + \lambda A v$, in the weak sense of Definition 4. For $\mu > 0$ set u^μ and v_μ the sup-convolutions defined in [53]:

$$u^\mu(x) = \sup_{y \in \mathbb{G}} \left(u(y) - \frac{1}{\mu} d_0^{2r!}(x, y)\right) \quad v_\mu := \inf_{y \in \mathbb{G}} \left(v(y) + \frac{1}{\mu} d_0(x, y)^{2r!}\right)$$

In view of Wang's results in [53] one has that u^μ is a super-solution of (3.1) and is semiconvex. For each $\alpha, \mu > 0$ we can then apply Jensen's lemma (see [38] or [16, Theorem 3.2]) to the semi-convex function in $\mathbb{G} \times \mathbb{G}$,

$$\phi_{\mu,\alpha}(x, y) := u^\mu(x) - v_\mu(y) - \frac{\alpha}{2}|x - y|_E^2,$$

where $|\cdot|_E$ denotes the Euclidean norm, and obtain a sequence of points $p_j^{\mu,\alpha} = (x_j^{\mu,\alpha}, y_j^{\mu,\alpha}) \to p^{\mu,\alpha} = (x^{\mu,\alpha}, y^{\mu,\alpha})$ with $p^{\mu,\alpha}$ a maximum point for the function $\phi_{\mu,\alpha}$ which in turn is twice differentiable at $p_j^{\mu,\alpha}$ and satisfies

$$|\nabla_E \phi_{\mu,\alpha}(p_j^{\mu,\alpha})| = o(1) \text{ and } -\gamma I \le D_E^2 \phi^{\mu,\alpha}(p_j^{\mu,\alpha}) \le o(1) I \text{ as } j \to \infty,$$

where ∇_E and D^2_E denote respectively the Euclidean gradient and Hessian, for some choice of $\gamma > 0$. This implies that for every $i, h = 1, ..., m$

$$\lim_{j\to\infty} |X_i u^\mu(x_j^{\mu,\alpha}) - X_i v_\mu(y_j^{\mu,\alpha})| = 0 \text{ and } \lim_{j\to\infty} X_i X_h u^\mu(x_j^{\mu,\alpha}) - X_i X_h v_\mu(y_j^{\mu,\alpha}) \leq 0.$$

Since by definition u and v are solution of the PDE's $f = u + \lambda A u$ and $w = v + \lambda A v$ respectively, so that u^μ and v_μ satisfy

$$u^\mu - f \leq \lambda \sum_{i,k=1}^m a_{ik}(\nabla_0 u^\mu) X_i X_k u^\mu \text{ at } x_j^{\mu,\alpha} \text{ and } v_\mu - w \geq \lambda \sum_{i,k=1}^m a_{ik}(\nabla_0 v_\mu) X_i X_k v_\mu \text{ at } y_j^{\mu,\alpha}.$$

Subtracting the first inequality from the second and using the following fact proved in [16, Lemma 3.2]

$$\lim_{\alpha\to\infty,\mu\to 0} \alpha|x^{\mu,\alpha} - y^{\mu,\alpha}| = 0 \text{ and } \lim_{\alpha\to\infty,\mu\to 0} \sup_{\mathbb{G}\times\mathbb{G}} \phi_{\mu,\alpha}(x,y) = \sup_{x\in\mathbb{G}}(u(x) - v(x)),$$

one concludes that $||u - v|| \leq ||f - w||$, which is the first part in the definition of $m-$dissipative operator.

We now establish that $R(I + \lambda A) = \mathcal{B}$. Let $f \in \mathcal{B} \cap C^\infty(\mathbb{G})$ and for each $\epsilon > 0$ consider weak solutions u_ϵ of the approximating elliptic PDE (2.2). A simple variant of the argument above shows that for every $\epsilon > 0$ one has $||u_\epsilon|| \leq ||f||$ and, in view of the fact that the vector fields $X_i's$ are left-invariant, that

$$\forall y \in \mathbb{G} \ \sup_{x\in\mathbb{G}} |u_\epsilon(yx) - u(y)| \leq \sup_{x\in\mathbb{G}} |f(yx) - f(y)|.$$

The latter proves that $\{u_\epsilon\}_\epsilon$ is equi-continuous and equi-bounded. For a subsequence one has $u_{\epsilon_k} \to u \in \mathcal{B}$ as $\epsilon_k \to \infty$ and the argument in [9, Theorem 5.2] shows that u is a weak solution of (3.1). In view of the comparison principle we deduce that the range of $I + \lambda A$ is closed, hence it is the whole of $\mathcal{B}$.

□

The proof of the following proposition is exactly as the proof of the first part of [27, Theorem 2.4].

Proposition 8. *The domain of A is dense in $\mathcal{B}$.*

Propositions 7 and 8 allow us to invoke the previously recalled Generation Theorem of Crandall and Liggett (see (1.7) above), and ensure that $-A$ generates a non linear semi-group $M(t)(f)$.

Proof of Proposition 1

Let $\phi : \mathbb{G} \times (0,\infty) \to \mathbb{R}$ be a smooth function such that $u - \phi$ has a maximum point at $(x_0, t_0) \in \mathbb{G} \times (0,\infty)$. We can always assume that the maximum is strict adding a power of the gauge distance as in [9]. We set

$$u^s(x,t) = \Big((I + \frac{1}{s}A)^{-k} f\Big)(x) \text{ for all integers } s, k \text{ and } t \in \Big[\frac{k}{s}, \frac{k+1}{s}\Big)$$

and note that for a fixed $t > 0$, as $s \to \infty$ one has $k \to \infty$ and $k/s \to t$. By Crandall and Liggett's convergence result one has that $u^s \to u$ uniformly globally. As a consequence one can find a sequence $(x_s, t_s) \in \mathbb{G} \times (0,\infty)$ such that:

(i) $(x_s, t_s) \to (x_0, t_0)$;
(ii) For fixed $t_s > 0$ the function $(u^s - \phi)(\cdot, t_s)$ has a maximum at x_s;

(iii) $(u^s - \phi(x_s, t_s) + \frac{1}{s^2} \geq (u^s - \phi)(x,t)$ for all $(x,t) \in \mathbb{G} \times (0, \infty)$.

Following [27, Theorem 2.5] for each s we let k_s be such that $t_s \in [\frac{k_s}{s}, \frac{k_{s+1}}{s})$ and in particular $t_s - \frac{1}{s} \in [\frac{k_{s-1}}{s}, \frac{k_s}{s})$. This choice yields

$$u^s(x, t_s) = (I + \frac{1}{s}A)^{-1} u^s(\cdot, t_s - \frac{1}{s})(x),$$

in the weak sense of Definition 4. By virtue of (ii) and (iii) above one has

$$\frac{\phi(x_s, t_s) - \phi(x_s, t_s - \frac{1}{s})}{\frac{1}{s}} - \sum_{i,j=1}^{m} \left(\delta_{ij} - \frac{X_i \phi X_j \phi}{1 + |\nabla_0 \phi|^2} \right) X_i X_i \phi|_{(x_s, t_s)} \leq \frac{1}{s}.$$

As $s \to \infty$, the latter completes the proof that u is a sub-solution. In a similar fashion one can prove that u is also a super-solution and conclude the proof.

