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Ahlfors Type Estimates For Perimeter Measures In Carnot-carathéodory Spaces

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AHLFORS TYPE ESTIMATES FOR PERIMETER MEASURES IN CARNOT-CARATHÉODORY SPACES.

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Abstract. We study the relationship between the geometry of hypersurfaces in a Carnot-Carathéodory (CC) space and the Ahlfors regularity of the corresponding perimeter measure. To this end we establish comparison theorems for perimeter estimates between an hypersurface and its tangent space, and between a CC geometry and its "tangent" Carnot group structure.

1. Introduction.

Ahlfors type conditions for Borel measures play a crucial role in the analysis of boundary value problems for partial differential equations, both in the classical theory and in more recent studies, see [DS1], [CKL], [DGN2] and the references therein. A frequent instance of such condition is expressed by the following estimate

$$C^{-1}\, r^s \ \le\ \mu(B(x,r))\ \le\ C\, r^s\ , \tag{1.1}$$

where μ is a Borel measure supported in a closed set $F \subset \mathbb{R}^n$, and $r > 0$ is sufficiently small. Particularly important is the situation in which $F = \partial\Omega$, where $\Omega \subset \mathbb{R}^n$ is a given open set. In this context, estimates such as (1.1) reflect regularity properties of the set $\partial\Omega$. For instance, if Ω is a Lipschitz domain and μ represents the perimeter measure in the sense of De Giorgi relative to Ω, then (1.1) holds for $s = n-1$, see for instance [AFP]. The relevance of estimates such as (1.1) is also shown by their central role in the analysis of metric spaces, see [DS1] and [He].

Recently, there has been a growing interest in Ahlfors type estimates on lower dimensional manifolds in *Carnot-Carathéodory (CC) spaces*, especially in view of the key role played by such estimates in the development of boundary value problems and of geometric measure theory. For some of these aspects we refer the reader to the papers [CGN1], [DGN1], [CGN2], [CGN3], [DGN2], [DGN3], [DGN4]. In the present paper we establish estimates of Ahlfors type for hypersurfaces in CC spaces. Our estimates from above have been recently announced, and also used, in [DGN2]: here, we prove the relevant results. Also, we establish estimates from below which generalize to hypersurfaces in CC spaces those obtained in [DGN2] for Carnot groups of step $r = 2$.

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We recall that a CC space is a Riemannian manifold (M^n, g) endowed with a distance d different from the Riemannian one $d_{\mathcal{R}}$ generated by the metric tensor g. Such distance d is the control metric associated with a given subbundle HM^n of the tangent bundle TM^n. If $X = \{X_1, ..., X_m\}$ is a smooth distribution of vector fields (locally) describing HM^n, then the basic assumption is that X satisfy Hörmander's finite rank condition [H]

$$\text{rank } Lie[X_1, ..., X_m] \equiv n . \tag{1.2}$$

The class of CC spaces encompasses of course all Riemannian manifolds. Less trivial examples include Euclidean $\mathbb{R}^n$ with a system of C^∞ vector fields satisfying (1.2), but also the Gromov-Hausdorff limit of some sequences of Riemannian manifolds, see [Gro]. Moreover, tangent spaces of CC spaces are themselves CC spaces endowed with a special non-Abelian structure. They are graded nilpotent Lie groups, also known as *Carnot groups*, or quotient spaces of Carnot groups.

Given a CC space we will denote by $B(x,r) = \{y \in M^n \mid d(x,y) < r\}$ the open ball centered at x with radius r in the control metric d. If v_g indicates the volume form on M^n, attached to the metric tensor g, we let $|E| = \int_E dv_g$ denote the ordinary Lebesgue measure of the measurable set $E \subset M^n$. We recall that the Lebesgue measure $|B(x,r)|$ of the CC balls was studied in a fundamental paper by Nagel, Stein and Wainger [NSW]. Their main result states that for every bounded set $K \subset M^n$ there exist $C, R_o > 0$, depending on K such that for every $x \in K$ and $r < R_o$ one has

$$C\ \Lambda(x,r) \ \le\ |B(x,r)| \ \le\ C^{-1}\ \Lambda(x,r)\ . \tag{1.3}$$

Here, the Nagel-Stein-Wainger polynomial

$$\Lambda(x,r) \ =\ \sum_I\ |a_I(x)|\ r^{d(I)}\ , \tag{1.4}$$

is defined as follows: For every $x \in M^n$ denote by $Y_1, ..., Y_l$ the collection of the X_j's and of those commutators which are needed to generate $T_x M^n$. A "degree" is assigned to each Y_i, namely the corresponding order of the commutator. If $I = (i_1, ..., i_n), 1 \le i_j \le l$, is a n-tuple of integers, one defines $d(I) = \sum_{j=1}^n deg(Y_{i_j})$, and $a_I(x) = \det\ (Y_{i_1}, ..., Y_{i_n})$.

As one can easily infer from (1.4), apart from the situation of Carnot groups, which are CC spaces endowed with a homogeneous structure of dilations, for general CC spaces the Lebesgue measure does not satisfy (1.1) for any choice of s. This observation led in [DGN1], [DGN2] to consider a modified version of the Ahlfors type estimates. Given a CC space (M^n, g, d), denote by $\mathcal{B}$ the class of non-negative Borel measures on it.

Definition 1.1. *Given $s \ge 0$, a measure $\mu \in \mathcal{B}$ is called an* upper s-Ahlfors measure with respect to the CC distance *if there exist $M, R_o > 0$, such that for $x \in M^n$, $0 < r \le R_o$, one has*

$$\mu(B(x,r)) \le\ M\ \frac{|B(x,r)|}{r^s}. \tag{1.5}$$

One says that μ is a lower s-Ahlfors measure*, if for some $M, R_o > 0$ one has for x and r as above*

$$\mu(B(x,r)) \ \geq \ M^{-1} \ \frac{|B(x,r)|}{r^s}. \tag{1.6}$$

When μ is both an upper and lower s-Ahlfors measure, then it is called a s-Ahlfors measure on M^n with respect to the CC distance.

Next, we recall the definition of perimeter measure in a CC space, see [CDG1] and [GN1]. Let M^n be a CC space with respect to a given subbundle $HM^n \subset TM^n$, which we assume locally generated by a system of smooth vector fields $X = \{X_1, ..., X_m\}$. Given an open set $\Omega \subset M^n$, we denote by $\mathcal{F}(\Omega)$ the set of all vector fields $\zeta \in C^1_o(\Omega, HM^n)$ such that $|\zeta| \leq 1$. If $f \in L^1(\Omega)$, then the X-variation of f is defined by

$$Var_X(f;\Omega) \ = \ \sup_{\zeta \in \mathcal{F}(\Omega)} \int_\Omega f \ \sum_{j=1}^m X_j^* \zeta_j \ dv_g \ .$$

