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Conformality and Q-harmonicity in Carnot Groups

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CONFORMALITY AND Q-HARMONICITY IN CARNOT GROUPS

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Abstract

We show that if f is a 1-quasiconformal map defined on an open subset of a Carnot group G, then composition with f preserves Q-harmonic functions. We combine this with a regularity theorem for Q-harmonic functions and an algebraic regularity theorem for maps between Carnot groups to show that f is smooth. We give some applications to the study of rigidity.

1. Introduction

The theory of quasiconformal maps weaves together analysis and geometry, revealing fundamental aspects of both disciplines and links between them. Quasiconformal maps may be studied in very general settings (see, e.g., [18] and [24]) and with minimal smoothness assumptions. For instance, if Ω_1 and Ω_2 are open subsets of a metric space (Ω, ϱ), then the distortion of a homeomorphism $f: \Omega_1 \to \Omega_2$ is defined by

$$H_f(p,t) = \frac{\sup\{\varrho(f(q), f(p)) \mid \varrho(q,p) \leq t\}}{\inf\{\varrho(f(q), f(p)) \mid \varrho(q,p) \geq t\}}; \tag{1.1}$$

f is said to be λ-quasiconformal if $\limsup_{t\to 0} H_f(p,t) \leq \lambda$ for all $p \in \Omega_1$.

Typically, constraints on the distortion of the map or the geometry of the underlying space yield extra regularity and rigidity properties. Liouville's 1850 theorem, which states that C^4-conformal maps between domains in $\mathbb{R}^3$ are products of translations, dilations, and inversions, is the prototype. In modern terms, the group $\mathrm{O}(1,4)$ acts conformally on the sphere S^3 and hence locally on $\mathbb{R}^3$ by stereographic projection, and any conformal map between domains in $\mathbb{R}^3$ is the restriction of the action of an element of $\mathrm{O}(1,4)$. The same result also holds in $\mathbb{R}^n$ when $n > 3$ (see, e.g., R. Nevanlinna [27]). A major advance in the theory was the passage from smoothness assumptions to metric assumptions (see F. W. Gehring [14] and Yu. G. Reshetnyak [32]): the conclusion of Liouville's theorem holds for 1-quasiconformal maps. A useful reference for the classical theory is the book by J. Väisälä [36].

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Similar rigidity theorems hold if the ambient space is not Riemannian. Analogous results with the Heisenberg group instead of Euclidean space and the sphere in $\mathbb{C}^n$ with its CR structure instead of S^n have been long known (see, e.g., É. Cartan [4], S.-S. Chern and J. K. Moser [5], N. Tanaka [34], and A. Korányi and H. M. Reimann [22], [23]). In this work, the notion of conformality involves the Levi metric: a conformal map is a differentiable contact map whose differential, restricted to the contact plane, is a multiple of a unitary map. The relevant result for this article is that sufficiently smooth conformal maps come from the action of the group $\mathrm{SU}(1, n)$. Capogna [2, Corollary 1.4, page 869] and P. Tang [35, Theorem 2.3] extended this to 1-quasiconformal maps by establishing that 1-quasiconformal maps are smooth and appealing to previous results; their regularity theorems are immediate ancestors of this article.

An important generalization of the results for CR manifolds is the study of quasiconformal maps in Carnot groups. These are nilpotent Lie groups with a left-invariant sub-Riemannian metric on a left-invariant subbundle of the tangent bundle, called the horizontal bundle, and they admit a group $\{\delta_t \mid t \in \mathbb{R}^+\}$ of automorphisms which expand when $t > 1$ and contract when $t < 1$. *We define Carnot groups to exclude* $\mathbb{R}$.

In his celebrated work [25], G. D. Mostow introduced quasiconformal maps on certain Carnot groups to prove rigidity theorems for rank-one symmetric spaces. He showed that certain quasi-isometries extend to quasiconformal maps of the ideal boundary and then that these are conformal.

P. Pansu [28] made the next breakthrough: he proved that quasiconformal maps* on general Carnot groups are differentiable in a suitable sense and applied this to study rigidity. A map $f : \Omega_1 \to \Omega_2$ between open subsets of Carnot groups is said to be Pansu differentiable at p in Ω_1 if $\delta_t^{-1}[f(p)^{-1} f(p\delta_t x)]$ converges locally uniformly in x as $t \to 0$. The limit $Df(p)$ is called the Pansu differential at p. In [28] Pansu showed that a quasiconformal map f is Pansu differentiable almost everywhere and that $Df(p)$ is a group homomorphism that intertwines the dilations δ_t. In particular, $Df(p)$ induces a Lie algebra homomorphism $df(p)$ that sends horizontal vectors to horizontal vectors. Pansu used this form of differentiability and a study of Lie algebra automorphisms to prove that in the quaternionic and octonionic Heisenberg groups, there is a natural generalized contact structure of codimension greater than one, and the analogue of Liouville's theorem holds for global maps that preserve the contact structure.

These ideas have been extended to many more Carnot groups. For example, the space of smooth, locally defined quasiconformal maps is finite-dimensional for the so-called groups of Heisenberg type (or, H-type groups; see [21], [8]) when the center has dimension at least three (see [30]; see also the work of Reimann and F. Ricci [31],

*Actually, Pansu's definition of quasiconformality was different but equivalent (see Definition 4.5).

Cowling and Reimann [9], and Cowling, F. De Mari, A. Korányi, and Reimann [7]). These results, however, need some smoothness. Capogna [3] followed the approach of Gehring [14] and proved that 1-quasiconformal maps in H-type groups are smooth. The point of this article is that 1-quasiconformal maps are smooth in *all* Carnot groups.

THEOREM 1.1
Suppose that Ω_1 and Ω_2 are open subsets of Carnot groups G_1 and G_2, and suppose that $f : \Omega_1 \to \Omega_2$ is 1-quasiconformal. Then f is smooth.

In $\mathbb{R}^n$, our proof is simpler than Gehring's [14], as we do not need such a careful study of the capacity of rings. It may be very hard to generalize Gehring's approach to all Carnot groups since there is no explicit Green's function for the Q-Laplacian (see (2.3)). However, Gehring's method does extend to some noncommutative groups (see [1], [3], [6], [10], and [38]).

The first main step of our proof is to use the deep results of Pansu [28] and Heinonen and Koskela [18] to prove that the components of the first layer of f are Q-harmonic (see (2.3)). The regularity theory in [3, Main Theorem, page 264, Theorem 4.6, page 282] then implies that the first layer is smooth. The second main step is an algebraic regularity argument (see Theorem 6.1) to show that the higher layers are smooth; we study how the Pansu differential of the map links the layers of its components.

Theorem 1.1 can be used to prove rigidity theorems for quasiconformal maps between open subsets in certain classes of groups without any a priori smoothness assumption. For instance, in Corollary 7.4 we show that if G is a two-step Carnot group whose only dilation-preserving automorphisms are dilations (see [28, Proposition 13.1] for a large class of examples), then the only quasiconformal maps between open subsets of G are translations composed with dilations. Combining Theorem 1.1 with the results in [30], we can show a local version of one of the main results in [28].

THEOREM 1.2
Suppose that G is the nilpotent group in the Iwasawa decomposition of the isometry group of the quaternionic or Cayley hyperbolic space. Quasiconformal maps between two open subsets of G form a finite-dimensional subset of the space of smooth, generalized contact maps.

In conclusion, two more of our contributions should be highlighted. First, the metric definition and Pansu's definition of 1-quasiconformal maps are equivalent (see Corollary 7.2), and second, the morphism property of 1-quasiconformal maps needs no extra regularity assumptions (see Theorem 5.7).

2. Preliminary results

In this section we recall the relevant definitions, fix notation, and recall some results from the literature on which our arguments rest.

A grading of step R of a Lie algebra $\mathfrak{g}$ is a decomposition of $\mathfrak{g}$ as a vector-space direct sum $\sum_{k=1}^{R} V_k$ such that $[V_i, V_j] \subseteq V_{i+j}$ when $1 \leq i, j \leq R$; here we interpret V_i as $\{0\}$ when $i > R$ and require that $V_R \neq \{0\}$.