4. A discrete time-step operator based on the heat flow

4.1. **Flow of subgraphs of functions belonging to $\mathcal{B}$.** Let $(\mathcal{B}, ||\cdot||)$ be the Banach space introduced in Definition 4. If $f \in \mathcal{B}$, we call subgraph of $\mathcal{B}$ the subset of $\mathbb{G} \times \mathbb{R}$ defined as

$$S = \{\tilde{x} = (x, x_{n+1}) : x_{n+1} \leq f(x)\} \subset \mathbb{G} \times \mathbb{R}. \tag{4.1}$$

We give here the definition of heat flow of a subgraph S defined in (4.1), as an extension of the analogous notion given by Evans in the Euclidean case [27]. For all $t > 0$ denote $w(x,t)$ the solution of the Cauchy problem

$$\begin{cases} \Delta w &= \partial_t w \\ w(.,0) &= \chi_S \end{cases}$$

and define

$$\mathcal{H}(t)(S) = \{\tilde{x} \in \mathbb{G} \times \mathbb{R} : w(\tilde{x}, t) \geq \frac{1}{2}\} \tag{4.2}$$

In order to prove that this functional induces a flow on $\mathcal{B}$, we need to show that if the subgraph of a periodic, continuous function $f : \mathbb{G} \to \mathbb{R}$ is evolved through the flow $\mathcal{H}(t)$ then the new set is also the subgraph of a periodic continuous function.

Lemma 1. Let $f \in \mathcal{B}$. For any $t > 0$ there exists a function $g \in \mathcal{B}$ such that

$$\mathcal{H}(t)\{x_{n+1} < f(x)\} = \{x_{n+1} < g(x)\}.$$

Proof. Let $(y, y_{n+1}) \in \partial\{\mathcal{H}(t)\{x_{n+1} < f(x)\}$ so that

$$\frac{1}{2} = \int_{\{x_{n+1} < f(x)\}} \Gamma(\tilde{x}^{-1}\tilde{y}, t) d\tilde{x} = \int_{\mathbb{G}} \int_{-\infty}^{f(x)} \Gamma(x^{-1}y, y_{n+1} - x_{n+1}, t) dx_{n+1} dx.$$

If we denote

$$F_f(y, y_{n+1}) := \int_{\mathbb{G}} \int_{-\infty}^{f(x)} \Gamma(x^{-1}y, y_{n+1} - x_{n+1}, t) dx dx_{n+1}. \tag{4.3}$$

differentiating along the variable y_{n+1} yields

$$\partial_{y_{n+1}} F_f(y, y_{n+1}) = \int_{\mathbb{G}} \Gamma(x^{-1}y, y_{n+1} - f(x), t) dx < 0.$$

The implicit function theorem then implies that in a small neighborhood of every point $z \in \mathbb{G}$, there exists a continuous function $g : B(z,r) \subset \mathbb{G} \to \mathbb{R}$ such that $F_f(y, g(y)) = 1/2$ for all $y \in B(z,r)$. On the other hand, by the strict monotonicity of the function F_f as a function of its last variable, we immediately deduce that the function g is defined on the whole group $\mathbb{G}$ with real values.

□

This lemma allows to give the following definition of flow of functions in $\mathcal{B}$:

Definition 6. If $f \in \mathcal{B}$ we define the operator

$$H(t) : \mathcal{B} \to \mathcal{B}, \quad H(t)(f) = g,$$

where g is the unique function satisfying

$$\mathcal{H}(t)\{x_{n+1} < f(x)\} = \{x_{n+1} < g(x)\}.$$

Thanks to the explicit expression of H we can establish comparison principle properties of the flow H.

Theorem 3. *For each $t \geq 0$ the flow $H(t) : \mathcal{B} \to \mathcal{B}$ just defined has the following properties*

1. *If $f \leq g$ then $H(t)f \leq H(t)g$*
2. *If C is a real constant, then $H(t)(f + C) = H(t)f + C$*
3. *$H(t)$ is a contraction on $\mathcal{B}$*

$$||H(t)f - H(t)g|| \leq ||f - g||$$

Proof. Let us show assertion (1). If $f, g \in \mathcal{B}$, and $f \geq g$ we can use Lemma 1, and represent the function F_g defined in (4.3) as

$$F_g(y, y_{n+1}) = F_f(y, y_{n+1}) + \int_{\mathbb{G}} \int_{f(x)}^{g(x)} \Gamma(x^{-1}y, y_{n+1} - x_{n+1}, t) dx dx_{n+1}.$$

Since from the definition of F_f and $H(t)f$ we have

$$F_f(x, H(t)f(x)) = F_g(x, H(t)g(x)) = 1/2,$$

we immediately get

$$F_f(x, H(t)f(x)) = F_g(x, H(t)g(x)) \geq F_f(x, H(t)g(x)).$$

By the strict monotonicity of F_f in its last variable we deduce:

$$H(t)f \leq H(t)g.$$

In order to prove assertion (2) we note that

$$F_f(y, y_{n+1}) = \int_{\mathbb{G}} \int_{-\infty}^{f(x)} \Gamma(x^{-1}y, y_{n+1} - x_{n+1}, t) dx dx_{n+1} =$$

with the change of variable $z_{n+1} = x_{n+1} + C$,

$$= \int_{\mathbb{G}} \int_{-\infty}^{f(x)+C} \Gamma(x^{-1}y, y_{n+1} + C - z_{n+1}, t) dx dx_{n+1} = F_{f+C}(y, y_{n+1} + C)$$

For $y_{n+1} = H(t)f(y)$ we get

$$F_{f+C}(y, H(t)f(y) + C) = 1/2,$$

which implies by the uniqueness that $H(t)(f)(y) + C = H(t)(f + C)(y)$.

Finally we prove assertion (3). Let us choose x such that

$$H(t)f(x) - H(t)g(x) > ||H(t)f - H(t)g|| - \epsilon$$

and call

$$H(t)f(x) = \mu \quad H(t)g(x) = \lambda.$$

By assertion (2) we have

$$H(t)f(x) - H(t)(g - \lambda + \mu - \epsilon)(x) > 0$$

By assertion (1) this implies that there exists a point y such that

$$f(y) - (g(y) - \lambda + \mu - \epsilon) > 0,$$

which by definition of λ and μ implies

$$||H(t)f - H(t)g|| < H(t)f(x) - H(t)g(x) + \epsilon < f(y) - g(y) + 2\epsilon \leq ||f - g|| + 2\epsilon$$

Assertion (3) follows. □

5. Sub-Riemannian diffusion of smooth non-characteristic graphs

In this section we prove the geometrical core of the paper, which is the equality of the normal speed of a graph evolving by mean curvature and the normal speed of the same graph evolving under the heat flow defined in (4.2). According to the notation introduced in section 2, for $f \in \mathcal{B}$ we will denote by $d\sigma_1$ the surface measure element on $\partial S = \{(x, f(x)) \in \mathbb{G}\times\mathbb{R} | x \in \mathbb{G}\}$ induced by the metric g_ϵ, with $\epsilon = 1$ and by $d\sigma_0$ the horizontal perimeter measure $d\sigma_0 = \frac{\sqrt{1+|\nabla_0 u|^2}}{\sqrt{1+|\nabla_1 u|^2}} d\sigma_1$. The outer horizontal unit normal ν_0 to S is expressed in the frame $X_1, ..., X_{n+1}$ as $\nu_0 = (-\nabla_0 f, 1)/\sqrt{1 + |\nabla_0 f|^2}$.