Given a measurable set $E \subset \mathbb{R}^n$ we define the X-*perimeter* of E with respect to Ω as

$$P_X(E;\Omega) \ = \ Var_X(\chi_E;\Omega) \ ,$$

where χ_E denotes the characteristic function of E. We also refer the reader to the papers [BM] and [FSS1] where related definitions of variation and perimeter were independently set forth. In the Euclidean geometry, if Ω is a smooth set then the perimeter is equivalent to the surface measure. The situation is quite different in the CC case. Let $\Omega = \{x \in M^n | \ \phi(x) < 0\}$, where $\phi : M^n \to \mathbb{R}$ is a C^1 *defining* function, *i.e.* $\nabla\phi \neq 0$ in a neighborhood of $\partial\Omega$. The sub-gradient of ϕ along the system X is defined by $X\phi = (X_1\phi, ..., X_m\phi)$, so that $|X\phi| = (\sum_{j=1}^m (X_j\phi)^2)^{1/2}$. Define a new measure supported on $\partial\Omega$ by letting for every Borel set $E \subset M^n$

$$\mu(E) \ \stackrel{def}{=} \ \int_{E\cap\partial\Omega} |X\phi| \ d\sigma \ , \tag{1.7}$$

where $\sigma = H_{n-1}\lfloor\partial\Omega$, and as before H_{n-1} indicates the $(n-1)$-dimensional Hausdorff measure on M^n constructed with the Riemannian distance $d_{\mathcal{R}}$. A key fact, see Theorem 5.8 in [DGN2], is the existence of $C = C(\Omega) > 0$ such that for every $g \in \partial\Omega$ and $r > 0$ one has

$$C \ \mu(B(x,r)) \ \leq \ P_X(\Omega; B(x,r)) \ \leq \ C^{-1} \ \mu(B(x,r)) \ . \tag{1.8}$$

It is clear that when $M^n = \mathbb{R}^n$, if $X = \{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\}$ is the standard basis of $\mathbb{R}^n$, then $d(x,y) = |x-y|$, and $d\mu = |\nabla\phi| d\sigma$ is equivalent to $d\sigma$. In the sub-Riemannian case, however, the *angle function* $|X\phi|$ vanishes on a subset of $\partial\Omega$, the so-called *characteristic set* of $\partial\Omega$. The existence of characteristic points makes controlling the measure $d\mu$, and thereby $P_X(\Omega;\cdot)$, a very delicate task.

The objective of this paper is to study the interplay between the geometry of a minimally smooth hypersurface $S = \partial\Omega \subset M^n$ in a CC space and the $s-$Ahlfors regularity of its perimeter measure $P_X(\Omega;\cdot)$. Such study is mainly motivated by the desire of:

(1) Constructing large classes of domains in CC spaces for which the perimeter is a (locally) upper $1-$Ahlfors measure. As we have mentioned above, this property find numerous applications to the study

of Dirichlet and Neumann problems for sub-Laplacians [CGN1], [CGN2], [CGN3], [DGN2], [DGN5], while general Ahlfors regularity plays a crucial role in the development of function spaces and potential theory on lower dimensional manifolds in CC spaces [DGN1], [DGN2], and geometric measure theory in CC spaces [DGN3], [DGN6].

(2) Studying the effect of linear approximation on perimeter estimates. Our proofs are based on a series of linear approximations: In the setting of Carnot group we approximate the domain with its tangent space. In the setting of general CC manifolds we approximate the whole geometry with its linearization, i.e. the *tangent free nilpotent Lie group.* The possibility of having an initial system of vector fields which is not free is dealt with through the *lifting* technique introduced by Rothschild and Stein [RS].

Carnot groups. From the point of view of the CC geometry, Euclidean smoothness is not relevant (see for instance [HH]), hence one should not expect $1-$Ahlfors regularity in general, even for C^∞ domains. One should rather aim at establishing some kind of $s-$Ahlfors regularity, where the exponent $s = s(x)$ is a real-valued function defined on the boundary of the domain. In order to describe the function $s(x)$ we introduce the notion of *type*. Henceforth, to distinguish the model setting of a Carnot group $\boldsymbol{G}$ from that of a general CC manifold M^n, we will indicate points in $\boldsymbol{G}$ with the letters g, g_o, etc., whereas points in M^n will be denoted by x, x_o, etc. With this in mind, let $\boldsymbol{G}$ be a Carnot group, and let $\phi \in C^1(\boldsymbol{G})$, such that $|\nabla\phi| \neq 0$ in a neighborhood of its zero level set. Consider the C^1 domain

$$\Omega \ = \ \{g \in \boldsymbol{G} \ \mid \phi(g) < 0\} \ . \tag{1.9}$$

As before, $d\sigma$ will denote the standard surface measure on $\partial\Omega$. We define the *type* of a point $g_o \in \partial\Omega$ as the smallest order of commutators which are transversal to the $\partial\Omega$ at g_o, see Definition 2.2. We stress that this definition depends only on the first order Taylor polynomial at $g_o \in \partial\Omega$ of the defining function ϕ. It will be helpful to the reader to keep in mind the following example. If $\boldsymbol{G}$ has step r, with Lie algebra $\mathfrak{g} = V_1 \oplus ... \oplus V_r$, let $m_j = dim(V_j)$ and denote by $\xi_j = (x_{j,m_1}, ..., x_{j,m_j})$, $j = 1, ..., r$, the projection of the exponential coordinates onto the j-th layer of the Lie algebra of $\mathfrak{g}$, see Section 2.1. For a fixed $j \in \{1, ..., r\}$ consider the "hyperplane" passing through the group identity e

$$\Pi_j \ = \ \{x_{j,m_s} \ = \ 0\} \ , \tag{1.10}$$

where $s \in \{1, ..., m_j\}$ is fixed. An elementary calculation shows that the point e is of type j. Thus for instance for any of the m_1 hyperplanes Π_1 the identity is of type 1, and therefore it is non-characteristic (one can easily recognize that, in fact, such hyperplanes do not possess any characteristic point). This example shows that the type of a point can be any integer ranging from 1 to the step r of the group. In the sequel, we will call "hyperplane" any manifold in $\boldsymbol{G}$ which in exponential coordinates is expressed by the zero set of a linear polynomial.

Having introduced the notion of type in a Carnot group, we now state the Ahlfors estimates in this setting. The reason for starting with this situation is twofold. First, the Ahlfors estimates in Carnot groups are more precise than those in a general CC manifold. Secondly, the analysis of this special situation constitutes the backbone of the general case. Hereafter, the number Q will denote the so-called homogeneous dimension of the group $\boldsymbol{G}$, see (2.9).

Theorem 1.2. *Let $\boldsymbol{G}$ be a Carnot group. One has:*

(a) *Let $\Pi \subset \boldsymbol{G}$ be an hyperplane of type k_o, containing the group identity e, and let σ denote its surface measure. There exist $M = M(\boldsymbol{G}, \Pi) > 0$ and $R_o = R_o(\boldsymbol{G}, \Pi) > 0$ such that for any $0 < R < R_o$ one has*

$$
\begin{aligned}
M^{-1}\ R^{Q-k_o} \ &\le\ \sigma(\Pi \cap B(e,R)) \ \le\ M R^{Q-k_o}\ , \\
M^{-1}\ R^{Q-1} \ &\le\ P_X(\Pi; B(e,R)) \ \le\ M R^{Q-1}\ .
\end{aligned}
\tag{1.11}
$$

(b) *If Ω is a bounded, open set as in (1.9), with $\phi \in C^{1,1}(\boldsymbol{G})$, then for every $g_o \in \partial\Omega$, let $k_o = k_o(g_o)$ be the type of g_o. There exist $M = M(\boldsymbol{G}, \Omega, g_o) > 0$ and $R_o = R_o(\boldsymbol{G}, \Omega, g_o) > 0$ depending continuously on g_o, such that for for any $0 < R < R_o$ one has*

$$
\sigma(\partial\Omega \cap B(g_o,R)) \ \le\ M\ R^{Q-k_o}, \qquad \text{and} \qquad P_X(\Omega; B(g_o,R)) \ \le\ M\ R^{Q-s(g_o)},
\tag{1.12}
$$

with

$$
s(g_o) = \begin{cases} k_o - 1, & \text{if } k_o \ge 3\ , \\ 1, & \text{if } \partial\Omega \text{ is real analytic near } g_o \text{ or if } k_o = 1, 2. \end{cases}
\tag{1.13}
$$

(c) *If $\phi \in C^2(\boldsymbol{G})$, there exist $M = M(\boldsymbol{G}, \Omega, g_o) > 0$ and $R_o = R_o(\boldsymbol{G}, \Omega, g_o) > 0$ as above such that if $g_o \in \partial\Omega$ is of type ≤ 2, then we also have*

$$
\sigma(\partial\Omega \cap B(g_o,R)) \ \ge M^{-1} R^{Q-1}, \qquad \text{and} \qquad P_X(\Omega; B(g_o,R)) \ \ge\ M^{-1}\ R^{Q-1}\ ,
\tag{1.14}
$$

for $0 < R < R_o$.