A stratified Lie group is a simply connected, necessarily nilpotent group G whose Lie algebra $\mathfrak{g}$ has a grading with the additional property that V_1 generates $\mathfrak{g}$. The map on $\mathfrak{g}$ which multiplies the elements of the kth layer V_k by k is a derivation. This gives rise to a group of automorphic dilations $\{\delta_t \mid t \in \mathbb{R}^+\}$ of $\mathfrak{g}$ defined by

$$\delta_t(X_1 + \cdots + X_R) = tX_1 + \cdots + t^R X_R,$$

where $X_k \in V_k$, such that $\delta_s \delta_t = \delta_{st}$ and δ_1 is the identity map (thus $\delta_{t^{-1}} = \delta_t^{-1}$). We also write δ_t for the corresponding automorphisms $\exp \circ \delta_t \circ \log$ of G. (Here, exp denotes the exponential map, which is bijective, and log denotes its inverse.) We identify the Lie algebra with the tangent space TG_e to G at the identity e, and for X in $\mathfrak{g}$, we write $\tilde{X}$ for the left-invariant vector field that agrees with X at e. The vector fields in $\tilde{V}_k$ determine a subspace $T_k G_p$ of the tangent space at each point p. In particular, when $k = 1$, we obtain the *horizontal subspace*. These subspaces vary smoothly from point to point and give rise to subbundles $T_k G$ of the tangent bundle. The hypothesis of stratification implies that the sections of the horizontal tangent bundle, called the horizontal vector fields, generate all possible vector fields with linear combinations of commutators of order up to R.

A *Carnot group* is a stratified group G, *other than* $\mathbb{R}$, whose Lie algebra $\mathfrak{g}$ carries an inner product for which the different layers V_k are orthogonal. Henceforth, G denotes a Carnot group with Lie algebra $\mathfrak{g}$. Denote by π_k the orthogonal projection onto V_k by m_k, the vector space dimension of the layer V_k, when $k = 1, \ldots, R$, and denote by m the dimension $m_1 + \cdots + m_R$. We identify V_k isometrically with Euclidean space $\mathbb{R}^{m_k}$, and $\mathfrak{g}$ with $\mathbb{R}^m$. We write Q for the *homogeneous dimension* $\sum_{k=1}^{R} km_k$ of G (see [13]). Since exp is a bijection, we may parametrize G by $\mathfrak{g}$. The measure on G obtained by pushing forward the Lebesgue measure on $\mathfrak{g}$ is translation invariant. We denote by $|E|$ the measure of a set E; then $|\delta_t E| = t^Q |E|$.

The simplest noncommutative Carnot group is the Heisenberg group $\mathbb{H}^1$, for which $R = 2$, $m_1 = 2$, and $m_2 = 1$. We use it as an example to illustrate some of the main ideas in a simple setting. A basis of its Lie algebra is $\{X_1, X_2, X_3\}$, where $[X_1, X_2] = X_3$ and $[X_1, X_3] = [X_2, X_3] = 0$. Then $V_1 = \operatorname{span}\{X_1, X_2\}$, while $V_2 = \operatorname{span}\{X_3\}$. We identify $\mathbb{H}^1$ as a set with $\mathbb{R}^3$, writing x or (x_1, x_2, x_3) for $\exp(x_1 X_1 + x_2 X_2 + x_3 X_3)$. The group law is

$$xy = \left(x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1 y_2 - x_2 y_1)\right).$$

In coordinates, the left-invariant vector fields $\tilde{X}_1$, $\tilde{X}_2$, and $\tilde{X}_3$ are given by

$$\tilde{X}_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3}, \qquad \tilde{X}_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3}, \qquad \text{and} \qquad \tilde{X}_3 = \frac{\partial}{\partial x_3}.$$

Note that $[\tilde{X}_1, \tilde{X}_2]$ is $\tilde{X}_3$ and that $\tilde{X}_1$, $\tilde{X}_2$, and $\tilde{X}_3$ span the Lie algebra of all left-invariant vector fields.

The inner product on V_1 induces a left-invariant inner product on each horizontal space. This inner product allows us to define a (left-invariant) Carnot-Carathéodory metric ϱ on G, as follows. A smooth curve is said to be *horizontal* if its tangent vectors are horizontal, that is, they lie in the horizontal tangent space, and the length of a horizontal curve is the integral of the lengths of its tangent vectors. The distance between two points is then the infimum of the lengths of the horizontal curves joining them. The metric is left-translation-invariant and is related to the dilations by the formula $\varrho(\delta_t x, \delta_t y) = t\,\varrho(x, y)$ for all $t \in \mathbb{R}^+$ and all $x, y \in G$. We denote by $B(x, t)$ the open ball $\{y \in G \mid \varrho(y, x) < t\}$; then $|B(x, t)| = C_G t^Q$. We write $S(x, t)$ for the *sphere* $\{y \in G \mid \varrho(y, x) = t\}$. If $X \in V_1$, then the curve $s \mapsto \exp(sX)$ is horizontal and minimizes lengths, so that

$$\varrho(\exp X, e) = |X|, \quad \forall X \in V_1. \tag{2.1}$$

There is a natural pseudonorm $\|\cdot\|$ on the stratified Lie algebra $\mathfrak{g}$, given by $\|X\|^{2R!} = \sum_{k=1}^{R} |\pi_k X|^{2R!/k}$, and an associated left-invariant pseudometric ϱ_{NSW}, given by $\varrho_{\mathrm{NSW}}(x, y) = \|\log(y^{-1}x)\|$ for all $x, y \in G$. According to A. Nagel, E. M. Stein, and S. Wainger [26], this pseudometric is equivalent to the Carnot-Carathéodory metric, in that

$$C_1\,\varrho(x, y) \le \varrho_{\mathrm{NSW}}(x, y) \le C_2\,\varrho(x, y), \quad \forall x, y \in G \tag{2.2}$$

(for suitable constants C_1 and C_2).

We use the symbol $\simeq$ in the following sense. For functions A and B defined on S, we write $A(s) \simeq B(s)$ for all s in S if and only if there exist constants C_1 and C_2 (all constants are taken to be positive) such that $C_1 A(s) \le B(s) \le C_2 A(s)$ for all $s \in S$. In particular, $\varrho \simeq \varrho_{\mathrm{NSW}}$. Constants are denoted by $C, C', C'', C_1, C_2, \dots$; their values may vary from formula to formula and may depend on anything not explicitly quantified.

We take the standard basis $\{X_1, \dots, X_m\}$ for $\mathfrak{g}$, so that $\{X_1, \dots, X_{m_1}\}$ is a basis for V_1, while $\{X_{m_1+1}, \dots, X_{m_1+m_2}\}$ is a basis for V_2, and so on. Then $\{\tilde{X}_1, \dots, \tilde{X}_m\}$ is a basis for the left-invariant vector fields on G, and $\{\tilde{X}_1, \dots, \tilde{X}_{m_1}\}$ is a basis for the left-invariant horizontal vector fields. The functions $X_i \cdot \log\colon G \to \mathbb{R}$ are called *exponential coordinates* on G. The Lie algebra element X_i is in the first layer when

$1 \le i \le m_1$, and accordingly, we call $(X_1 \cdot \log, \dots, X_{m_1} \cdot \log)\colon G \to \mathbb{R}^{m_1}$ the *first layer* of coordinates.

Suppose that V is a finite-dimensional inner product space. For a function $u\colon G \to V$, we write $\nabla_H u$ for the vector $(\tilde{X}_1 u, \dots, \tilde{X}_{m_1} u)$; then $\nabla_H^h u$ is a V-valued tensor when $h = 2, 3, \dots$. We consider a multi-index $(i_1, i_2, \dots, i_l)$ for which $1 \le i_1, i_2, \dots, i_l \le m$; we abbreviate this to I and define its homogeneous length $|I|$ to be $\sum_{j=1}^{l} \kappa_{i_j}$, where the integer κ_i is specified by requiring that $m_1 + \cdots + m_{\kappa_i - 1} < i \le m_1 + \cdots + m_{\kappa_i}$ when $1 \le i \le m$. We write $\nabla^I u$ for $\tilde{X}_{i_1} \tilde{X}_{i_2} \cdots \tilde{X}_{i_l} u$ and consider this to be a derivative of u of order $|I|$ since $\nabla^I (u \circ \delta_s)(p) = s^{|I|} (\nabla^I u)(\delta_s(p))$.