Lemma 2. If $f : \mathbb{G} \to \mathbb{R}$ is a smooth function, with sub-graph S, then for every $i = 1, ..., m$ one has

$$\int_S v(x)X_i u(x)dx = -\int_S u(x)X_i v(x)dx + \int_{\partial S} u(x)v(x)\nu_i^0 d\sigma_0(x),$$

for any smooth functions u, v for which the integrals are defined.

Proof. Integration by parts yields

$$\int_S v(x)X_i u(x)dx = -\int_S u(x)X_i v(x)dx + \int_{\partial S} u(x)v(x)\frac{X_i f}{\sqrt{1 + |\nabla_1 u|^2}} d\sigma_1(x)$$
$$= -\int_S u(x)X_i v(x)dx + \int_{\partial S} u(x)v(x)\frac{X_i f}{\sqrt{1 + |\nabla_0 f|^2}} d\sigma_0(x).$$

□

Lemma 3. Let $\mathcal{B}$ be the space of periodic functions defined in Definition 4. If $f \in \mathcal{B} \cap C^2(\mathbb{G})$ let S its sub-graph (defined in (4.1), and let $\tilde{x} = (x, x_{n+1}) = (x, f(x))$ be a fixed point in ∂S. Given $L > 0$ for every $w \in Lip(\mathbb{G} \times \mathbb{R})$ with Lipschitz norm bounded by L, one has

$$\int_{\partial S} \Gamma(\tilde{x}^{-1}\tilde{y}, \tau)w(\tilde{y})\, d\sigma_0(\tilde{y}) = \frac{w(\tilde{x})}{\sqrt{\tau}} \int_\Pi \Gamma(z, 1)\, d\sigma_0(z) + O(1)$$

as $\tau \to 0$ uniformly for $\tilde{x} \in \partial S$. Here Π is the intrinsic tangent plane $y_{n+1} = \Pi(y) := f(x) + \sum_{d(i)=1} X_i f(x)(y_i - x_i)$ to ∂S at $\tilde{x}$.

Proof. Since ∂S is a graph, the integration variables are of the form $\tilde{y} = (y, f(y))$. Note that

$$(\tilde{x}^{-1}\tilde{y}) = (x^{-1}y, f(y) - f(x)) = (x^{-1}y, \sum_{d(i)=1} X_i f(x)(y-x)_i + O(|y^{-1}x|^2)).$$

Consequently, setting $\tilde{z} = \delta_{\tau^{-1/2}}(\tilde{x}^{-1}\tilde{y})$ one obtains

$$\begin{aligned}\tilde{z} &= (z, z_{n+1}) = (\delta_{\tau^{-1/2}}(x^{-1}y), \tau^{-1/2}(f(y) - f(x))) \\ &= (\delta_{\tau^{-1/2}}(x^{-1}y), \sum_{d(i)=1} X_i f(x)(\delta_{\tau^{-1/2}}(x^{-1}y))_i + O(\sqrt{\tau}|\tilde{z}|^2) = (z, \Pi(z) + O(\sqrt{\tau}|z|^2)).\end{aligned} \tag{5.1}$$

In view of the latter and using the scaling properties of the heat kernel one has

$$\begin{aligned}\Gamma(\tilde{x}^{-1}\tilde{y}, \tau) &= \tau^{-Q/2-1/2}\Gamma(\tilde{z}, 1) = \tau^{-Q/2-1/2}\,\Gamma((z, \Pi(z) + O(\sqrt{\tau}|z|^2), 1) \\ &= \tau^{-Q/2-1/2}\Big(\Gamma((z, \Pi(z)), 1) + R(z)\Big)\end{aligned} \tag{5.2}$$

where R is a remainder satisfying $\int_{\mathbb{G}} R dz = O(\sqrt{\tau}|z|^2)$. Consider the surface integral

$$\begin{aligned}&\int_{\partial S} \Gamma(\tilde{x}^{-1}\tilde{y}, \tau) w(\tilde{y})\, d\sigma_0(\tilde{y}) \\ &= w(\tilde{x}) \int_{\partial S} \Gamma(\tilde{x}^{-1}\tilde{y}, \tau)\, d\sigma_0(\tilde{y}) + O\Big(\int_{\mathbb{G}\times\mathbb{R}} |\tilde{x}^{-1}\tilde{y}| \Gamma(\tilde{x}^{-1}\tilde{y}, \tau)\, d\sigma_0(\tilde{y})\Big) \\ &= w(\tilde{x}) \int_{\mathbb{G}} \Gamma(x^{-1}y, f(y) - x_{n+1}, \tau)\sqrt{1 + |\nabla_0 f(y)|^2}\, dy + \\ &\quad + O\Big(\int_{\mathbb{G}} |(x^{-1}y, f(y) - x_{n+1})|\, \Gamma(x^{-1}y, f(y) - x_{n+1}, \tau)\sqrt{1 + |\nabla_0 f(y)|^2}\, dy\Big)\end{aligned} \tag{5.3}$$

Applying the change of variable $z = \delta_{\sqrt{\tau}}(x^{-1}\, y)$ we have $dy = \tau^{Q/2} dz$ and obtain

$$\begin{aligned}&\int_{\mathbb{G}} \Gamma(x^{-1}y, f(y) - x_{n+1}, \tau)\sqrt{1 + |\nabla_0 f(y)|^2} dy \\ &= \frac{1}{\sqrt{\tau}} \int_{\mathbb{G}} \Big(\Gamma((z, \Pi(z)), 1) + R(z)\Big) \sqrt{1 + |\nabla_0 f(x + \delta_{\sqrt{\tau}}(z))|^2} dz \\ &= \frac{1}{\sqrt{\tau}} \int_{\mathbb{G}} \Big(\Gamma((z, \Pi(z)), 1) + R(z)\Big) \sqrt{1 + |\nabla_0 \Pi|^2} dz \; + O(1) \\ &= \frac{1}{\sqrt{\tau}} \int_{\Pi} \Gamma(\tilde{z}, 1) d\sigma_0(\tilde{z}) + O(1).\end{aligned} \tag{5.4}$$

where in the last line we have used the approximation $\nabla f(x + \delta_{\sqrt{\tau}}(z)) = \nabla f(x) + O(\sqrt{\tau}|z|)$ and we have denoted $\tilde{z} = (z, \Pi(z))$. The proof follows immediately from the latter and from (5.3). □

We are now ready to establish the key geometric identity needed in the proof of Proposition 3 and eventually to prove Theorem 2.