In the setting of the Heisenberg group, the upper 1-Ahlfors regularity of $P_X(\Omega; \cdot)$ was established in [DGN1]. The same result was subsequently generalized to Carnot groups of step 2 in [CGN2]. The lower 1-Ahlfors regularity for C^2 domains in Carnot groups of step 2 has been recently established in [DGN2]. Our proof is based on comparison between perimeter estimates for the domain and for its tangent space, and it builds on the arguments first introduced in [CGN2] and [DGN2]. We also recall that a geometric measure theory proof of the lower bounds for Carnot groups of step two can be found in [DGN1], Theorems 1.7 and 1.8, and [DGN2], see the proof of Theorem 7.1 in Section 7.1. Such proof uses the relative isoperimetric inequality in [GN1], but also uniform density estimates at the boundary for the relevant domain. Despite its seemingly elementary character, and the fact that it applies to $C^{1,1}$ domains (whereas in (1.14) we need to assume C^2 smoothness) this proof has the disadvantage that, since the perimeter measure does not see the characteristic set, the crucial role played by the latter is not transparent. Also, the necessary density estimates are hard to come by in the case of groups of higher step. We mention here that for the lower Ahlfors estimates the $C^{1,1}$ smoothness is best possible. In fact, in the Heisenberg group $\mathbb{H}^n$, for any $0 < \alpha < 1$ there exist $C^{1,\alpha}$ domains for which the lower 1-Ahlfors regularity of $P_X(\Omega; \cdot)$ in Theorem 1.2 is not true, see Section 7.4 in [DGN2]. The "*type assumption*" is necessary. In fact, in Section 4 we construct an example of an analytic domain of type 3 in a Carnot group of step 3 for which the lower $1-$Ahlfors regularity of the X-perimeter fails.

General CC setting. Having discussed the model setting of Carnot groups, we now turn to that of a general CC manifold M^n. Given a CC manifold with generating distribution $\{X_1, ..., X_m\}$, consider a C^1 domain $\Omega = \{x \in M^n \mid \phi(x) < 0\}$. We say that a point $x_o \in \partial\Omega$ is of type ≤ 2 if either there exists $j_o \in \{1, ..., m\}$ such that $X_{j_o}\phi(x_o) \neq 0$, i.e., x_o is non-characteristic, or there exist indices $i_o, j_o \in \{1, ..., m\}$ such that $[X_{i_o}, X_{j_o}]\phi(x_o) \neq 0$. We say that Ω is of type ≤ 2 if every point $x_o \in \partial\Omega$ is of type ≤ 2. It is important to stress that when M^n is a CC space of rank $r \leq 2$, then every C^1 domain is automatically of type ≤ 2.

Theorem 1.3. *Let (M^n, d) be a CC space and consider a bounded $C^{1,1}$ domain $\Omega \subset M^n$. For every point $x_o \in \partial\Omega$ of type ≤ 2 there exist $M = M(\Omega, x_o) > 0$ and $R_o = R_o(\Omega, x_o) > 0$, depending continuously on x_o, such that for any $0 < r < R_o$ one has*

$$P_X(\Omega; B(x_o, r)) \ \leq \ M \ \frac{|B(x_o, r)|}{r} \ . \tag{1.15}$$

The same conclusion holds if $\partial\Omega$ is real anlytic in a neighborhood of x_o, regardless of the type of x_o.

If $\Omega \subset M^n$ is a bounded C^2 domain, then for every point $x_o \in \partial\Omega$ of type ≤ 2 there exist $M = M(\Omega, x_o) > 0$ and $R_o = R_o(\Omega, x_o) > 0$ depending continuously on x_o, such that for any $0 < r < R_o$, one has

$$P_X(\Omega; B(x_o, r)) \ \geq \ M^{-1} \ \frac{|B(x_o, r)|}{r} \ . \tag{1.16}$$

The following immediate corollary applies to several important examples such as CR manifolds (see [Krantz]).

Corollary 1.4. *Let (M^n, d) be a CC space with step less or equal than two. For every bounded $C^{1,1}$ hypersurface $\partial\Omega \subset M^n$ the perimeter measure $P_X(\Omega; \cdot)$ is a $1-$Ahlfors measure.*

Final remarks and open problems. (a) Theorems 1.2 and 1.3 provide two-sided $1-$Ahlfors estimates near points in $\Delta_2 = \{x_o \in \partial\Omega \mid \ type(x_o) \ \leq \ 2\}$, while the closed set $\partial\Omega \setminus \Delta_2$, is essentially the set where such estimates fail. This set is small in the following sense:

$$H_{n-1}(\partial\Omega \setminus \Delta_2) \ = \ 0 \ . \tag{1.17}$$

When M^n is a Carnot group the following stronger information is available

$$\mathcal{H}^{Q-1}(\partial\Omega \setminus \Delta_2) \ = \ 0 \ . \tag{1.18}$$

Here we denote by H_s the s-dimensional Hausdorff measure constructed with the Riemannian distance $d_{\mathcal{R}}$ of M^n, and with the notation $\mathcal{H}^s$ we indicate instead the s-dimensional Hausdorff measure constructed with the CC distance d. Equation (1.17) is essentially due to Derridj [De]. Although he actually proved that the complement of the characteristic set has zero H_{n-1}-dimensional measure for C^∞ domains, his ideas can be adapted to cover the case of C^2 domains. Equation (1.18) instead, follows from the recent work of Magnani [Ma].

(b) In Section 7 we analyze the connection between the 1-Ahlfors regularity of the X-perimeter and boundary value problems for sub-Laplacians. We show that the former property implies the regularity of the relevant domain with respect to the Dirichlet problem. This fact, combined with some examples of Hansen and Hueber [HH], gives another (indirect) proof of the impossibility of the 1-Ahlfors estimates when the domain is of type ≥ 3.

(c) If the step of a CC space is less or equal than two then every C^1 hypersurface is of type less or equal than two. In the higher step case we do not know of any examples of such domains. In analogy with the Heisenberg group one can expect the existence of special tori, possibly of higher genus, which are non-characteristic or of type two. It seems difficult to construct such domains and their existence is an open problem at the moment.

(d) In the C^∞ category, in a CC space, Monti and Morbidelli [MM1] have recently proved the $1-$Ahlfors regularity of the ordinary surface measure $d\sigma$ away from characteristic points. The approach in [MM1] substantially differs from ours and fails to work in a neighborhood of characteristic points. For other regularity results in this vein, see [MM2], and [MM3], where the type condition is also implicitly used.

(e) One of the underlying goals in the present paper is to construct explicit classes of domains in which boundary value problems for sub-Laplacians are well-posed. Since every closed hypersurface must have non-empty characteristic set Σ, then it becomes necessary to include characteristic points in our analysis. This consideration accounts for the relatively strong (Euclidean) regularity assumption near Σ.

On the other hand, it would be interesting to study Ahlfors regularity for the perimeter of intrinsically regular domains, as defined in [FSS2]-[FSS3]. Such domains by definition cannot contain characteristic points and may be rather irregular from the Euclidean point of view (see the recent work of Kircheim and Serra Cassano [KSC]).