If $0 \le \alpha \le 1$, then the space $C^{0,\alpha}(\Omega, V)$ is defined to be the set of all continuous functions $u\colon \Omega \to V$ such that $|u(p) - u(q)| \le C\, \varrho(p, q)^\alpha$ for all $p, q \in \Omega$, and when $k \ge 1$, the spaces $C^{k,\alpha}(\Omega, V)$ are defined by

$$C^{k,\alpha}(\Omega, V) = \left\{ u\colon \Omega \to V \;\middle|\; \tilde{X}_i u \in C^{k-1,\alpha}(\Omega, V) \text{ when } 1 \le i \le m_1 \right\}.$$

Because every vector field in $\tilde{\mathfrak{g}}$ arises as a commutator of vector fields in $\tilde{V}_1$, if $u \in C^{k,\alpha}(\Omega, V)$, then $\nabla^I u \in C^{0,\alpha}(\Omega, V)$ when $0 \le |I| \le k$. In particular, if $\tilde{X}_i u \in C^\infty(\Omega, V)$ whenever $1 \le i \le m_1$, then $u \in C^\infty(\Omega, V)$.

The local versions of these spaces $C^{k,\alpha}_{\mathrm{loc}}(\Omega, V)$ are defined in the standard way. Similarly, the subscript 0 on a function space indicates that the functions are compactly supported. When V is $\mathbb{R}$ or $\mathbb{C}$, we omit V in the notation.

By definition, the Lie bracket is bilinear. Clearly, there exists a constant C such that $|[X, Y]| \le C |X|\, |Y|$ for all $X, Y \in \mathfrak{g}$. It follows that if U and V lie in $C^{k,\alpha}_{\mathrm{loc}}(\Omega, \mathfrak{g})$, then $[U, V] \in C^{k,\alpha}_{\mathrm{loc}}(\Omega, \mathfrak{g})$.

Let Ω be an open subset of G. When $k \in \mathbb{N}$ and $1 \le P < \infty$, we define the *horizontal Sobolev spaces* (see [12] and [13]) by

$$S^{k,P}(\Omega) = \left\{ u\colon \Omega \to \mathbb{C} \;\middle|\; |\nabla_H^h u| \in L^P(\Omega) \text{ when } 0 \le h \le k \right\}.$$

The differentiation here must be interpreted in the distributional sense.

A real-valued function u in $S^{1,Q}_{\mathrm{loc}}(\Omega)$ is said to be a weak solution of the Q-Laplace equation (Q denotes the homogeneous dimension of G)

$$\nabla_H \cdot (|\nabla_H u|^{Q-2} \nabla_H u) = 0 \tag{2.3}$$

if $\int_\Omega |\nabla_H u(x)|^{Q-2} \nabla_H u(x) \cdot \nabla_H \phi(x)\, dx = 0$ for all $\phi \in C_0^\infty(\Omega)$; such functions are called Q-harmonic. By a standard density argument, we can change the requirement that $\phi \in C_0^\infty(\Omega)$ to the requirement that $\phi \in C_0^{0,1}(\Omega)$.

The following regularity theorem, proved in [3, Corollary, page 264], plays a crucial role in the proof of Theorem 1.1.

THEOREM 2.1 (Regularity theorem for the Q-Laplacian)
Suppose that u is Q-harmonic, and suppose that $|\nabla_H u| \simeq 1$ in a subset of Ω of full measure. Then, after changing u on a null set, $u \in C^\infty(\Omega)$.

The Baker-Campbell-Hausdorff (BCH) formula (see [20]) is useful for computations in Carnot groups. For X and Y in $\mathfrak{g}$, the product $e^X e^Y$ is equal to $\exp(\mathrm{BCH}(X, Y))$, where $\mathrm{BCH}(X, Y)$ is a polynomial in X and Y; more precisely, $\mathrm{BCH}(X, Y) = \mathrm{BCH}_1(X, Y) + \mathrm{BCH}_2(X, Y) + \mathrm{BCH}_3(X, Y) + \cdots$, where each term BCH_i is a homogeneous Lie algebra polynomial of degree i, that is, a weighted sum of commutators of order $i - 1$. For instance,

$$\begin{aligned}
\mathrm{BCH}_1(X, Y) &= X + Y,\\
\mathrm{BCH}_2(X, Y) &= \frac{1}{2}[X, Y],\\
\mathrm{BCH}_3(X, Y) &= \frac{1}{12}\big(\big[X, [X, Y]\big] - \big[Y, [X, Y]\big]\big),\\
\mathrm{BCH}_4(X, Y) &= -\frac{1}{48}\big(\big[Y, [X, [X, Y]]\big] + \big[X, [Y, [X, Y]]\big]\big),
\end{aligned}$$

and so on. This series terminates with BCH_R, as G is nilpotent of step R.

Suppose that $T : G_1 \to G_2$ is a continuous (hence, smooth; see [20, Theorem 2.6, page 117]) homomorphism of Carnot groups with dilations δ_t^1 and δ_t^2. We say that T *intertwines dilations* (and call T a *Carnot homomorphism*) if $\delta_t^2 T = T\delta_t^1$. Then the (Riemannian) differential $T_*(e)$ of T at the identity is a Lie homomorphism from $\mathfrak{g}_1$ to $\mathfrak{g}_2$ which respects the layers of the two groups. From the definition of the exponential map, $\exp T_*(e)W = T \exp W$ for all $W \in V_1$. Similarly, if $\gamma : [0, 1] \to G_1$ is a horizontal curve, then $\exp T_*(\gamma(s))(\dot\gamma(s)) = T \exp \dot\gamma(s)$. In the rest of this article, we abuse notation by writing δ_t for the dilations on all Carnot groups.

LEMMA 2.2
Suppose that $T : G_1 \to G_2$ is a Carnot homomorphism. Then T is Lipschitz; that is, $\varrho(Tx, Ty) \le C\,\varrho(x, y)$ for all $x, y \in G$, where

$$\begin{aligned}
C &= \max\big\{\varrho(Tz, e) \,\big|\, z \in S(e, 1)\big\}\\
&= \max\big\{\varrho(Ty, e) \,\big|\, y \in \exp(V_1) \cap S(e, 1)\big\}.
\end{aligned}$$

Proof
First, since $\varrho(Tx, Ty) = \varrho(T(y^{-1}x), e)$ and $\varrho(T\delta_t z, e) = t\,\varrho(z, e)$ for all x, y, and z in G_1,

$$\varrho(Tx, Ty) \le \max\big\{\varrho(Tz, e) \,\big|\, z \in S(e, 1)\big\}\,\varrho(x, y).$$

Next, if $z \in S(e, 1)$ and $\epsilon > 0$, then there exists a horizontal curve $\gamma : [0, 1] \to G_1$ such that $|\dot{\gamma}(s)| \le 1 + \epsilon$ for all s in $[0, 1]$, and $\gamma(0) = e$ and $\gamma(1) = z$. Then $T\gamma(0) = e$ and $T\gamma(1) = Tz$, while

$$\begin{aligned} |(T\gamma)^{\cdot}(s)| &= \big|T_*\big(\gamma(s)\big)\big(\dot{\gamma}(s)\big)\big| \\ &\le \max\Big\{\big|T_*\big(\gamma(s)\big)(W)\big| \;\Big|\; W \in V_1, |W| \le 1 + \epsilon\Big\} \\ &= (1 + \epsilon) \max\Big\{\varrho(T \exp W, e) \;\Big|\; W \in V_1, |W| \le 1\Big\}, \end{aligned}$$

from (2.1), whence $\varrho(Tz, e) \le \max\{\varrho(Tw, e) \mid w \in \exp(V_1) \cap S(e, 1)\}$. □

3. Calculus on Carnot groups

In Sections 3–6, we show that the first layer of a 1-quasiconformal map is smooth. Some of the material is often described as well known, but we had difficulty finding detailed references for all the results. We therefore review some features of calculus on Carnot groups and quasiconformal maps.

Throughout this section, we suppose that Ω_1, Ω_2, and Ω_3 are open subsets of Carnot groups G_1, G_2, and G_3, and we suppose that f is a map from Ω_1 to Ω_2. We write $f \in C^{0,\alpha}(\Omega_1, \Omega_2)$ if $\varrho(f(p), f(q)) \le C\, \varrho(p, q)^\alpha$ for all $p, q \in \Omega_1$.

There is an important notational issue: for functions whose values lie in multiplicative groups, expressions such as $f^{-1}(p)$ are ambiguous. We always write $f^{-1}(p)$ for the value of the inverse function f^{-1} at p, and we write $f(p)^{-1}$ for $[f(p)]^{-1}$, the multiplicative inverse of the group element $f(p)$.