Remark 2. The previous lemma indicates that it would be plausible to have an identity of the form

$$\int_{\partial S} |\nabla\Gamma(\tilde{x}^{-1}\tilde{y},\tau)| w(\tilde{y})\, d\sigma_0(\tilde{y}) = \frac{Cw(\tilde{x})}{\tau} + O(\frac{1}{\sqrt{\tau}})$$

However in the next lemma we show that for integrands with a special structure one can improve on such an identity an obtain a better decay rate.

Lemma 4. Consider $f \in \mathcal{B}$ a C^3 function and suppose that the horizontal mean curvature h_0 of its graph is bounded by a positive constant L. Let S denote its sub-graph, defined in (4.1), and consider $\tilde{x} = (x, x_{n+1}) = (x, f(x))$ a fixed point in ∂S. Then one has

$$\int_{\partial S} \langle \nabla_0\Gamma(\tilde{x}^{-1}\tilde{y},\tau), \nu_0(\tilde{y})\rangle_{\tilde{g}_0} d\sigma_0(\tilde{y}) = -\frac{h_0(\tilde{x})}{2\sqrt{\tau}}\int_{\Pi} \Gamma(\tilde{x}^{-1}\tilde{z},1) d\sigma_0(\tilde{z}) + O(1) \tag{5.5}$$

as $\tau \to 0$, and the convergence is uniform with respect to the variable $\tilde{x} \in \partial S$, but depends on the parameter L. Here Π is the intrinsic tangent plane $y_{n+1} = \Pi(z) := f(x) + \sum_{d(i)=1} X_i f(x)(y_i - x_i)$ to S at the point $\tilde{x}$, $d\sigma_0$ denotes the horizontal perimeter measure of Π associated to $X_1, ..., X_m, \partial_{n+1}$.

Remark 3. Note that if $\tilde{x} \notin \partial S$ then there exists $\alpha = \alpha(S, \tilde{x}) > 0$ such that

$$\int_{\partial S} \langle \nabla_0\Gamma(\tilde{x}^{-1}\tilde{y},\tau), \nu_0(\tilde{y})\rangle_{\tilde{g}_0} d\sigma_0(\tilde{y}) = O(e^{-\alpha/\tau}). \tag{5.6}$$

In particular the left-hand side is uniformly bounded by $O(\tau^{-1/2})$ for any point $\tilde{x} \in \mathbb{G} \times \mathbb{R}$.

Proof. The proof proceeds in three steps. We will start considering the extremely simplified case in which the group $\mathbb{G}$ is the Euclidean space and the horizonal tangent plane coincides with the whole tangent plane, so that the sub-Riemannian metric reduces to a Riemannian one, and $m = n$. For reader convenience we will keep also in this case the same notations used in the rest of the paper. In the first step the metric will be constant, in the second step we consider an arbitrary Riemannian metric, while the subriemannian case will be studied in the third step as limit of the Riemannian ones.

Step 1 Let us first assume that $\mathbb{G} = \mathbb{R}^n$ with the Euclidean group structure and that we have a constant coefficient Riemannian metric $\tilde{g}_0$ defined on the whole tangent space of $\mathbb{G} \times \mathbb{R}$. Up to a change of variables we can assume that $\tilde{x}$ is the origin, that $(X_1, \cdots X_n) = (\partial_{y_1}, \cdots, \partial_{y_n})$ is an orthonormal frame for $T_0\partial S$ and X_{n+1} coincides with the outer unit normal ν_0 to S at 0. In these coordinates the matrix associated to the metric $\tilde{g}_0$ is the identity, the heat operator (defined in Section 2) simply reduces to the standard Euclidean one $\partial_t - X_i^2$.

There exists a neighborhood $U \subset \mathbb{G} \times \mathbb{R}$ of the origin where $U \cap \partial S$ can be represented as a graph of a smooth function over $U \cap T_0\partial S = U \cap \{\tilde{y}_{n+1} = 0\}$. We will still denote by f this function and by $(y, f(y))$ its graph. The normal is expressed $\nu_0(\tilde{y})d\sigma_0(\tilde{y}) = (-\nabla_0 f(y), 1)dy$ and, by the choice of coordinates, $\nabla_0 f(0) = 0$. It follows that

$$\int_{\partial S} \langle \nabla_0\Gamma(\tilde{x}^{-1}\tilde{y},\tau), \nu_0(\tilde{y})\rangle_{\tilde{g}_0} d\sigma_0(\tilde{y}) = \int_{\partial S\cap U} \langle \nabla_0\Gamma(\tilde{x}^{-1}\tilde{y},\tau), \nu_0(\tilde{y})\rangle_{\tilde{g}_0} d\sigma_0(\tilde{y}) + O(e^{-\alpha/t})$$

(where $\alpha > 0$ is a suitable constant)

$$= -\int_{U\cap T_0\partial S} \frac{\langle (y, f(y)), (-\nabla_0 f(y), 1)\rangle_I}{2t}\Gamma(y, f(y), t)dy + O(e^{-\alpha/t})$$

where $\langle \cdot, \cdot \rangle_I$ denotes the Euclidean scalar product. Extending in a periodic fashion f to all of $T_0 \partial S$ and identifying the latter with $\mathbb{R}^n$ one obtains

$$= - \int_{\mathbb{R}^n} \frac{f(y) - y\nabla_0 f(y)}{2t} \Gamma(y, f(y), t) dy + O(e^{-\alpha/t})$$

The integrand function can be expanded as follows:

$$f(y) - y\nabla_0 f(y) = -\frac{1}{2} y_i y_j \partial_{ij} f(0) + O(|y|^3)$$

Applying the change of variable $y = \sqrt{t} z$, and the scaling property of the heat kernel in $\mathbb{R}^{n+1}$, we arrive at

$$\int_{\partial S} \langle \nabla_0 \Gamma(\tilde{y}, \tau), \nu_0(\tilde{y}) \rangle_{\tilde{g}_0} d\sigma_0(\tilde{y}) = \frac{\partial_{ij} f(0)}{2\sqrt{t}} \left(\int_{\mathbb{R}^n} \frac{z_i z_j}{2} \Gamma(z, 0, 1)\, dz + O(\sqrt{t}) \right)$$
$$= \frac{\partial^2_{ii} f(0)}{2\sqrt{t}} \int_{\Pi} \Gamma(z, 0, 1) d\sigma_0(z) + O(1) = -\frac{h(\tilde{x})}{2\sqrt{\tau}} \int_{\Pi} \Gamma(\tilde{z}, 1) d\sigma_0(z) + O(1),$$

here we have used the fact that in this system of coordinates $h_0(0) = -\partial_{ii} f(0)$. This concludes the proof in the constant metric case. We explicitly remark that in the setting of step 1 the hyperplane Π coincides with the tangent plane to S at $\tilde{x}$.