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2. **Carnot-Carathéodory spaces**

In this section we recall the definition of CC manifolds and of their tangent spaces, the class of Carnot groups. The relation between CC spaces and Carnot groups is described in a series of papers by Rothschild and Stein [RS], Folland [F], Nagel, Stein and Wainger [NSW], and Sanchez-Calle [SC]. One should also see [Mi], [F], [FS], [FSC], [Str], [P], [Be], [Gro], and [Mon].

Let (M^n, g) be a smooth Riemannian manifold, with $n \geq 3$, with volume form dv_g. Denote by $d_{\mathcal{R}}$ the Riemannian distance on M^n, and by $|E| = \int_E dv_g$ the standard Lebesgue measure of a measurable set

$E \subset M^n$. We consider a given subbundle $HM^n \subset TM^n$ of the tangent bundle. Let $X = \{X_1, ..., X_m\}$ be a system of C^∞ vector fields which locally generate HM^n, and consider the system of differential equations

$$\gamma' \ = \ \sum_{j=1}^{m} u_j(t) \ X_j(\gamma) \ , \tag{2.1}$$

where the *control* $u = (u_1, ..., u_m)$ is assumed to belong to $L^1([a,b], \mathbb{R}^m)$. If the path $\gamma : [a,b] \to M^n$ solves the above system and if $\gamma(a) = x$, $\gamma(b) = y$, then one says that the control u steers the system from the state x to the state y. The length of γ is defined by

$$l(\gamma) \ = \ \int_a^b \sqrt{u_1(t)^2 \ + \ ... \ + \ u_m(t)^2} \ dt \ .$$

Next, for $x \in M^n$ and $v \in T_x M^n$ we let

$$h_x(v) \ = \ \inf \ \{||u||^2 = u_1^2 + ... + u_m^2 \mid u_1 X_1(x) + ... + u_m X_m(x) = v\} \ .$$

If v lies outside $H_x M^n$, then one lets $h_x(v) = +\infty$. In this way, on each section $H_x M^n$ of the subbundle $HM^n \subset TM^n$ we have defined a quadratic form h_x. The *sub-Riemannian metric* associated with the subbundle HM^n is given by the assignment $x \to h_x$. We set $||v||_{H,x} \ = \ \sqrt{h_x(v)}$, and define the *horizontal length* of an absolutely continuous path $\gamma : [a,b] \to M^n$ as $l_H(\gamma) \ = \ \int_a^b ||\gamma'(t)||_{H,\gamma(t)} \ dt$. An absolutely continuous path γ is called *horizontal* (or *controlled*), if it satisfies (2.1) for a measurable control $u(t) = (u_1(t), ..., u_m(t))$. Given an open set $\Omega \subset M^n$, and two points $x, y \in \Omega$, we denote by $\mathcal{H}_\Omega(x,y)$ the collection (possibly empty) of all horizontal paths $\gamma : [a,b] \to \Omega$ joining x to y. The accessibility Theorem of Chow (see [NSW]) states that if at every $x \in M^n$ the system $X = \{X_1, ..., X_m\}$ which locally describes HM^n satisfies the finite rank condition (1.2), then if $\Omega \subset M^n$ is connected one has $\mathcal{H}_\Omega(x,y) \neq \varnothing$ for every $x, y \in \Omega$. This basic result allows to define the *Carnot-Carathéodory* (or *control*) *distance* between x and y as

$$d_\Omega(x,y) \ = \ \inf \ \{l_H(\gamma) \mid \gamma \in \mathcal{H}_\Omega(x,y)\} \ .$$

When $\Omega = M^n$, we write $d(x,y)$ instead of $d_{M^n}(x,y)$.

2.1. **Carnot groups.** Next, we describe in detail a special subclass of CC spaces which plays a basic role in the development of the general theory. A *Carnot group* of step r is a connected, simply connected Lie group $\boldsymbol{G}$ whose Lie algebra $\mathfrak{g}$ admits a stratification $\mathfrak{g} \ = \ V_1 \oplus ... \oplus V_r$ which is r-nilpotent, i.e., $[V_1, V_j] \ = \ V_{j+1}$, $\quad j = 1, ..., r-1$, and $[V_j, V_r] \ = \ \{0\}$, $j = 1, ..., r$. By these assumptions one immediately sees that any basis of the *horizontal layer* V_1 satisfies the finite rank condition (1.2). A trivial example of (an abelian) Carnot group is $\boldsymbol{G} = \mathbb{R}^n$, whose Lie algebra admits the trivial stratification $\mathfrak{g} = V_1 = \mathbb{R}^n$. The simplest non-abelian example is the Heisenberg group $\mathbb{H}^n$, already described in the introduction, whose Lie algebra is given by $\mathfrak{h}_n = V_1 \oplus V_2$, with $V_1 = \mathbb{C}^n$, $V_2 = \mathbb{R}$.

We assume that a scalar product $< \cdot, \cdot >$ is given on $\mathfrak{g}$ for which the $V_j' s$ are mutually orthogonal. Let $\pi_j : \mathfrak{g} \to V_j$ denote the projection onto the j-th layer of $\mathfrak{g}$. Since the exponential map $exp : \mathfrak{g} \to \boldsymbol{G}$ is a global analytic diffeomorphism [V], we can define analytic maps $\xi_j : \boldsymbol{G} \to V_j$, $j = 1, ..., r$, by letting

$\xi_j = \pi_j \circ \exp^{-1}$. As a rule, we will use letters g, g', g'', g_o for points in $\boldsymbol{G}$, whereas we will reserve the letters $\xi, \xi', \xi'', \xi_o, \eta$, for elements of the Lie algebra $\mathfrak{g}$. We let $m_j = dim\ V_j$, $j = 1, ..., r$, and denote by $n = m_1 + ... + m_r$ the topological dimension of $\boldsymbol{G}$. The notation $\{X_{j,1}, ..., X_{j,m_j}\}$, $j = 1, ..., r$, will indicate a fixed orthonormal basis of the $j - th$ layer V_j. For $g \in \boldsymbol{G}$, the projection of the *exponential coordinates* of g onto the layer V_j, $j = 1, ..., r$, are defined as follows

$$x_{j,s}(g) = < \xi_j(g), X_{j,s} >, \qquad s = 1, ..., m_j. \tag{2.2}$$

The vector $\xi_j(g) \in V_j$, $j = 1, ..., r$, will be routinely identified with the point

$$(x_{j,1}(g), ..., x_{j,m_j}(g)) \in \mathbb{R}^{m_j} .$$

It will be easier to have a separate notation for the horizontal layer V_1. For simplicity, we set $m = m_1$, and let

$$X = \{X_1, \ ... \ , X_m\} = \{X_{1,1}, \ ... \ , X_{1,m_1}\} . \tag{2.3}$$

We indicate with

$$x_i(g) = < \xi_1(g), X_i >, \qquad i = 1, ..., m , \tag{2.4}$$

the projections of the exponential coordinates of g onto V_1. Whenever convenient, we will identify $g \in \boldsymbol{G}$ with its exponential coordinates

$$x(g) \stackrel{def}{=} (x_1(g), ..., x_m(g), x_{2,1}(g), ..., x_{2,m_2}(g), ..., x_{r,1}(g), ..., x_{r,m_r}(g)) \in \mathbb{R}^n , \tag{2.5}$$

and we will ordinarily drop in the latter the dependence on g, i.e., we will write $g = (x_1, ..., x_{r,m_r})$.