Definition 3.1

Given $f : \Omega_1 \to \Omega_2$ and p in Ω_1, we define

$$\begin{aligned} \operatorname{Lip}_f(p) &= \limsup_{q \to p} \frac{\varrho(f(q), f(p))}{\varrho(q, p)}, \\ \operatorname{lip}_f(p) &= \liminf_{q \to p} \frac{\varrho(f(q), f(p))}{\varrho(q, p)}. \end{aligned}$$

Suppose that $f : \Omega_1 \to \Omega_2$ is a homeomorphism, and suppose that $p \in \Omega_1$. Then

$$\operatorname{lip}_f(p) = \big[\operatorname{Lip}_{f^{-1}}\big(f(p)\big)\big]^{-1}, \tag{3.1}$$

and for small t, the distortion (1.1) is given by

$$H_f(p, t) = \frac{\max\{\varrho(f(q), f(p)) \mid \varrho(q, p) = t\}}{\min\{\varrho(f(q), f(p)) \mid \varrho(q, p) = t\}} \tag{3.2}$$

because G_1 and G_2 are manifolds.

Definition 3.2
A continuous function $f\colon \Omega_1 \to \Omega_2$ is said to be Pansu differentiable at a point p in Ω_1 if the functions $x \mapsto \delta_t^{-1}[f(p)^{-1} f(p\delta_t x)]$ converge locally uniformly to a Lie group homomorphism $Df(p)$ as $t \to 0$.

The homomorphism $Df(p)\colon G_1 \to G_2$, called the Pansu differential, is a Carnot homomorphism. The Lie derivative of $Df(p)$, written $df(p)$, is a grading-preserving Lie homomorphism from $\mathfrak{g}_1$ to $\mathfrak{g}_2$.

LEMMA 3.3
If $f\colon \Omega_1 \to \Omega_2$ is Pansu differentiable at p, then

$$\begin{aligned}\operatorname{Lip}_f(p) &= \max\left\{\varrho\left(Df(p)x, e\right) \,\middle|\, x \in S(e,1)\right\}\\ &= \max\left\{|df(p)X| \,\middle|\, X \in V_1,\ |X| = 1\right\},\end{aligned}$$

and

$$\operatorname{lip}_f(p) = \min\left\{\varrho\left(Df(p)x, e\right) \,\middle|\, x \in S(e,1)\right\}.$$

If, in addition, $\operatorname{Lip}_f(p) \neq 0$*, then*

$$\limsup_{t\to 0} H_f(p,t) = \frac{\operatorname{Lip}_f(p)}{\operatorname{lip}_f(p)}.$$

Proof
Since f is Pansu differentiable at p,

$$\begin{aligned}\lim_{t\to 0} \frac{\varrho(f(p\delta_t x), f(p))}{t} &= \lim_{t\to 0} \varrho\left(\delta_t^{-1}[f(p)^{-1} f(p\delta_t x)], e\right)\\ &= \varrho\left(Df(p)x, e\right)\end{aligned}$$

uniformly for x in $S(e,1)$. Since the function $x \mapsto Df(p)x$ is continuous, it has a maximum and minimum on $S(e,1)$. Therefore

$$\begin{aligned}\operatorname{Lip}_f(p) &= \limsup_{q\to p} \frac{\varrho(f(q), f(p))}{\varrho(q,p)}\\ &= \limsup_{t\to 0} \max_{x\in S(e,1)} \frac{\varrho(f(p\delta_t x), f(p))}{t}\\ &= \max_{x\in S(e,1)} \varrho\left(Df(p)x, e\right).\end{aligned}$$

The second equality for $\operatorname{Lip}_f(p)$ follows from Lemma 2.2. To prove the equality for $\operatorname{lip}_f(p)$, we exchange suprema and maxima with infima and minima in the argument above. Finally, from (3.2),

$$\begin{aligned}\limsup_{t\to 0} H_f(p,t) &= \limsup_{t\to 0} \frac{\max\{\varrho(f(q), f(p)) \mid \varrho(q,p)=t\}}{\min\{\varrho(f(q), f(p)) \mid \varrho(q,p)=t\}} \\ &= \limsup_{t\to 0} \frac{\max\{\varrho(f(p\delta_t x), f(p)) \mid x \in S(e,1)\}}{\min\{\varrho(f(p\delta_t x), f(p)) \mid x \in S(e,1)\}} \\ &= \frac{\lim_{t\to 0} \max\{\varrho(f(p\delta_t x), f(p)) \mid x \in S(e,1)\}/t}{\lim_{t\to 0} \min\{\varrho(f(p\delta_t x), f(p)) \mid x \in S(e,1)\}/t} \\ &= \frac{\operatorname{Lip}_f(p)}{\operatorname{lip}_f(p)},\end{aligned}$$

unless the last expression is of the form $0/0$. □

Since $df(p)$ respects the gradings, the matrix representing $df(p)$ is block diagonal, and since it is a Lie algebra homomorphism, the top block, which represents the action on V_1, determines the other blocks. We now show that the restrictions to V_1 of the Pansu differential df and of the Riemannian differential coincide when they both exist (cf. [17, Section 3]).

LEMMA 3.4
Suppose that $f : \Omega_1 \to \Omega_2$ is Pansu differentiable at p in Ω_1. Write $f(p)^{-1} f(px)$ as $\exp(F(x))$, and write $F(x)$ as $F_1(x) + \cdots + F_R(x)$, where $F_j(x) \in V_j$. If $1 \le k \le j \le R$ and $X \in V_k$, then $\tilde{X} F_j(e)$ exists, and

$$\tilde{X} F_j(e) = \begin{cases} df(p)X & \text{if } k = j, \\ 0 & \text{if } k < j. \end{cases}$$

In addition, $F_k \circ \exp|_{V_k}$ is differentiable in the usual sense at the origin, with differential $(F_k \circ \exp|_{V_k})_*(0)$, which is equal to $df(p)|_{V_k}$.

Proof
Since f is Pansu differentiable at p,

$$\lim_{t\to 0} \delta_t^{-1}[f(p)^{-1} f(p\delta_t \exp X)] = Df(p)(\exp X)$$

uniformly for X in V_k of norm at most 1. Thus there exists a function $\epsilon : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{t\to 0} \epsilon(t) = 0$ and

$$\varrho\left(\delta_t^{-1}[f(p)^{-1} f(p\delta_t \exp X)], Df(p)(\exp X)\right) \le \epsilon(t)$$

for all $X \in V_k$ such that $|X| \le 1$. Now $df(p)X \in V_k$, and thus

$$\begin{aligned}
&\varrho\left(\delta_t^{-1}[f(p)^{-1}f(p\delta_t \exp X)], Df(p)(\exp X)\right)\\
&\quad = t^{-1}\,\varrho\left([f(p)^{-1}f(p\delta_t \exp X)], \delta_t Df(p)(\exp X)\right)\\
&\quad = t^{-1}\,\varrho\left(\exp(F(\delta_t \exp X)), \exp(\delta_t df(p)X)\right)\\
&\quad = t^{-1}\,\varrho\left(\exp(-df(p)t^k X)\exp(F(\exp t^k X)), e\right)\\
&\quad = t^{-1}\,\varrho\left(\exp(\mathrm{BCH}(-df(p)t^k X, F(\exp t^k X))), e\right)\\
&\quad \simeq t^{-1}\Big(\sum_{j=1}^{R}\left|\pi_j \mathrm{BCH}\big(-df(p)t^k X, F(\exp t^k X)\big)\right|^{2R!/j}\Big)^{1/2R!},
\end{aligned}$$

whence $|\pi_j \mathrm{BCH}(-df(p)t^k X, F(\exp t^k X))| \le C[t\epsilon(t)]^j$ uniformly for X in V_k of norm at most 1; or equivalently,

$$\left|\pi_j \mathrm{BCH}\big(-df(p)Y, F(\exp Y)\big)\right| \le C\left(|Y|^{1/k}\epsilon(|Y|^{1/k})\right)^j \tag{3.3}$$

uniformly for Y in V_k, by (2.2).

If $1 \le j < k$ and $Y \in V_k$, then $\mathrm{BCH}(-df(p)Y, F(\exp Y)) = F_j(\exp Y) + R$, where $R \in V_{j+1}\oplus\cdots\oplus V_R$, so $\pi_j \mathrm{BCH}(-df(p)Y, F(\exp Y)) = F_j(\exp Y)$, and hence,

$$|F_j(\exp Y)| \le C\left(|Y|^{1/k}\epsilon(|Y|^{1/k})\right)^j = C_j|Y|^{j/k}\epsilon_j(|Y|),$$

where $\epsilon_j(t) \to 0$ as $t \to 0$.