Step 2 Let us now assume that the metric $\tilde{g}_0 = \tilde{E}^T \tilde{E}$ is an arbitrary Riemannian metric on $\mathbb{R}^{n+1}$, invariant with respect to a Lie group $\mathbb{G}$. We can define an orthonormal frame in $\mathbb{R}^{n+1}$ by setting $X_i = (\tilde{E}^{-1})^T_{ij} \partial_j$, with $i, j = 1, ..., n+1$. The associated heat operator now contains first order terms

$$L = \partial_t - \sum_{i=1}^{n+1} X_i^2 = \partial_t - \tilde{g}^{kj} \partial^2_{jk} - \tilde{E}^{-1}_{ji} \partial_j (\tilde{E}^{-1})^T_{ik} \partial_k = \partial_t - \tilde{g}^{kj} \partial^2_{jk} - \tilde{a}_k \partial_k$$

where $\{\tilde{g}^{ij}\}$ denotes the inverse of g, and $\tilde{a}_k = \tilde{E}^{-1}_{ji} \partial_j (\tilde{E}^{-1})^T_{ik}$. As in step 1 we can assume without loss of generality that $\nabla_0 f(\tilde{x}) = 0$. Consequently $-a_k(\tilde{x}) = 0$ and the associated constant coefficient operator, obtained evaluating the coefficients at the point $\tilde{x}$, reduces to:

$$L_{\tilde{x}} = \partial_t - \sum_{i=1}^{n+1} X^2_{\tilde{x},i} = \partial_t - g^{kj}(\tilde{x}) \partial^2_{jk},$$

where we have denoted $X_{\tilde{x},i}$ the vector fields with coefficients evaluated at the point $\tilde{x}$. We will denote by $\nabla_{\tilde{x}} = (X_{\tilde{x},1}, \cdots X_{\tilde{x},n+1})$ the gradient along such frozen coefficients vector fields. We will also denote by Γ and $\Gamma_{\tilde{x}}$ the heat kernels of L and $L_{\tilde{x}}$. Finally we denote by h_0 and $h_{0,\tilde{x}}$ the mean curvatures of the graph with respect to the metric g_0 and with respect to the frozen metric $g(\tilde{x})$. Note that at the point $\tilde{x}$ one has $h_{\tilde{x}}(\tilde{x}) = h(\tilde{x})$.

We want to emphasize that in this step of the proof we are never going to use the invariance properties of the heat operator L and its fundamental solution Γ. Because of this, the group structure itself becomes irrelevant and the fundamental solutions will be considered as a function defined on $(\mathbb{R}^{n+1} \times \mathbb{R}^+)^2$, thus requiring the notation $\Gamma((\tilde{y}, t), (\tilde{x}, \tau))$ (see also discussion at the beginning of Section 2 and (2.3)).

Using the Levi's parametrix method (see for instance [43], [29]), it is known that one has

$$\Gamma((\tilde y,t),(\tilde x,\tau))=\Gamma_{\tilde x}((\tilde y,t),(\tilde x,\tau))+O\Big(\frac{1}{(t-\tau)^{n/2}}e^{-C_2\frac{|\tilde y-\tilde x|^2}{t-\tau}}\Big). \tag{5.7}$$

More precisely

$$\Gamma((\tilde y,t),(\tilde x,\tau))=\Gamma_{\tilde x}((\tilde y,t),(\tilde x,\tau))+J((\tilde y,t),(\tilde x,\tau)) \tag{5.8}$$

where

$$J((\tilde y,t),(\tilde x,\tau))=\int_{\mathbb{R}^{n+1}\times[\tau,t]}\Gamma_{\tilde z}((\tilde y,t),(\tilde z,s))Z_1((\tilde z,s),(\tilde x,\tau))d\tilde z ds+O\Big(\frac{1}{(t-\tau)^{(n-1)/2}}e^{-C_2\frac{|\tilde y-\tilde x|^2}{t-\tau}}\Big)$$

and

$$Z_1((\tilde y,t),(\tilde x,\tau))=\sum_{ij}(\tilde g^{ij}(\tilde y)-\tilde g^{ij}(\tilde x))\partial_{ij}\Gamma_{\tilde x}((\tilde y,t),(\tilde x,\tau))+\sum_i(\tilde a^i(\tilde y)-\tilde a^i(\tilde x))\partial_i\Gamma_{\tilde x}((\tilde y,t),(\tilde x,\tau)),$$

so that, for some constant $C=C(\mathbb{G},g)>0$,

$$|Z_1((\tilde y,t),(\tilde x,\tau))|\le C\frac{\Gamma((\tilde y,t),(\tilde x,\tau))}{\sqrt{t-\tau}}.$$

Clearly one also has

$$\nabla_0\Gamma((\tilde y,t),(\tilde x,\tau))=\nabla_0\Gamma_{\tilde x}((\tilde y,t),(\tilde x,\tau))+\nabla_0 J((\tilde y,t),(\tilde x,\tau)). \tag{5.9}$$

Let us consider the integral

$$\int_{\partial S}\langle\nabla_0 J((\tilde y,t),(\tilde x,\tau)),\nu_0(\tilde y)\rangle_{\tilde g_0}d\sigma_0(\tilde y)= \tag{5.10}$$

$$=\int_{\partial S}\langle\Big(\int_{\mathbb{R}^{n+1}\times[\tau,t]}\nabla\Gamma_{\tilde z}((\tilde y,t),(\tilde z,s))Z_1((\tilde z,s),(\tilde x,\tau))d\tilde z ds\Big),\ \nu_0(\tilde y)\rangle_{\tilde g_0}d\sigma_0(\tilde y)=$$
$$=\int_{\mathbb{R}^{n+1}\times[\tau,t]}\Big(\int_{\partial S}\langle\nabla_0\Gamma_{\tilde z}((\tilde y,t),(\tilde z,s)),\nu_0(\tilde y)\rangle_{\tilde g_0}d\sigma_0(\tilde y)\Big)Z_1((\tilde z,s),(\tilde x,\tau))d\tilde z ds.$$

In view of Remark 3 there exists $C=C(\mathbb{G},g,S)>0$ such that

$$\Big|\int_{\partial S}\langle\nabla_0 J((\tilde y,t),(\tilde x,\tau)),\nu_0(\tilde y)\rangle_{\tilde g_0}d\sigma_0(\tilde y)\Big|\le C\int_{\mathbb{R}^{n+1}\times[\tau,t]}\frac{|Z_1((\tilde z,s),(\tilde x,\tau))|}{\sqrt{t-s}}d\tilde z ds+O(e^{-\alpha/(t-\tau)})=$$
$$\le C\int_\tau^t\frac{1}{\sqrt{(t-s)(s-\tau))}}\int_{\mathbb{R}^{n+1}}\Gamma_{\tilde z}((\tilde y,t),(\tilde z,s))d\tilde z ds+O(e^{-\alpha/(t-\tau)})=$$
$$=O\Big(\int_\tau^t\frac{ds}{\sqrt{(t-s)(s-\tau)}}\Big)+O(e^{-\alpha/(t-\tau)})=O\Big(\int_0^1\frac{dr}{\sqrt{r(1-r)}}\Big),$$

where, in the last line, we have used the change of variable $r=(s-\tau)/(t-\tau)$.