Each element of the layer V_j is assigned the formal degree j. Accordingly, one defines dilations on $\mathfrak{g}$ by the rule

$$\Delta_\lambda \xi = \lambda\ \xi_1 + ... + \lambda^r\ \xi_r ,$$

provided that $\xi = \xi_1 + ... + \xi_r \in \mathfrak{g}$. Using the exponential mapping $\exp : \mathfrak{g} \to \boldsymbol{G}$, these dilations are then tansferred to the group

$$\delta_\lambda(g) = \exp \circ \Delta_\lambda \circ \exp^{-1} g .$$

We will denote by

$$L_{g_o}(g) = g_o\ g , \qquad R_{g_o}(g) = g\ g_o , \tag{2.6}$$

respectively, the left- and right-translations on $\boldsymbol{G}$ by an element $g_o \in \boldsymbol{G}$. We continue to denote by X the corresponding system of left-invariant vector fields on $\boldsymbol{G}$ defined by

$$X_j(g) = (L_g)_*(X_j) , \qquad j = 1, ..., m ,$$

where $(L_g)_*$ denotes the differential of L_g. The system X defines a basis for the so-called *horizontal subbundle* $\boldsymbol{HG}$ of the tangent bundle $\boldsymbol{TG}$. If we keep in mind that the integral curve of X_j passing through

$g = exp(\xi)$ is given by $exp(\xi)\ exp(tX_j)$, then given a function $u : \boldsymbol{G} \to \mathbb{R}$, the action of X_j on u is specified by the equation

$$X_j u(g) \ = \ \lim_{t\to 0} \frac{u(g\ exp\ (tX_j)) \ - \ u(g)}{t} \ = \ \frac{d}{dt}\, u(g\ exp\ (tX_j))\big|_{t=0} \ . \tag{2.7}$$

A similar formula holds for any left-invariant vector field. We now recall the Baker-Campbell-Hausdorff formula, see, e.g., sec.2.15 in [V],

$$exp(\xi)\ exp(\eta) \ = \ exp\bigg(\xi + \eta + \frac{1}{2}\ [\xi,\eta] + \frac{1}{12}\{[\xi,[\xi,\eta]] - [\eta,[\xi,\eta]]\} + ...\bigg) \ , \tag{2.8}$$

where the dots indicate commutators of order four and higher. Using (2.8) we can express (2.7) using the coordinates (2.5), obtaining the following lemma.

Lemma 2.1. *For each* $i = 1, ..., m$, *and* $g = (x_1, ..., x_{r,m_r})$, *we have*

$$\begin{aligned} X_i \ = \ X_i(g) \ &= \ \frac{\partial}{\partial x_i} \ + \ \sum_{j=2}^{r}\sum_{s=1}^{m_j} \ b^s_{j,i}(x_1, ..., x_{j-1,m_{(j-1)}})\ \frac{\partial}{\partial x_{j,s}} \\ &= \ \frac{\partial}{\partial x_i} \ + \ \sum_{j=2}^{r}\sum_{s=1}^{m_j} \ b^s_{j,i}(\xi_1, ..., \xi_{j-1})\ \frac{\partial}{\partial x_{j,s}} \ , \end{aligned}$$

where each $b^s_{j,i}$ *is a homogeneous polynomial of* weighted degree $j-1$.

By weighted degree we mean that, as previously mentioned, the layer V_j, $j = 1, ..., r$, in the stratification of $\mathfrak{g}$ is assigned the formal degree j. Correspondingly, each homogeneous monomial $\xi_1^{\alpha_1}\xi_2^{\alpha_2}...\xi_r^{\alpha_r}$, with multi-indices $\alpha_j = (\alpha_{j,1}, ..., \alpha_{j,m_j}), j = 1, ..., r$, is said to have weighted degree k if

$$\sum_{j=1}^{r} j|\alpha_j| \ = \ \sum_{j=1}^{r} j\ (\sum_{s=1}^{m_j} \alpha_{j,s}) \ = \ k \ .$$

Throughout the paper we will indicate by dg the bi-invariant Haar measure on $\boldsymbol{G}$ obtained by lifting via the exponential map exp the Lebesgue measure on $\mathfrak{g}$. One easily checks that

$$(d \circ \delta_\lambda)(g) \ = \ \lambda^Q\ dg, \qquad \text{where} \quad Q \ = \ \sum_{j=1}^{r} j\ dim(V_j). \tag{2.9}$$

The number Q, called the *homogeneous dimension* of $\boldsymbol{G}$, plays an important role in the analysis of Carnot groups. In the non-abelian case $r > 1$, one clearly has $Q > n$.

We denote by $d(g, g')$ the *CC distance* on $\boldsymbol{G}$ associated with the system X. It is well-known that $d(g, g')$ is equivalent to the *gauge pseudo-metric* $\rho(g, g')$ on $\boldsymbol{G}$, i.e., there exists a constant $C = C(\boldsymbol{G}) > 0$ such that

$$C\ \rho(g, g') \ \le \ d(g, g') \ \le \ C^{-1}\ \rho(g, g'), \qquad\qquad g, g' \in \boldsymbol{G}, \tag{2.10}$$

see [NSW], [VSC]. The pseudo-distance $\rho(g,g')$ is defined as follows. Let $|\cdot|$ denote the Euclidean distance to the origin on $\mathfrak{g}$. For $\xi = \xi_1 + \cdots + \xi_r \in \mathfrak{g}$, $\xi_i \in V_i$, one lets

$$(2.11) \qquad |\xi|_{\mathfrak{g}} \ = \ \left(\sum_{j=1}^{r} |\xi_j|^{2r!/j} \right)^{\frac{1}{2r!}} , \qquad |g|_{\boldsymbol{G}} \ = \ |\exp^{-1} \ g|_{\mathfrak{g}}, \qquad g \in \boldsymbol{G},$$

and defines

$$(2.12) \qquad \rho(g,g') \ = \ |g^{-1} \ g'|_{\boldsymbol{G}}.$$

Both d and ρ are invariant under left-translations

$$(2.13) \qquad d(L_g(g'),L_g(g'')) \ = \ d(g',g'') \ , \qquad \rho(L_g(g'),L_g(g'')) \ = \ \rho(g',g'') \ .$$

and dilations

$$(2.14) \qquad d(\delta_\lambda(g'),\delta_\lambda(g'')) \ = \ \lambda \ d(g',g'') \ , \qquad \rho(\delta_\lambda(g'),\delta_\lambda(g'')) \ = \ \lambda \ \rho(g',g'') \ .$$

Denoting with

$$(2.15) \qquad B(g,R) \ = \ \{g' \in \boldsymbol{G} \mid d(g',g) < R\}, \qquad B_\rho(g,R) = \{g' \in \boldsymbol{G} \mid \rho(g',g) < R\},$$

respectively the CC ball and the gauge pseudo-ball centered at g with radius R, by (2.14) and a rescaling one easily recognizes that there exist $\omega = \omega(\boldsymbol{G}) > 0$, and $\alpha = \alpha(\boldsymbol{G}) > 0$ such that

$$(2.16) \qquad |B(g,R)| \ = \ \omega \ R^Q, \qquad |B_\rho(g,R)| \ = \ \alpha \ R^Q, \qquad g \in \boldsymbol{G}, R > 0.$$

The first equation in (2.16) shows, in particular, that for a Carnot group the Nagel-Stein-Wainger polynomial in (1.4) is simply the monomial ωR^Q.