Next, we consider the case where $k = j$. For Y in V_k, direct computation shows that $\pi_k \mathrm{BCH}(-df(p)Y, F(\exp Y)) = F_k(\exp Y) - df(p)Y$, whence

$$|F_k(\exp Y) - df(p)Y| \le C\,|Y|\epsilon(|Y|^{1/k})^k = C_k|Y|\epsilon_k(|Y|),$$

where $\epsilon_k(t) \to 0$ as $t \to 0$. Thus $F_k \circ \exp|_{V_k}$ is differentiable with derivative $df(p)|_{V_k}$, as required. Consequently, $|F_k(\exp Y)| \le C_k|Y|\epsilon'_k(|Y|)$, where $\epsilon'_k(t)$ remains bounded as $t \to 0$.

Now, we consider the case where $j > k$. For Y in V_k,

$$\pi_{k+1}\mathrm{BCH}\big(-df(p)Y, F(\exp Y)\big) = F_{k+1}(\exp Y) - \frac{1}{2}\,[df(p)Y, F_1(\exp Y)],$$

and from the estimates above, if $|Y| \le 1$, then

$$\begin{aligned}
\left|[df(p)Y, F_1(\exp Y)]\right| &\le C\,|df(p)Y|\,|F_1(\exp Y)|\\
&\le \begin{cases} C'|Y|^{1+1/k}\epsilon(|Y|^{1/k}) & \text{if } k > 1,\\ C''|Y|^2 & \text{if } k = 1.\end{cases}
\end{aligned}$$

It follows that $|F_{k+1}(\exp Y)| \le C_{k+1}|Y|^{(k+1)/k}\epsilon_{k+1}(|Y|)$, where $\epsilon_{k+1}(t)$ remains bounded as $t \to 0$. Continuing inductively, it is easy to show that if $j > k$, then $|F_j(\exp Y)| \le C_j|Y|^{j/k}\epsilon_j(|Y|)$, where $\epsilon_j(t)$ remains bounded as $t \to 0$, whence $F_j \circ \exp|_{V_k}$ is differentiable, and its derivative is zero. □

COROLLARY 3.5
Suppose that $u\colon \Omega_1 \to V$ is a function from an open subset Ω_1 of the Carnot group G_1 to a vector space V. Then if $Du(p)$ exists, so does $\nabla_H u(p)$, and $\tilde{X}u(p) = Du(p)\exp X$ for all $X \in V_1$. Consequently, $\mathrm{Lip}_u(p) = |\nabla_H u(p)|$.

LEMMA 3.6
Suppose that $u\colon \Omega_1 \to \mathbb{R}$ is Q-harmonic. Then (after correction on a null set) u is continuous and Pansu differentiable almost everywhere in Ω_1.

Proof
The proof has two steps. First, one shows that for any bounded ball Ω_0 such that $\bar{\Omega}_0 \subset \Omega_1$, there exists P in (Q, ∞) such that $\nabla_H u$ is in $L^P(\Omega_0)$. This requires a standard argument based on Gehring's reverse Hölder inequality (see [17, (6.2)]). It follows from a Sobolev-type embedding theorem that (after correction on a null set) u is continuous.

Next, one shows that $S^{1,P}(\Omega_0)$-functions are Pansu differentiable almost everywhere. By the L^P-version of Lebesgue's differentiation theorem,

$$\lim_{t\to 0} \frac{1}{|B(p,2t)|} \int_{B(p,2t)} |\nabla_H u(x) - \nabla_H u(p)|^P \, dx = 0 \tag{3.4}$$

for almost every $p \in \Omega_0$. Fix any such point p, and define the function v by $v(q) = u(q) - u(p) - \langle \nabla_H u(p), \pi_1(\log p^{-1}q)\rangle$. (We write π_1 for the orthogonal projection of $\mathfrak{g}$ onto V_1.) To show that u is Pansu differentiable at p, we must show that $|v(q)| = o(\varrho(q,p))$ as $q \to p$. For this, we use a Morrey-type estimate (see, e.g., [17, Lemma 6.10]):

$$\begin{aligned}
|v(q)| &= |v(q) - v(p)| \\
&\le C\,\varrho(q,p)^{1-Q/P}\Big(\int_{B(p,2\varrho(q,p))} |\nabla_H v(x)|^P \, dx\Big)^{1/P} \\
&= C\,\varrho(q,p)^{1-Q/P}\Big(\int_{B(p,2\varrho(q,p))} |\nabla_H u(x) - \nabla_H u(p)|^P \, dx\Big)^{1/P} \\
&= C'\,\varrho(q,p)\Big(\frac{1}{|B(p,2\varrho(q,p))|}\int_{B(p,2\varrho(q,p))} |\nabla_H u(x) - \nabla_H u(p)|^P \, dx\Big)^{1/P} \\
&= o\big(\varrho(q,p)\big),
\end{aligned}$$

by (3.4), uniformly in q such that $B(p, 2\varrho(q, p)) \subset \Omega_0$. □

LEMMA 3.7 (Chain rule)
Suppose that $f\colon \Omega_1 \to \Omega_2$ is Pansu differentiable at p, and suppose that $g\colon \Omega_2 \to \Omega_3$ is Pansu differentiable at $f(p)$. Then $g \circ f$ is Pansu differentiable at p, and $D(g \circ f)(p) = (Dg)(f(p))Df(p)$.

Proof
Fix a compact set K_1 in G_1. Since f is Pansu differentiable at p, if $t \le 1$ and $x \in K_1$, then $\delta_t^{-1}[f(p)^{-1}f(p\delta_t x)]$ lies in a compact set K_2 in G_2. Further, $(Dg)(f(p))y = \lim_{t\to 0}\delta_t^{-1}[g(f(p))^{-1}g(f(p)\delta_t y)]$ uniformly for $y \in K_2$, as g is Pansu differentiable at $f(p)$. Take y to be $\delta_t^{-1}[f(p)^{-1}f(p\delta_t x)]$. Then if $t \le 1$,

$$
\begin{aligned}
&\varrho\left(\delta_t^{-1}[g(f(p))^{-1}g(f(p\delta_t x))], (Dg)(f(p))Df(p)x\right)\\
&\quad\le \varrho\left(\delta_t^{-1}[g(f(p))^{-1}g(f(p\delta_t x))], (Dg)(f(p))\delta_t^{-1}[f(p)^{-1}f(p\delta_t x)]\right)\\
&\qquad + \varrho\left((Dg)(f(p))\delta_t^{-1}[f(p)^{-1}f(p\delta_t x)], (Dg)(f(p))Df(p)x\right)\\
&\quad\le \sup_{y\in K_2}\varrho\left(\delta_t^{-1}[g(f(p))^{-1}g(f(p)\delta_t y)], (Dg)(f(p))y\right)\\
&\qquad + C\,\varrho\left(\delta_t^{-1}[f(p)^{-1}f(p\delta_t x)], Df(p)x\right)
\end{aligned}
$$

since $(Dg)(f(p))$ is Lipschitz, and this converges as needed. □

4. Quasiconformal maps

In this section, we suppose that Ω_1 and Ω_2 are open subsets of Carnot groups G_1 and G_2 and that $f\colon \Omega_1 \to \Omega_2$ is a homeomorphism. We define the distortion $H_f(p, t)$ of f by (1.1) or (3.2).

Definition 4.1
The map $f\colon \Omega_1 \to \Omega_2$ is said to be *quasiconformal* if there exists a constant λ such that $\limsup_{t\to 0} H_f(p, t) \le \lambda$ for all $p \in \Omega_1$.

Next, f is said to be *weakly quasisymmetric* if there exists a constant H such that $H_f(p, t) \le H$ for all $p \in \Omega_1$ and $t \in \mathbb{R}^+$.

Finally, f is said to be *quasisymmetric* if there exists an increasing homeomorphism $\eta\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varrho(f(p), f(r)) \le \eta(s)\,\varrho(f(p), f(q))$ whenever $p, q, r \in \Omega_1$ and $\varrho(p, r) \le s\,\varrho(p, q)$.

Each of these definitions has a local version; for instance, f is locally quasiconformal if it is quasiconformal in a neighborhood of each point.

THEOREM 4.2
Suppose that $f : \Omega_1 \to \Omega_2$ is a quasiconformal map. Then f is locally quasisymmetric.

For quasiconformal maps on $\mathbb{R}^n$, this result is now classical. Heinonen and Koskela proved it for Carnot groups (see [18, Theorem 1.3]) and later for more general manifolds (see [19, Theorem 4.7]). Väisälä [37, Theorem 2.9] had already shown the equivalence of weak quasisymmetry and quasisymmetry.