Consequently,

$$\int_{\partial S}\langle\nabla_0\Gamma((\tilde y,t),(\tilde x,\tau)),\nu_0(\tilde y)\rangle_{\tilde g_0}d\sigma_0(\tilde y)=$$
$$=\int_{\partial S}\langle\nabla_0\Gamma_{\tilde x}((\tilde y,t),(\tilde x,\tau)),\nu_0(\tilde y)\rangle_{\tilde g_0}d\sigma(\tilde y)+$$

$$+\int_{\partial S}\langle\nabla_0 J((\tilde y,t),(\tilde x,\tau)),\nu_0(\tilde y)\rangle_{\tilde g_0}d\sigma_0(\tilde y)+O\Bigg(\int_{\partial S}\frac{1}{(t-\tau)^{n/2}}e^{-C_2\frac{|\tilde x-\tilde y|^2}{t-\tau}}d\sigma(\tilde y)\Bigg).$$

Denoting by $d\sigma_{g(\tilde x)}$ the surface measure corresponding to the frozen metric $g(\tilde x)$ and using the statement in Step 1, one has

$$\begin{aligned}\int_{\partial S}\langle\nabla_0\Gamma((\tilde y,t),(\tilde x,\tau)),\nu_0(\tilde y)\rangle_{\tilde g_0}d\sigma_0(\tilde y) &= -\frac{h(\tilde x)}{2\sqrt{t-\tau}}\int_{\Pi}\Gamma_{\tilde x}((\tilde y,1),(\tilde x,0))d\sigma_{g(\tilde x)}(\tilde y)+O(1)\\ \text{(using Euclidean rescaling)}\qquad &= -\frac{h(\tilde x)}{2}\int_{\Pi}\Gamma_{\tilde x}((\tilde y,t-\tau),(\tilde x,0))d\sigma_{g(\tilde x)}(\tilde y)+O(1)=\end{aligned}\tag{5.11}$$

(by estimate (5.7))

$$-\frac{h(\tilde x)}{2}\int_{\Pi}\Gamma((\tilde y,t-\tau),(\tilde x,0))d\sigma(\tilde y)+O(1)\tag{5.12}$$

We explicitly remark that also in the setting of step 2 the hyperplane Π coincides with the tangent plane to S at $\tilde x$.

Step 3 Let us now assume that Γ is a sub-Riemannian heat kernel corresponding to a sub-Riemannian metric g_0 in the Carnot group $\mathbb{G}$. From Section 2 one has a sequence of left-invariant Riemannian metrics g_ϵ in $\mathbb{G}$ such that $(\mathbb{G},d_\epsilon)\to(\mathbb{G},d_0)$ in the Gromov-Hausdorff topology and corresponding sequence of Riemannian heat kernels Γ_ϵ satisfying the uniform estimates in Proposition 6. Applying step 2 at every level $\epsilon>0$ and using the fact that g_ϵ are left-invariant, one has the identities

$$\int_{\partial S}\langle\nabla_\epsilon\Gamma_\epsilon(\tilde x^{-1}\tilde y,\tau),\nu_\epsilon(\tilde y)\rangle_{\tilde g_\epsilon}d\sigma_\epsilon(\tilde y)=-\frac{h_\epsilon(\tilde x)}{2}\int_{T_{\tilde x}\partial S}\Gamma_\epsilon(\tilde x^{-1}\tilde y,\tau)d\sigma_\epsilon(\tilde y)+O(1),\ \text{ as }\tau\to0,\tag{5.13}$$

where the bounds in $O(1)$ are uniform in ϵ. If we denote by $\Gamma_{\tilde x,\epsilon}$ the heat kernel corresponding to the frozen Riemannian metric $g_\epsilon(\tilde x)$ (no longer left-invariant with respect to $\mathbb{G}$) then (5.11) and (5.12) yield

$$\int_{\partial S}\langle\nabla_\epsilon\Gamma_\epsilon(\tilde x^{-1}\tilde y,\tau),\nu_\epsilon(\tilde y)\rangle_{\tilde g_\epsilon}d\sigma_\epsilon(\tilde y)=-\frac{h_\epsilon(\tilde x)}{2\sqrt\tau}\int_{T_{\tilde x}\partial S}\Gamma_{\tilde x,\epsilon}((\tilde z,1),(\tilde x,0))d\sigma_{g_\epsilon(\tilde x)}(\tilde z)+O(1),\ \text{ as }\tau\to0,\tag{5.14}$$

where the bounds in $O(1)$ are uniform in ϵ.

In view of Proposition 6 and the dominated convergence theorem it follows that the left-hand side integrals

$$\int_{\partial S}\langle\nabla_\epsilon\Gamma_\epsilon(\tilde x^{-1}\tilde y,\tau),\nu_\epsilon(\tilde y)\rangle_{\tilde g_\epsilon}d\sigma_\epsilon(\tilde y)$$

converge to the left-hand side of (5.5) as $\epsilon\to0$.

Next we turn our attention to the right-hand side of (5.5). First we recall that at every point $h_\epsilon\to h_0$ as $\epsilon\to0$ and $d\sigma_\epsilon\to d\sigma_0$ as measures. Using the uniform bounds in Proposition 6 along with dominated convergence we deduce that

$$\begin{aligned}\lim_{\epsilon\to0}\int_{T_0\partial S}\Gamma_\epsilon(\tilde y,t)d\sigma_\epsilon(\tilde y) &= \int_{T_0\partial S}\Gamma(\tilde y,t)d\sigma_0(\tilde y)\\ \text{(applying Lemma 3)}\qquad &= \frac{1}{\sqrt\tau}\int_{\Pi}\Gamma(\tilde z,1)d\sigma_0(\tilde z)+O(1),\end{aligned}$$

□

which completes the proof.

Remark 4. The result in Lemma 4 seems to be new even in the Riemannian setting (Step 2 in the previous proof). Since we find that such extension may be of independent interest we state it explicitly.

Corollary 1. Consider a Riemannian manifold (N, g) and a smooth embedded hypersurface $M \subset N$ endowed with the induced metric. Denote by Γ be the heat kernel on (N, g), and for $\tilde{x} \in N$ denote by $\Gamma_{\tilde{x}}$ the heat kernel in N corresponding to the frozen metric $g(\tilde{x})$ viewed as a function defined on $(T_{\tilde{x}}N \times \mathbb{R}^+)^2$. We also consider $d\sigma_g$, the induced volume element on M, the unit vector field ν normal to M, and by h the mean curvature of M. For For every $\tilde{x} \in M$ and $t > s > 0$ one has

$$\int_M \langle \nabla_g \Gamma((\tilde{x}, s), (\tilde{y}, t)), \nu(\tilde{y}\rangle_G d\sigma_g(\tilde{y}) = -\frac{h(\tilde{x})}{2\sqrt{t-s}} \int_{T_{\tilde{x}}M} \Gamma_{\tilde{x}}((\tilde{x}, 0), (z, 1)) d\sigma_{\tilde{x}}(z) + O(1) \tag{5.15}$$

as $t \to s$ uniformly for $\tilde{x} \in M$. Here we have denoted by $d\sigma_{\tilde{x}}$ the volume element on $T_{\tilde{x}}M$ induced by the metric $g(\tilde{x})$ on $T_{\tilde{x}}N$.