We conclude the section with an important definition:

Definition 2.2. *Let $\boldsymbol{G}$ be a Carnot group of step r with homogeneous dimension Q. Consider a bounded, open set $\Omega = \{g \in \boldsymbol{G} \ \mid \ \phi(g) < 0\}$, where $\phi \in C^{1,1}(\boldsymbol{G})$ is such that $|\nabla\phi| \neq 0$ in a neighborhood of $\partial\Omega$. For any $g_o \in \partial\Omega$, we define the "type" of g_o to be the smallest $j = 1, ..., r$ such that there exists $s = 1, ..., m_j$ for which $X_{j,s}\phi(g_o) \neq 0$. We will denote by type(g_o) the type of g_o, and if for every $g_o \in \partial\Omega$ we have that type(g_o) $\leq k \in \mathbb{N}$ then we will say that Ω has type $\leq k$.*

2.2. **Free Lie algebras and groups.** In section four we will work with special systems $X = \{X_1, ..., X_m\}$ of vector fields of Hörmander type, for which both the X_j's and their commutators satisfy the minimal amount of relations. Such systems give rise to CC metrics for which the formula (1.4) is greatly simplified. More importantly, the corresponding CC geometry is locally well approximated by particular stratified Lie algebras.

Definition 2.3. *A free Lie algebra $\mathfrak{g}_{m,s}$ is a nilpotent Lie algebra of step s having m generators, but otherwise as few relations among the commutators as possible.*

The precise definition of such algebra, as quotients of the infinite dimensional free Lie algebra on m generators is given in detail in [RS], Example 4, page 256.

Definition 2.4. *Denote by $n_{m,s}$ the dimension (as a vector space) of the free nilpotent Lie algebra $\mathfrak{g}_{m,s}$. Let $X_1, ..., X_m$ be a set of smooth vector fields defined in an open neighborhood of a point $x_o \in M^n$, and let n_s be the dimension of the space generated by all commutators of the X_j's of length $\leq s$ evaluated at the point x_o. We shall say that $X_1, ..., X_m$ are free up to step r if for any $1 \leq s \leq r$ we have $n_{m,s} = n_s$.*

Remark 2.5. *We observe that if the vector fields $X_1, ..., X_m$ are free up to step r in an open set $\Omega \subset M^n$, then commutators of different lengths are linearly independent, while commutators of the same length may be linearly dependent only because of anti-symmetry, or of the Jacobi identity. Consequently, any $n-$tuple $Y_{i_1}, ..., Y_{i_n}$ of commutators which is a basis for $\mathbb{R}^n$, must have the same cumulative degree*

$$Q \;=\; \sum_{k=1}^{n} d_{i_k} = \sum_{j=1}^{r} j(n_{m,j} - n_{m,j-1}) \ .$$

This simple observation implies that for any $K \subset\subset M^n$ there exists $R(K) > 0$, such that for any $x \in K$, and $0 < r < R(K)$, the polynomial in the right-hand side of (1.4) is actually a monomial

$$\Lambda(x,r) \;=\; r^Q \sum_I |a_I(x)| \ ,$$

and

$$C_1 \;\leq\; \frac{|B(x,r)|}{r^Q \ \sum_I |a_I(x)|} \;\leq\; C_2 \ . \tag{2.17}$$

From this point on, we will denote by $Y_1, ..., Y_m$ the generators of the Lie algebra $\mathfrak{g}_{m,s}$.

Consider $X_1, ..., X_m$ smooth vectors field in M^n which are free up to step r in the open set $\Omega \subset M^n$, and let $\xi \in \Omega$. For each $k \in \mathbb{N}$, $1 \leq k \leq r$, choose $\{X_{k,i}\}$, commutators of length k with $X_{1,i} = X_i$ such that the system $\{X_{k,i}\}$, $k = 1, ..., r$, $i = 1, ..., m_k$ evaluated at ξ is a basis of $\mathbb{R}^n$. Then we can define a system of coordinates (*canonical coordinates*) associated to $\{X_{k,i}\}$, based at the point ξ, as follows

$$(u_{k,j}) \leftrightarrow \exp(\Sigma u_{k,j} X_{k,j}) \cdot \xi \tag{2.18}$$

where $\exp(\cdot) \cdot \xi : T_\xi M^n \to M^n$ denotes the exponential map based at ξ.

Remark 2.6. *By virtue of Theorems 1-7 in* [NSW], *we know that there exist $R_o > 0$, and one particular collection of commutators $\{X_{k,i}\}$, the one corresponding to the largest of the monomials on the right hand side of (1.4), for which the box-like set, that in canonical coordinates (u_{ik}) is expressed by*

$$Box(\delta) \;=\; \{u_{k,i} \in \mathbb{R},\ k = 1, ..., r\ |u_{k,i}| \leq \delta^k\} \ , \tag{2.19}$$

is equivalent to the metric ball $B(\xi, \delta)$ for any $0 < \delta < R_o$. Since we are considering vectors fields which are free up to step r at ξ, then all monomials in the right hand side of (1.4) are of the same degree, hence they are locally equivalent and give rise to equivalent sets of coordinates. Consequently we can state that for any compact set $K \subset\subset \Omega$, there exist constants $C_1, C_2 > 0$ such that

$$Box(C_1\delta) \;\subset\; B(\xi,\delta) \;\subset\; Box(C_2\delta) \ , \tag{2.20}$$

for any $\xi \in K$, *and* $0 < \delta < R_o$.

Following Rothschild and Stein [RS], pg. 273, we want to approximate the free vector fields $X_1, ..., X_m$ with left-invariant vector fields $\{Y_k\}$, $k = 1, ..., m$ generating the free nilpotent Lie algebra $\mathfrak{g}_{m,r}$. Let $\boldsymbol{G}_{m,r}$ denote the Lie group associated to $\mathfrak{g}_{m,r}$. For $k = 1, ..., r$ and $i = 1, ..., m_k$, denote by $\{Y_{k,i}\}$ a basis of the space V_k in the stratification $\mathfrak{g}_{m,r} = V_1 \oplus ... \oplus V_r$, and by $y_{k,i}$ the corresponding exponential coordinates in the group $\boldsymbol{G}_{m,r}$. We indicate by $Y_{1,i} = Y_i$, $i = 1, ..., m_1$ the algebra generators. If α denotes the multi-index $\{k, i\}$, then its degree is defined to be $|\alpha| = k$.

Our arguments will depend crucially upon the following fundamental result (see [RS], Theorem 5, page 273).

Theorem 2.7. *Let* $X_1, ..., X_m$ *be a system of smooth vector fields in* M^n *such that*

(i) $X_1, ..., X_m$ *satisfy (1.2) with rank* r.

(ii) $X_1, ..., X_m$ *are free up to step* r *at* $\xi \in M^n$.

There exists a neighborhood V *of* ξ, *and a neighborhood* U *of the identity in* $\boldsymbol{G}_{m,r}$, *such that:*

(A) *Let* $\eta = \exp(\Sigma u_{jk} X_{jk}) \cdot \xi$, *denote the canonical coordinate chart* $\eta \to u_{jk}$ *for* V *centered at* ξ. *The map* $\theta : V \times V \to U \subset \boldsymbol{G}_{m,r}$ *defined by*

$$\theta_\xi(\eta) = \theta(\xi, \eta) = \exp(\Sigma u_{jk} Y_{jk}) \cdot \xi \tag{2.21}$$

is a diffeomorphism onto its image.

(B) *In the coordinate system given by* θ_ξ *one can write*

$$X_i = Y_i + \mathcal{R}_i, \quad i = 1, ..., m \tag{2.22}$$

where $\mathcal{R}_i$ *is a vector field of local degree less or equal than zero, depending smoothly on* ξ, *i.e. for any smooth* f,

$$X_i\bigg(f(\theta_\xi(\cdot))\bigg) = (Y_i f + \mathcal{R}_i f)(\theta_\xi(\cdot)).$$

More in general, if α *denotes the multi-index* $\{k, i\}$, *then we have*

$$X_\alpha = Y_\alpha + \mathcal{R}_\alpha,$$

with $\mathcal{R}_\alpha$ *a vector fields of degree less or equal than* $|\alpha| - 1$.