Obviously, quasisymmetry implies weak quasisymmetry (and $H = \eta(1)$), which in turn implies quasiconformality, so that the local versions of these properties are equivalent. The next result is elementary.

LEMMA 4.3
Suppose that $f : \Omega_1 \to \Omega_2$ is an η-quasisymmetric map. Then $f^{-1} : \Omega_2 \to \Omega_1$ is $\tilde{\eta}$-quasisymmetric, where $\tilde{\eta}(s) = [\eta^{-1}(s^{-1})]^{-1}$.

COROLLARY 4.4
Suppose that $f : \Omega_1 \to \Omega_2$ is a quasiconformal map. Then $f^{-1} : \Omega_2 \to \Omega_1$ is locally quasiconformal.

Pansu [28] uses a different definition of quasiconformality.

Definition 4.5
A homeomorphism $f : \Omega_1 \to \Omega_2$ is said to be *Pansu quasiconformal* if f and f^{-1} are locally quasisymmetric.*

In light of Theorem 4.2, Lemma 4.3, and Corollary 4.4, Pansu quasiconformality is equivalent to quasiconformality, at least locally. We now focus on the connections between Pansu differentiability and quasiconformality. The next fundamental result was proved by Pansu [28, Théorème 2, Proposition 7.3].

THEOREM 4.6
Suppose that $f : \Omega_1 \to \Omega_2$ is locally quasiconformal. Then f is Pansu differentiable almost everywhere. Moreover, the maps f and f^{-1} are absolutely continuous; that is, $|f(E)| = 0$ if and only if $|E| = 0$.

Suppose that $f : \Omega_1 \to \Omega_2$ is quasiconformal. Then f and f^{-1} are absolutely continuous, and we may define a measure μ on Ω_1 by the formula $\mu(E) = |f(E)|$. Clearly, μ is absolutely continuous, and so it has a Radon-Nikodym derivative, say, f'. By

*Pansu apparently requires a uniform choice of η; this uniformity does not seem to play a significant role.

Lebesgue's differentiation theorem,

$$f'(p) = \lim_{t\to 0} \frac{|f(B(p,t))|}{|B(p,t)|}$$

for almost all p in Ω_1. The change of variables formula

$$\int_{\Omega_1} u\big(f(x)\big) f'(x)\,dx = \int_{f(\Omega_1)} u(y)\,dy$$

follows (see Federer [11, Section 2.9]). If $Df(p)$ and $f'(p)$ exist, then

$$f'(p) \le \lim_{t\to 0} \frac{|(B(f(p), \mathrm{Lip}_f(p)t))|}{|B(p,t)|} = \mathrm{Lip}_f(p)^Q,$$

and similarly, $f'(p) \ge \mathrm{lip}_f(p)^Q$. From Lemma 3.3,

$$\begin{aligned} \mathrm{Lip}_f(p)^Q &\ge f'(p) \ge \limsup_{t\to 0} H_f(p,t)^{-Q}\mathrm{Lip}_f(p)^Q, \\ \mathrm{lip}_f(p)^Q &\le f'(p) \le \limsup_{t\to 0} H_f(p,t)^{Q}\mathrm{lip}_f(p)^Q. \end{aligned} \tag{4.1}$$

Since f' is locally integrable, $\mathrm{Lip}_f \in L^Q_{\mathrm{loc}}(\Omega_1)$.

5. 1-Quasiconformal maps

Again, we assume throughout this section that Ω_1 and Ω_2 are open subsets of Carnot groups G_1 and G_2.

Definition 5.1

A homeomorphism $f\colon \Omega_1 \to \Omega_2$ is said to be 1-quasiconformal if $\lim_{t\to 0} H_f(p,t) = 1$ for all $p \in \Omega_1$.

Further, f is said to be 1-quasiconformal in the sense of Pansu (which we shorten to $P1$-quasiconformal) if f is locally quasiconformal, and $Df(p)$ is a similarity (i.e., a product of a dilation and an isometry) for almost all p.

LEMMA 5.2

Suppose that $f\colon \Omega_1 \to \Omega_2$ is Pansu differentiable at p, and suppose that $Df(p)$ is a similarity. Then $df(p)|_{V_1}$ is a similarity. More precisely, for all $X \in V_1$, we have $|df(p)X| = \mathrm{Lip}_f(p)|X|$.

Proof

Since the Pansu derivative $Df(p)$ is a similarity, from Lemma 3.3 we infer that $\varrho(Df(p)x, e) = \mathrm{Lip}_f(p)\,\varrho(x,e)$ for all x in G. From Lemma 2.2, if $X \in V_1$, then $\varrho\big(\exp(df(p)X), e\big) = \mathrm{Lip}_f(p)\,\varrho(\exp X, e)$. □

LEMMA 5.3
Suppose that $f\colon \Omega_1 \to \Omega_2$ *is* 1*-quasiconformal. Then* f *is also* $P1$*-quasiconformal.*

Proof
Take p in Ω_1, where f is Pansu differentiable. In view of Lemma 3.3 and of the fact that $\limsup_{t\to 0} H_f(p,t) = 1$,

$$\min\bigl\{\varrho\bigl(Df(p)x, e\bigr) \bigm| x \in S(e,1)\bigr\} = \max\bigl\{\varrho\bigl(Df(p)x, e\bigr) \bigm| x \in S(e,1)\bigr\}.$$

Consequently, for all y such that $\varrho(y,e) = t$,

$$\varrho\left(Df(p)y, e\right) = t\,\varrho\left(Df(p)x, e\right) = t\mathrm{Lip}_f(p)\,\varrho(x,e) = \mathrm{Lip}_f(p)\,\varrho(y,e),$$

where $x = \delta_t^{-1} y$, which shows that $Df(p)$ is indeed a similarity. □

The converse is less obvious. If f is not smooth, then we may lack information about $H_f(p,t)$ for p in a null set. In Corollary 7.2, we show that the two definitions are in fact equivalent.

LEMMA 5.4
Suppose that $f\colon \Omega_1 \to \Omega_2$ *is* $P1$*-quasiconformal. Then* f^{-1} *is also* $P1$*-quasiconformal. Furthermore,* $\mathrm{Lip}_f{}^{Q} = f'$ *almost everywhere.*

Proof
Compare with [28, Lemme 7.11]. By Theorem 4.6, f and f^{-1} are absolutely continuous. By the chain rule, $D(f^{-1})(f(p))\,Df(p) = I$ for almost all p, and the inverse of a similarity is a similarity. Hence $D(f^{-1})$ is a similarity almost everywhere. The second statement follows from (4.1). □

The following crucial result is due to Pansu [28, Lemme 11.4] (see the more recent article [1, Theorem 6.6] for a different proof).

THEOREM 5.5
Suppose that $f\colon \Omega_1 \to \Omega_2$ *is* $P1$*-quasiconformal. Then* f *and* f^{-1} *are locally Lipschitz regular; that is, for any compact subset* Ω_0 *of* Ω_1*,*

$$\varrho(p,q) \simeq \varrho\left(f(p), f(q)\right), \quad \forall p,q \in \Omega_0.$$

Before our next result, recall that $(X_1 \cdot \log f, \ldots, X_{m_1} \cdot \log f)$ is the first layer of f in exponential coordinates.

PROPOSITION 5.6
Suppose that $f : \Omega_1 \to \Omega_2$ is $P1$-quasiconformal. For each compact subset Ω_0 of Ω_1, and for all $i = 1, \ldots, m_1$,

$$|\nabla_H (X_i \cdot \log f)(p)| \simeq 1 \quad \text{for a.a. } p \text{ in } \Omega_0. \tag{5.1}$$

Proof
From (3.1) and Theorem 5.5 applied to f and f^{-1}, Lip_f and $\mathrm{Lip}_f{}^{-1}$ are locally bounded. The map $df(p)|_{V_1}$ is an orthogonal map, multiplied by dilation by a factor of $\mathrm{Lip}_f(p)$. The norm of any row (or column) of a positive multiple of an orthogonal matrix is the multiplying factor. Thus if we represent $df(p)$ as a matrix, the norm of any column is $\mathrm{Lip}_f(p)$, which is bounded and bounded away from zero. □

The following result was known (see [17]) for quasiregular maps f, satisfying the further regularity condition that the distributional derivatives $\tilde{X}_i f$ (where $1 \leq i \leq m$, not just $1 \leq i \leq m_1$) lie locally in $L^m(G_1)$. A priori, 1-quasiconformal maps need not satisfy this condition. N. S. Dairbekov [10] found a clever way to eliminate the regularity assumption in the step-two case.