The proof follows closely the argument in Step 2 of the previous Lemma 4. As we already noted, in that argument we never used the group law structure, and Step 2 can be applied for any Riemannian metric and independently of the presence of a group structure. The result is local and we used the splitting of the space in $\mathbb{G} \times \mathbb{R}$ only to express the boundary of S as a graph. Since this can be always done, under suitable regularity assumptions, the result in Step 2 holds in any Riemannian metric.

We can now prove the asymptotic expansion stated in the Proposition 3 in the introduction

Proof of Proposition 3 Let ρ^ϵ be a smooth mollification of χ_S. For any interval $(t_1, t_2) \subset \mathbb{R}^+$, from the definition of heat kernel,

$$\begin{aligned} 0 = \lim_{R\to\infty} \int_{t_1}^{t_2} \int_{\{|\tilde{x}^{-1}\tilde{y}| \le R\}} (\partial_t - \mathcal{L}) \Gamma(\tilde{x}^{-1}\tilde{y}, \tau) \rho^\epsilon(\tilde{y}) d\tilde{y} = \Big(\int_{\mathbb{G}\times\mathbb{R}} \Gamma(\tilde{x}^{-1}\tilde{y}, \tau) \rho^\epsilon(\tilde{y}) d\tilde{y} \Big)_{t_1}^{t_2} \\ - \lim_{R\to\infty} \int_{t_1}^{t_2} \int_{\{|\tilde{x}^{-1}\tilde{y}| = R\}} \langle \nabla_0 \Gamma(\tilde{x}^{-1}\tilde{y}, \tau), \nu_0(\tilde{y}) \rangle_0 \, \rho^\epsilon(\tilde{y}) d\sigma_0(\tilde{y}) d\tau \\ + \int_{t_1}^{t_2} \int_{\mathbb{G}\times\mathbb{R}} \langle \nabla_0 \Gamma(\tilde{x}^{-1}\tilde{y}, \tau), \nabla_0 \rho^\epsilon(\tilde{y}) \rangle_0 d\tilde{y} d\tau, \end{aligned} \tag{5.16}$$

Letting $t_1, \epsilon \to 0$, one obtains

$$\begin{aligned} 0 = \int_{\mathbb{G}\times\mathbb{R}} \Gamma(\tilde{x}^{-1}\tilde{y}, t_2) \chi_S(\tilde{y}) d\tilde{y} - \lim_{t_1\to 0} \int_{\mathbb{G}\times\mathbb{R}} \Gamma(\tilde{x}^{-1}\tilde{y}, t_1) \chi_S(\tilde{y}) d\tilde{y} \\ - \lim_{R\to\infty} \int_0^{t_2} \int_{\{|\tilde{x}^{-1}\tilde{y}| = R\}} \langle \nabla_0 \Gamma(\tilde{x}^{-1}\tilde{y}, \tau), \nu_0(\tilde{y}) \rangle_0 \chi_S(\tilde{y})(\tilde{y}) d\sigma_0(\tilde{y}) d\tau \\ + \int_0^{t_2} \int_{\partial S} \langle \nabla_0 \Gamma(\tilde{x}^{-1}\tilde{y}, \tau), \nu_0(\tilde{y}) \rangle_0 d\sigma(\tilde{y}) d\tau, \end{aligned} \tag{5.17}$$

Let us note that since $\widetilde{x} \in \partial S$ then

$$\lim_{t_1\to 0}\int_{\mathbb{G}\times\mathbb{R}} \Gamma(\tilde{x}^{-1}\tilde{y}, t_1)\chi_S(\tilde{y})d\tilde{y} = \frac{1}{2}$$

Hence

$$\int_{G\times\mathbb{R}} \Gamma(\tilde{x}^{-1}\tilde{y}, t_2)\chi_S(\tilde{y})d\tilde{y} = \frac{1}{2} - \lim_{R\to\infty}\int_0^{t_2}\int_{\{\ |\tilde{x}^{-1}\tilde{y}|=R\}} \langle \nabla_0\Gamma(\tilde{x}^{-1}\tilde{y},\tau), \nu_0(\tilde{y})\rangle_0\ \chi_S(\tilde{y})d\sigma_0(\tilde{y})d\tau$$
$$+\int_0^{t_2}\int_{\partial S}\langle \nabla_0\Gamma(\tilde{x}^{-1}\tilde{y},\tau), \nu_0(\tilde{y})\rangle_0 d\sigma(\tilde{y})d\tau,$$

Next we show that

$$\lim_{R\to\infty}\int_0^{t_2}\int_{\{\ |x^{-1}y|=R\}} \langle \nabla_0\Gamma(\tilde{x}^{-1}\tilde{y},\tau), \nu_0(\tilde{y})\rangle_0\ \chi_S(\tilde{y})d\sigma_0(\tilde{y})d\tau = 0. \tag{5.18}$$

To see this we recall that the $d\sigma_0$ perimeter of $\partial B(0,R)$ in $\mathbb{G}\times\mathbb{R}$ is given by $\int_{\partial B(0,R)} d\sigma_0 = C_{\mathbb{G}}R^Q$. From this and from the heat kernel estimates one has

$$\int_{\{|\tilde{x}^{-1}\tilde{y}|=R\}} \langle \nabla_0\Gamma(\tilde{x}^{-1}\tilde{y},\tau), \nu_0(\tilde{y})\rangle_0\ \chi_S(\tilde{y})d\sigma_0(\tilde{y})d\tau \le C_{\mathbb{G}}R^{-2}\int_0^{t_2}\left(\frac{R^2}{\tau}\right)^{(Q+2)/2} e^{-\frac{R^2}{c\tau}}d\tau,$$

which implies (5.18).

Applying Lemma 4 one concludes

$$\int_{\mathbb{G}\times\mathbb{R}} \Gamma(\tilde{x}^{-1}\tilde{y}, t_2)\chi_S(\tilde{y})d\tilde{y} = \frac{1}{2} - \int_0^{t_2}\left(\frac{h_0(\tilde{x})}{2\sqrt{\tau}}\int_\Pi \Gamma(z,1)d\sigma_0(z) + O(1)\right)d\tau \tag{5.19}$$

$$= \frac{1}{2} - h_0(\tilde{x})\int_\Pi \Gamma(z,1)d\sigma_0(z)\int_0^{t_2}\frac{1}{2\sqrt{\tau}}d\tau + O(t_2) \tag{5.20}$$

$$= \frac{1}{2} - h_0(\tilde{x})\sqrt{t_2}\int_\Pi \Gamma(z,1)d\sigma_0(z) + O(t_2). \tag{5.21}$$

Theorem 4. *Let $f\in\mathcal{B}$ be a smooth function, and denote S its subgraph, defined in (4.1). If $x\in\mathbb{G}$, $\tilde{x}=(x,f(x))$ $\alpha\in\mathbb{R}^{m+1}$ and $t>0$, denote*

$$q(t) = (x, f(x))\exp(\sum_{d(i)=1}\alpha_i(t)\,X_i)$$

Then we have

$$q(t)\ \in\ \partial(\mathcal{H}(t)S) \Longrightarrow \sum_{d(i)=1}\alpha_i\nu_0^i = -h_0(\tilde{x})\,t + O(t^{3/2})\ \text{as}\ t\to 0,$$

where $h_0(\tilde{x})$ is the horizontal mean curvature of ∂S in $\tilde{x}$ and $\nu_0 = (\nu_0^1,...,\nu_0^{n+1})$.