Let us recall that a vector field on a Carnot group $\boldsymbol{G}$ has local degree less or equal than $d \in \mathbb{N}$ if, after taking the Taylor expansion at the origin of its coefficients, each term so obtained is an homogeneous operator of degree less or equal than d. More explicitly, denote by $\{y_\alpha\}$, $\alpha = (k, i)$, the exponential coordinates in $\boldsymbol{G}_{m,r}$ associated to the vector fields $Y_{k,i}$. We say that the vector field R_i has degree less or equal than $d \in \mathbb{N}$ if for any $N \in \mathbb{N}$, and any multi-index $\alpha = (k, i)$ one can find a function $g_{\alpha,i,N} \in C^\infty(\boldsymbol{G})$, with growth $g_{\alpha,i,N}(y) = O(||y||^N)$ such that

$$\mathcal{R}_i = \sum_{l=1}^{r} \sum_{|\alpha|=l} \bigg(p_{\alpha,i,N}(y)\partial_{y_\alpha} + g_{\alpha,i,N}(y)\partial_{y_\alpha}\bigg), \tag{2.23}$$

in a neighborhood of the origin. In (2.22), the functions $p_{\alpha,i,N}(y)$ depend on N and are homogeneous group polynomials (see [FS]) of degree less or equal than N and greater or equal than $|\alpha| - d$. The notation ∂_{y_α} indicates a first order derivative along one of the group coordinates whose formal degree is $|\alpha|$. In other words, modulo lower order terms, the operator R_i has order $|\alpha| - deg(p_{\alpha,i,N}) \leq |\alpha| - (|\alpha| - d) = d$.

2.3. **The lifting theorem of Rothschild and Stein.** Up to now we have seen how to locally approximate a system of free vector fields with its "tangent" Carnot group. Since not all systems of Hörmander type are free (for instance consider ∂_x, and $x\partial_y$ in $\mathbb{R}^2$), then there is need of some additional work in order to use the approximation scheme in the most general setting.

One of the main building blocks in the proof of Theorem 1.3 in section five is the Rothschild-Stein lifting theorem (see [RS], Theorem 4).

Theorem 2.8. *Let $X_1, ..., X_m$ be a system of smooth vector fields in M^n, satisfying (1.2) in an open set $U \subset M^n$. For any $\xi \in U$ there exists a connected open neighborhood of the origin $V \subset \mathbb{R}^{\tilde{n}-n}$, and smooth functions $\lambda_{kl}(x,t)$, with $x \in M^n$ and $t = (t_{n+1}, ..., t_{\tilde{n}}) \in V$, defined in a neighborhood $\tilde{U}$ of $\tilde{\xi} = (\xi, 0) \in U \times V$, such that the vector fields $\tilde{X}_1, ..., \tilde{X}_m$ given by*

$$\tilde{X}_k = X_k + \sum_{l=m+1}^{\tilde{m}} \lambda_{kl}(x,t)\partial_{t_l}$$

are free up to step r at every point in $\tilde{U}$.

Let us denote by $\tilde{B}((x,s),R)$ the Carnot-Carathéodory balls associated to the lifted vector fields $\tilde{X_1}, ..., \tilde{X_m}$. Let π_1 and π_2 denote the projections of $U \times V$ onto U and V respectively,

$$\pi_1(x,t) \;=\; x \;, \qquad \pi_2(x,t) \;=\; t \;.$$

The following lemma sums up some basic results from [RS], Lemma 3.1.

Lemma 2.9. *One has that $\pi_1 : \tilde{B}((x,t),R) \to B(x,R)$ and moreover, this map is onto. If $x, y \in U$ and $t, s \in V$ then $d(x,y) \leq \tilde{d}((x,s),(y,t))$*

The next estimate is crucial for our purposes, for its proof see [NSW], Lemma 3.2, and [SC], Theorem 4 and Lemma 7.

Lemma 2.10. *Let $E \subset\subset U$ be a compact set, and $v \in C_0^\infty(V)$. There is a constant $C = C(E, X, v) > 0$ such that if $x \in E$ and $y \in B(x,R)$, then*

$$C^{-1}\frac{|\tilde{B}((x,0),R)|}{|B(x,R)|} \leq \left| \int_V \chi_{\tilde{B}(x,0),R)}(y,s)\; v(s)ds \right| \leq C\frac{|\tilde{B}((x,0),R)|}{|B(x,R)|}.$$

Essentially this lemma says that even if the sets $\tilde{B}((x,0),R)$ are not the product of balls in R^n and $\mathbb{R}^{\tilde{n}}$, in terms of volume of sections they behave like such. We remark explicitly that the integral in the above formula simply represents the Lebesgue measure of the set

$$\pi_2\Big(\tilde{B}((x,0),R) \cap (\{y\} \times V)\Big) \tag{2.24}$$

in the projection onto the second factor. Lemma 2.10 implies that if $y_1, y_2 \in B(x,R)$, then

$$\begin{aligned} C^{-2}\left|\pi_2\Big(\tilde B((x,0),R)\cap(\{y_1\}\times V)\Big)\right| &\leq \left|\pi_2\Big(\tilde B((x,0),R)\cap(\{y_2\}\times V)\Big)\right| \\ &\leq C^2\left|\Big(\pi_2\Big(\tilde B((x,0),R)\cap(\{y_1\}\times V)\Big)\right|, \end{aligned} \tag{2.25}$$

i.e., the expression (2.24) is almost constant in y. Note that in (2.25) the symbol $|\cdot|$ denotes Lebesgue measure in different spaces.

3. **Carnot groups.**

In this section we prove the Ahlfors type estimates in the setting of Carnot groups. First we study the special case of hyperlanes passing through the origin. Next, we analyze how to approximate the perimeter of a surface ball on a hypersurface with the perimeter of a surface ball with the same radius on the tangent space. As a consequence, we will derive the desired estimates for generic domains.

3.1. **Hyperplanes: Upper bounds.** Let Π denote a hyperplane in the Lie algebra $\mathfrak{g}$ which contains the origin. Observe that the Euclidean metric on $\mathfrak{g}$, the gauge pseudo-metric and the Hausdorff measure are all invariant with respect to the action of the orthogonal group $O(\mathbb{R}^{m_i})$ on V_i. By a change of coordinates, performing a rotation inside each layer V_i, we can assume without loss of generality that there exist real numbers $a_1, ..., a_r$, such that the equation of the hyperplane Π is given by $\pi(\xi) = \sum_{j=1}^r a_j\, x_{j,1} = 0$. We will denote by $\mathcal{N}$ the set of indices $j = 1, ..., r$ such that $a_j \neq 0$, and by $\mathcal{N}^C$ the set of indices for which $a_j = 0$. Note that if the origin is non-characteristic, then $a_1 \neq 0$, while in general, the smallest index in $\mathcal{N}$ is simply the type of Π. We will denote such index as k_o, and assume without loss of generality that $a_{k_o} = 1$. The equation of the plane then will read as

$$\pi(\xi) \;=\; x_{k_o} - \sum_{j\in\mathcal{N}, j>k_o} a_j\, x_{j,1} \;=\; 0\ . \tag{3.1}$$

In view of (2.1) we obtain that for all $i = 1, ..., m_1$,

$$|X_i\pi(\xi)| \;\leq\; \sum_{j=1}^{r}\sum_{s=1}^{m_j} |b^s_{j,i}(\xi_1, ..., \xi_{j-1})||a_j|\delta_{s1} \;=\; \sum_{j\in\mathcal{N}} |b^1_{j,i}(\xi_1, ..., \xi_{j-1})|\; |a_j|. \tag{3.2}$$

To simplify our computations we replace the gauge pseudo-ball $B(0,R) \subset \mathfrak{g}$ with a simpler set, the anisotropic "box" centered at $0 \in \mathfrak{g}$ of radius R, introduced earlier in (2.19) and (2.20). From (3.2) and (2.20) we immediately deduce the following.