THEOREM 5.7 (Morphism property)
Suppose that $f : \Omega_1 \to \Omega_2$ is $P1$-quasiconformal. If u is Q-harmonic in Ω_2, then $u \circ f$ is Q-harmonic in Ω_1.

Proof
For any p for which $Df(p)$ is a similarity and any function w which is Pansu differentiable at $f(p)$, we have $\mathrm{Lip}_{w\circ f}(p) = \mathrm{Lip}_w(f(p))\mathrm{Lip}_f(p)$ and, by Corollary 3.5,

$$\mathrm{Lip}_{w\circ f}(p)^2 = \nabla_H(w \circ f)(p) \cdot \nabla_H(w \circ f)(p),$$

$$\mathrm{Lip}_w\big(f(p)\big)^2 = (\nabla_H w) \circ f(p) \cdot (\nabla_H w) \circ f(p),$$

whence $\nabla_H(w \circ f) \cdot \nabla_H(w \circ f) = \mathrm{Lip}_f{}^2 (\nabla_H w) \circ f \cdot (\nabla_H w) \circ f$ almost everywhere. Polarization of this identity shows that for almost everywhere Pansu differentiable functions u and v,

$$\nabla_H(u \circ f) \cdot \nabla_H(v \circ f) = \mathrm{Lip}_f{}^2 (\nabla_H u) \circ f \cdot (\nabla_H v) \circ f$$

almost everywhere.

Now we change variables. For any $\psi \in C_0^{0,1}(\Omega_1)$, we write ϕ for $\psi \circ f^{-1}$; then $\phi \in C_0^{0,1}(\Omega_2)$. By Lemma 3.6, every Q-harmonic function u is Pansu differentiable almost everywhere From Lemma 5.4, $f' = \mathrm{Lip}_f{}^Q$, and

$$\begin{aligned}
&\int_{\Omega_1} |\nabla_H(u\circ f)(x)|^{Q-2}\, \nabla_H(u\circ f)(x)\cdot \nabla_H\psi(x)\,dx\\
&\quad= \int_{\Omega_1} |\nabla_H(u\circ f)(x)|^{Q-2}\, \nabla_H(u\circ f)(x)\cdot \nabla_H(\phi\circ f)(x)\,dx\\
&\quad= \int_{\Omega_1} \mathrm{Lip}_f(x)^Q \left|(\nabla_H u)\big(f(x)\big)\right|^{Q-2} (\nabla_H u)\big(f(x)\big)\cdot(\nabla_H\phi)\big(f(x)\big)\,dx\\
&\quad= \int_{f(\Omega_1)} |(\nabla_H u)(y)|^{Q-2}\,(\nabla_H u)(y)\cdot(\nabla_H\phi)(y)\,dy\\
&\quad= 0
\end{aligned}$$

since u is a weak solution of (2.3). □

COROLLARY 5.8 (Regularity of the first layer of 1-quasiconformal maps)
Suppose that $f\colon \Omega_1 \to \Omega_2$ is P1-quasiconformal. Then $\pi_1 \log f \in C^\infty(\Omega_1)$.

Proof
When $1 \le i \le m_1$, the coordinate functions $X_i \cdot \log$ are Q-harmonic. Hence the functions $X_i \cdot \log f$ are too, by Theorem 5.7. The conclusion follows from (5.1) and Theorem 2.1. □

6. Algebraic regularity

THEOREM 6.1
Suppose that Ω_1 and Ω_2 are open subsets of Carnot groups G_1 and G_2; that $\pi_1 \log f \in C^{k,\beta}_{\mathrm{loc}}(\Omega_1, \mathfrak{g}_2)$*, where $k \ge 1$ and $\beta \in [0,1]$; and that $f \in C^{0,\alpha}_{\mathrm{loc}}(\Omega_1,\Omega_2)$, where $\alpha > 1/2$. Then* $\log f \in C^{k,\beta}_{\mathrm{loc}}(\Omega_1, \mathfrak{g}_2)$.

Note that the theorem does not hold if $\alpha = 1/2$; to see this, consider the map $f\colon (x_1,x_2,x_3) \mapsto (x_3,x_2,|x_1|)$ in the Heisenberg group $\mathbb{H}^1$. This map is not contact but is locally $C^{0,1/2}$.

To explain our ideas, we first present the proof of Theorem 6.1 in the setting of the Heisenberg group. Take open subsets Ω_1 and Ω_2 of $\mathbb{H}^1$ and $f\colon \Omega_1 \to \Omega_2$, as in the statement of the theorem. Now $f \in C^{0,\alpha}_{\mathrm{loc}}(\Omega_1,\Omega_2)$, so $\varrho(f(p\exp tX_1), f(p)) \le Ct^\alpha$ for small t. In coordinates, we write $f(p)$ as $(f_1(p), f_2(p), f_3(p))$. Using the Nagel-Stein-Wainger pseudometric and the group law, as in the proof of Lemma 3.4, we see that

$$\left| f_3(p\exp tX_1) - f_3(p) + \frac{f_1(p\exp tX_1)f_2(p) - f_2(p\exp tX_1)f_1(p)}{2} \right| \le Ct^{2\alpha}.$$

We approximate the $C^{1,\beta}$-numerator of the fraction, using its first-order Taylor polynomial in t, and find that

$$\left| f_3(p \exp tX_1) - f_3(p) + \frac{t}{2}\big(f_2(p)\tilde{X}_1 f_1(p) - f_1(p)\tilde{X}_1 f_2(p)\big)\right| = o(t).$$

Thus $\tilde{X}_1 f_3(p)$ exists and is equal to $(f_1(p)\tilde{X}_1 f_2(p) - f_2(p)\tilde{X}_1 f_1(p))/2$. By hypothesis, $f_1, f_2 \in C^{k,\beta}(\Omega_1)$, and so $\tilde{X}_1 f_3 \in C^{k-1,\beta}(\Omega_1)$. The same argument is applied to the vector field $\tilde{X}_2$ to complete the proof.

Proof of Theorem 6.1

We show by induction that if $\pi_j \log f \in C^{k,\beta}(\Omega_1, \mathfrak{g}_2)$ when $j = 1, 2, \dots, l-1$, where $l \geq 2$, then $\pi_l \log f \in C^{k,\beta}(\Omega_1, \mathfrak{g}_2)$.

Fix a point p in Ω_1, and fix X in V_1. Since $f \in C^{0,\alpha}_{\mathrm{loc}}(\Omega_1, \Omega_2)$, there exists an interval I containing zero, and a constant C, such that

$$\varrho\left(f(p \exp tX), f(p)\right) \leq C\,\varrho(p \exp tX, p)^\alpha = C\,\varrho(\exp tX, e)^\alpha = C|tX|^\alpha$$

for all $t \in I$. Write $f(q)$ as $\exp(F(q))$. Then

$$\begin{aligned}\varrho\left(f(p \exp tX), f(p)\right) &= \varrho\left(\exp(-F(p))\exp(F(p \exp tX)), e\right)\\ &= \varrho\left(\exp \mathrm{BCH}(-F(p), F(p \exp tX)), e\right)\\ &\simeq \left(\sum_{k=1}^{R} \left|\pi_k \mathrm{BCH}\big(-F(p), F(p \exp tX)\big)\right|^{2R!/k}\right)^{1/2R!}.\end{aligned}$$

From the hypothesis, it follows that

$$\left|\pi_l \mathrm{BCH}\big(-F(p), F(p \exp tX)\big)\right| \leq C|t|^{l\alpha} \quad \forall t \in I. \tag{6.1}$$