Proof. From the definition of the heat flow of sets one has

$$\frac{1}{2} = \int_S \Gamma(q(t)^{-1}\tilde{y}, t)\,dy$$

considering the Taylor expansion of the integrand with respect to v we obtain

$$= \int_S \Gamma(\tilde{x}^{-1}\tilde{y},t)d\tilde{y} - \sum_{d(i)=1} \alpha_i \int_S X_i|_{\tilde{x}}\Gamma(\tilde{x}^{-1}\tilde{y},t)\, d\tilde{y} + o(t)$$
$$= \int_S \Gamma(\tilde{x}^{-1}\tilde{y},t)d\tilde{y} - \sum_{d(i)=1} \alpha_i \int_S X_i|_{x}\Gamma(\tilde{x}^{-1}\tilde{y},t)\, d\tilde{y} + o(t)$$

moreover applying Lemma 2 we get

$$= \int_S \Gamma(\tilde{x}^{-1}\tilde{y},t)d\tilde{y} - \sum_{d(i)=1} \alpha_i \int_{\partial S} \Gamma(\tilde{x}^{-1}\tilde{y},t)\nu_i^0(\tilde{y})d\sigma_0(\tilde{y}) + o(t).$$

and applying Lemma 3

$$= \int_S \Gamma(\tilde{x}^{-1}\tilde{y},t)d\tilde{y} - \sum_{d(i)=1} \frac{\alpha_i\nu_0^i(\tilde{x})}{\sqrt{t}} \int_\Pi \Gamma(\tilde{z},1)d\sigma_0(\tilde{z}) + O(t)$$
$$= \frac{1}{2} - \left(\sqrt{t}h_0(\tilde{x}) + \frac{\langle\alpha,\nu_0\rangle_{g_0}}{\sqrt{t}}\right)\int_\Pi \Gamma(\tilde{z},1)d\sigma_0(\tilde{z}) + O(t))$$
$$= \frac{1}{2} - \frac{1}{\sqrt{t}}\int_\Pi \Gamma(\tilde{z},1)d\sigma_0(\tilde{z})\Big(th_0(\tilde{x}) + \langle\alpha,\nu_0\rangle_{g_0} + O(t\sqrt{t})\Big) \tag{5.22}$$

where we have applied Lemma 3 in the first integral, and Lemma 4 in the second, concluding the proof. □

Next we derive two important Corollaries from the previous theorem, which will be the main ingredients in the proof of Theorem 2.

Corollary 2. Choosing $\alpha = t\beta\nu_0$, with $\beta \in \mathbb{R}$ in the previous Theorem, we deduce that, if $q(t) = (\tilde{x})\exp(t\beta\nu_0) \in \partial(\mathcal{H}(t)S)$ then[2]

$$\beta = -h_0(\tilde{x}) + O(\sqrt{t}) \text{ as } t \to 0,$$

where $h_0(\tilde{x})$ is the horizontal mean curvature of ∂S in $\tilde{x}$

Proposition 9. *Let $f \in \mathcal{B}$ be a smooth function and for $t > 0$ denote by $H(t)f$ its flow defined in Definition 6. For every $x \in \mathbb{G}$ one has*

$$(H(t)f)(x) - f(x) = -th_0(x, f(x))\sqrt{1 + |\nabla_0 f(x)|^2} + o(t),$$

where the convergence $o(t)/t \to 0$ is uniform as $t \to 0$.

Proof. Choosing $\alpha = ((H(t)f)(x') - f(x'))X_{n+1}$ we have

$$(H(t)f)(x) - f(x) =< \alpha, \nu_0 >_{g_0} \sqrt{1 + |\nabla_0 f(x)|^2} = -th_0(x, f(x))\sqrt{1 + |\nabla_0 f(x)|^2} + O(t)$$

□

[2] Note the difference in sign with [27, Formula (60), Theorem 4.1], which arises because of our choice of unit normal

We can now conclude the proof of the main result of the paper, Theorem 2. As in [27] the key technical tool in the proof is the non-linear version of Chernoff's formula established by Brezis and Pazy (see Theorem 1) in the introduction.

Proof of Theorem 2. We only have to show (1.10) for $\lambda = 1$. To this end we set for $t > 0$ and $f \in \mathcal{B}$,

$$u^t := \Big(I + t^{-1}(I - H(t))\Big)^{-1} f, \quad \text{and} \quad A^t u := \frac{u - H(t)u}{t}.$$

In view of Proposition 3 the operator $-A^t$ is $m-$dissipative, thus implying that for all $y \in G$ and $t > 0$,

$$\sup_{x\in\mathbb{G}} |u^t(yx) - u^t(x)| \leq \sup_{x\in\mathbb{G}} |f(yx) - f(x)|$$

and consequently that $\{u^t\}_{t\in(0,1]}$ is a family bounded and equi-continuous.

Let $\phi \in C^\infty(G)$ such that $u - \phi$ has a positive maximum at $x_0 \in G$. We can always assume that the maximum is strict, adding a suitable power of the gauge distance, as for example in [9]. Since $u^{t_k} \to u$ uniformly on compact sets then one can find a sequence of points $x_k \to x_0$ as $k \to \infty$ such that $u^{t_k} - \phi$ has a positive maximum at x_k. In view of Proposition 3 one has

$$(H(t_k)u^{t_k})(x_k) - (H(t_k)\phi^{t_k})(x_k) \leq u^{t_k k}(x_k) - \phi(x_k), \text{ or equivalently } A^{t_k}\phi(x_k) \leq A^{t_k}u^{t_k}(x_k).$$

Since $u^t + A^t u^t = f$ then

$$u^{t_k}(x_k) + \frac{\phi(x_k) - (H(t_k)\phi)(x_k)}{t_k} \leq f(x_k). \tag{5.23}$$

Invoking Proposition 9 with ϕ in place of f, one obtains

$$u^{t_k}(x_k) - \frac{1}{2}\sum_{i,j=1}^{m}\left(\delta_{ij} - \frac{X_i\phi(x_k)X_j\phi(x_k)}{1 + |\nabla_0\phi(x_k)|^2}\right)X_iX_j\phi(x_k) + o(1) \leq f(x_k).$$

Letting $k \to \infty$ we establish that u is a weak sub solution of (3.2) with $\lambda = 1$. In a similar fashion one can prove that u is a weak super-solution, concluding the proof.

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