Write $F_1(q)$ for $\pi_1 F(q) + \cdots + \pi_{l-1}F(q)$, $F_2(q)$ for $\pi_l F(q)$, and $F_3(q)$ for $\pi_{l+1}F(q) + \cdots + \pi_R F(q)$. As $\mathfrak{g}_2$ is graded, $\pi_l \mathrm{BCH}(-F(p), F(p \exp tX))$ is equal to

$$F_2(p \exp tX) - F_2(p) + \pi_l \mathrm{BCH}\big(-F_1(p), F_1(p \exp tX)\big), \tag{6.2}$$

and $\pi_l \mathrm{BCH}(-F_1(p), F_1(p \exp tX))$ is a finite weighted sum of projections of commutators; assuming that $l \geq 4$, a typical term is

$$\pi_l\big[F_1(p), [F_1(p \exp tX), [F_1(p), F_1(p \exp tX)]]\big].$$

By hypothesis, $F_1 \in C^{k,\beta}(\Omega_1, \mathfrak{g}_2)$, so $F_1(p \exp tX) = F_1(p) + t\tilde{X}F_1(p) + o(t)$. Substituting this into our typical commutator, we obtain

$$t\pi_l\big[F_1(p), [F_1(p), [F_1(p), \tilde{X}F_1(p)]]\big] + o(t).$$

In general, we obtain t multiplied by a commutator involving $F_1(p)$ (perhaps more than once) and $\tilde{X}F_1(p)$ (once only), and higher-order terms. From (6.1), (6.2), and our discussion of $\pi_l \mathrm{BCH}(-F_1(p), F_1(p \exp tX))$, we deduce that $\pi_l \mathrm{BCH}(-F_1(p), F_1(p \exp tX))$ is equal to

$$t \frac{d}{dt} \pi_l \mathrm{BCH}\big(-F_1(p), F_1(p \exp tX)\big)\Big|_{t=0} + o(t),$$

and hence,

$$\left| F_2(p \exp tX) - F_2(p) + t \frac{d}{dt} \pi_l \mathrm{BCH}\big(-F_1(p), F_1(p \exp tX)\big)\Big|_{t=0} \right| = o(t).$$

From [20, Chapter II, Theorem 1.7],

$$\frac{d}{dt} \mathrm{BCH}(-A, A + tB)|_{t=0} = \frac{\exp(-\mathrm{ad} A) - I}{\mathrm{ad} A} B,$$

whence

$$\begin{aligned} \tilde{X}F_2(p) &= -\frac{d}{dt} \pi_l \mathrm{BCH}\big(-F_1(p), F_1(p \exp tX)\big)\Big|_{t=0} \\ &= -\sum_{n=1}^{\infty} \frac{(\mathrm{ad} F_1(p))^{n-1} X F_1(p)}{n!}. \end{aligned}$$

The latter, in turn, is a weighted sum of commutators such as (for instance) $\big[F_1(p), [F_1(p), [F_1(p), \tilde{X}F_1(p)]]\big]$, so that $\tilde{X}F_2 \in C^{k-1,\beta}(\Omega_1)$. Since X is an arbitrary element of V_1, we conclude that $F_2 \in C^{k,\beta}(\Omega_1)$. □

7. Proof of Theorem 1.1 and applications

We assume throughout this section that Ω_1 and Ω_2 are open subsets of Carnot groups G_1 and G_2.

Proof of Theorem 1.1

Take a $P1$-quasiconformal map $f : \Omega_1 \to \Omega_2$. Then $X_i \cdot \log f$ is smooth when $1 \le i \le m_1$, by Corollary 5.8, and f is locally Lipschitz, by Theorem 5.5. Apply Theorem 6.1. □

Definition 7.1

The diffeomorphism $f : \Omega_1 \to \Omega_2$ is said to be *conformal* if it is Pansu differentiable and df, restricted to V_1, is a similarity everywhere.

COROLLARY 7.2

For a map $f : \Omega_1 \to \Omega_2$, the following are equivalent:

(i) f *is* 1*-quasiconformal;*

(ii) f *is* $P1$*-quasiconformal;*

(iii) f *is smooth and conformal.*

Proof

Lemma 5.3 shows that (i) implies (ii). Conversely, suppose that f is $P1$-quasiconformal. By Lemma 5.4, so is f^{-1}. By Theorem 1.1, f and f^{-1} are Pansu differentiable everywhere, and so Df is invertible, by the chain rule. Apply Lemma 3.3. The equivalence of (ii) and (iii) follows from Theorem 1.1 and (2.1). □

Pansu [28, Proposition 13.1] gives examples of two-step Carnot groups whose Carnot automorphisms are all dilations. We show that all quasiconformal maps of domains in these groups are compositions of translations and dilations. Pansu was clearly aware of this result (see, e.g., [29] and [28, Théorème 4]).

LEMMA 7.3

Suppose that Ω_1 and Ω_2 are connected open subsets of a Carnot group G of step at most 2*. Suppose that $f : \Omega_1 \to \Omega_2$ is quasiconformal, and suppose that Df is a dilation almost everywhere in Ω_1. Then f is a translation composed with a dilation.*

Proof

By hypothesis, f is $P1$-quasiconformal, and hence, smooth. Thus Df is a dilation everywhere. Write $Df(p)$ as $\delta_{\alpha(p)}$. We show first that α is constant.

Take linearly independent X and Y in V_1. Then X and Y generate a two- or three-dimensional subalgebra $\mathfrak{n}(X, Y)$ of $\mathfrak{g}$, depending on whether or not $[X, Y] = 0$. Write $N(X, Y)$ for $\exp(\mathfrak{n}(X, Y))$. The vector fields $\tilde{Z}$ corresponding to Z in $\mathfrak{n}(X, Y)$ give rise to an integrable distribution on G, and G is foliated by the cosets of $N(X, Y)$. The differential of f maps $(\tilde{X})_p$ and $(\tilde{Y})_p$ to multiples of $(\tilde{X})_{f(p)}$ and $(\tilde{Y})_{f(p)}$, so f respects the foliation and maps connected components of leaves to connected components of leaves.

Fix p in Ω_1, and restrict f to the leaf through p. By composing f with translations, we may suppose that $p = e$ and $f(p) = e$. Thus, at least locally, f is a map of $N(X, Y)$ into itself, whose Pansu derivative is a dilation at each point. Suppose that $N(X, Y) = \mathbb{R}^2$. We write the restricted map f in coordinates, and then

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}.$$

It follows that $\partial\alpha/\partial x_1 = \partial^2 f_2/\partial x_1 \partial x_2 = 0$, and similarly, $\partial\alpha/\partial x_2 = 0$. We conclude that α is locally constant on the cosets of $N(X, Y)$ in this case.

If $\mathfrak{n}(X, Y)$ is three-dimensional, then the restriction of f to $N(X, Y)$ is a 1-quasiconformal map between domains of the Heisenberg group; hence by [22, Theorem 8], it is the restriction of the action of an element of SU(1, 2). These actions are described in [22, Section E], and those whose Pansu differential is a dilation at every point are made of dilations and translations. Thus α is locally constant on the cosets of $N(X, Y)$ in this case too.

It follows that $\tilde{X}\alpha = 0$ on Ω_1. But X was an arbitrary element of $V_1 \setminus \{0\}$, so $\nabla_H \alpha = 0$ and α is constant.

By composing with δ_α^{-1}, we may assume that $Df = \delta_1$ everywhere in Ω_1. By again considering the leaves of the foliation associated to $N(X, Y)$, we see that f acts as a translation on each leaf. This means that if $q \in \Omega_1$ and $f(q_1) = q_2$, then $f(q_1 \exp(tX)) = q_2 \exp(tX)$ for all sufficiently small t.

Suppose that $X_1, \ldots, X_N \in V_1$, and suppose that $f(q_1) = q_2$. By iterating the preceding argument, it may be seen that

$$f\big(q_1 \exp(t_1 X_1) \cdots \exp(t_N X_N)\big) = q_2 \exp(t_1 X_1) \cdots \exp(t_N X_N)$$

for all sufficiently small $t_1, \ldots, t_N$. From [12, Lemma 5.1], it follows that $f(q_1 x) = q_2 x$ for all x in some neighborhood of the identity in G_1.

Let Ω_0 be $\{p \in \Omega_1 \mid f(p) = q_2 q_1^{-1} p\}$. Then Ω_0 is clearly nonempty and closed and, by the argument above, is also open, so it is all Ω_1. This shows that f is a translation on Ω_1. □

COROLLARY 7.4
Suppose that G is a step-two Carnot group whose Carnot automorphisms are all dilations. Then the 1*-quasiconformal maps between domains of G are all translations composed with dilations.*

Before we state our final result, we recall that the so-called H-type groups (see [8], [21]) are important examples of Carnot groups.

COROLLARY 7.5
Suppose that G is an H-type group whose Lie algebra has center of dimension larger than 2*. The* 1*-quasiconformal maps between open subsets of G are smooth and form a finite-dimensional space.*

Proof
This follows from the work of Reimann [30], who established the corresponding infinitesimal result. Our main theorem allows us to obtain information about 1-quasiconformal maps by the standard method. □

For many H-type groups, all 1-quasiconformal maps are composed of translations and dilations. However, for the Iwasawa N-groups of real rank-one simple Lie groups, the group of 1-quasiconformal maps is like the Möbius group and contains inversions as well.

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