Lemma 3.1. *Using the notations introduced above, one has that there exist $C = C(\Pi, \boldsymbol{G}) > 0$ and $R_o = R_o(\Pi, \boldsymbol{G}) > 0$ such that if $0 < R < R_o$, then*

$$\sup_{Box(R)} |X\pi| \leq C\left(\sum_{j\in\mathcal{N}} |a_j|\; R^{j-1}\right) \leq CR^{k_o-1}\ .$$

Lemma 3.2. *Let $\Pi \subset \boldsymbol{G}$ denote a hyperplane passing through the origin, which is of type k_o. There exist constant $M = M(\pi, \boldsymbol{G}) > 0$ and $R_o = R_o(\Pi, \boldsymbol{G}) > 0$ such that, for any $0 < R < R_o$, one has*

$$\sigma(\Pi \cap B(0,R)) \leq M R^{Q-k_o}. \tag{3.3}$$

Proof. By an isometric linear map, we can transform

$$\text{Box}(R) \cap \Pi \ = \ \{(x_{1,1}, ..., x_{r,m_r}) \in \mathfrak{g} \approx \mathbb{R}^{\sum_{j=1}^r m_j} \ \mid \ \sum_{j=1}^r a_j x_{j,1} = 0, \ |\xi_j| \leq R^j, j = 1, ..., r\} \ , \tag{3.4}$$

into the set

$$\left(\Pi_{j \in \mathcal{N}^C} (-R^j, R^j)^{m_j}\right) \times \left(\Pi_{k \in \mathcal{N}} (-R^k, R^k)^{m_k - 1}\right) \times S \ . \tag{3.5}$$

Here, we have denoted by $|\mathcal{N}|$ the number of elements in $\mathcal{N}$, and by $\{\mathcal{N}_1, ..., \mathcal{N}_{|\mathcal{N}|}\}$ the elements themselves. We have let

$$S \ = \ \{s = (s_1, ..., s_{|\mathcal{N}|}) \in \mathbb{R}^{|\mathcal{N}|} \ \mid \ \sum_{\mathcal{N}_j \in \mathcal{N}} a_{\mathcal{N}_j} s_j = 0, \ |s_j| \leq R^{\mathcal{N}_j}, j = 1, ..., |\mathcal{N}|\} \ . \tag{3.6}$$

Consequently,

$$\begin{aligned} \sigma(\text{Box}(R) \cap \Pi) \ &\leq \quad C R^{\sum_{j \in \mathcal{N}^C} j m_j} \ R^{\sum_{j \in \mathcal{N}} j(m_j - 1)} \ H_{|\mathcal{N}|-1}(S) \\ &= \quad C \ R^Q \ R^{-\sum_{j \in \mathcal{N}} j} \ H_{|\mathcal{N}|-1}(S) \ . \end{aligned} \tag{3.7}$$

Next, we estimate from above the quantity in the right-hand side of (3.7). Now, it is not easy to compute $H_{|\mathcal{N}|-1}(S)$ exactly. However the following simple argument produces the bound (3.9), which will suffice for our purposes. We recall that if

$$\Sigma \ = \ \{s = (s_1, ..., s_{|\mathcal{N}|}) \mid \sum_{j \in \mathcal{N}} a_{\mathcal{N}_j} s_j = 0\}$$

is a hyperplane in $\mathbb{R}^{|\mathcal{N}|}$, and U represents the projection of a portion $\Delta \subset \Sigma$ onto the coordinate hyperplane $\{s_i = 0\}$, then the $(|\mathcal{N}| - 1)$-dimensional measure of Δ is given by

$$H_{|\mathcal{N}|-1}(\Delta) \ = \ \frac{\sqrt{\sum_{j \in \mathcal{N}} a_j^2}}{|a_{\mathcal{N}_i}|} \ H_{|\mathcal{N}|-1}(U) \ . \tag{3.8}$$

We now apply (3.8) with $\Delta = S$ to reach the crucial conclusion that $H_{|\mathcal{N}|-1}(S)$ is bounded from above by any of the quantities

$$\frac{\sqrt{\sum_{j \in \mathcal{N}} a_j^2}}{|a_{\mathcal{N}_i}|} \ (2R)^{\sum_{\{j \neq \mathcal{N}_i, j \in \mathcal{N}\}} j} \ , \qquad \mathcal{N}_i \in \mathcal{N} \ .$$

In fact, $(2R)^{\sum_{\{j\neq\mathcal{N}_i, j\in\mathcal{N}\}} j}$ is the $H_{|\mathcal{N}|-1}$ measure of the projection onto $\{s_i = 0\}$ of the box $\{|s_j| \leq R^{\mathcal{N}_j}, j = 1, ..., |\mathcal{N}|\} \subset \mathbb{R}^{|\mathcal{N}|}$. Consequently, we have

$$
\begin{aligned}
H_{|\mathcal{N}|-1}(S) \quad &\leq \min \left\{ \frac{\sqrt{\sum_{j\in\mathcal{N}} a_j^2}}{|a_{\mathcal{N}_i}|} (2R)^{\sum_{\{j\neq\mathcal{N}_i, j\in\mathcal{N}\}} j}, \ \mathcal{N}_i \in \mathcal{N} \right\} . \\
&\leq R^{\sum\limits_{j\in\mathcal{N}} j} \min \left\{ \frac{\sqrt{\sum_{j\in\mathcal{N}} a_j^2}}{|a_{k_o}| R^{k_o}}, ..., \frac{\sqrt{\sum_{j\in\mathcal{N}} a_j^2}}{|a_{\mathcal{N}_{|\mathcal{N}|}}| R^{|\mathcal{N}|}} \right\}. \qquad (3.9)
\end{aligned}
$$

Observe that the minimum in the above expression is achieved at the index k_o, that is the type of Π. In conclusion, recalling that $a_{k_o=1}$, one has

$$
\begin{aligned}
\sigma(\Pi \cap B(0,R)) \leq \sigma(\mathrm{Box}(R) \cap \Pi) \quad &\leq C R^Q R^{-\sum\limits_{j\in\mathcal{N}} j} \frac{\sqrt{\sum_{j\in\mathcal{N}} a_j^2}}{|a_{k_o}| R^{k_o}} \\
&\leq C \sqrt{\sum_{j\in\mathcal{N}} a_j^2} R^{Q-k_o}. \qquad (3.10)
\end{aligned}
$$

□

3.2. **Hyperplanes: Lower bounds.** We start with a simple lemma.

Lemma 3.3. *Let $n, r \in \mathbb{N}$, $n \leq r$, and consider a multi-index*

$$I \ = \ \{d_1, ..., d_n\} \ \in \ \mathbb{N}^n \ ,$$

with $1 \leq d_i \leq r$, $\ d_i < d_{i+1}$. Set $N = \sum_{i=1}^n d_i$. Cosider a $n-$tuple $(\alpha_1, ..., \alpha_n)$ of non-zero real numbers and for $R > 0$ define the portion of hyperplane

$$S_R = \ \{(s_1, ..., s_n) \in \mathbb{R}^n \text{ such that } \sum_{i=1}^n \alpha_i s_i = 0, \text{ and } |s_i| < R^{d_i}, \text{ for } i = 1, ..., n \ \}.$$

There exists $R_o = R_o(a_i, d_i, n, r)$ such that if $0 < R < R_o$, then

$$H_{n-1}(S_R) \ \geq \ \frac{\sqrt{\sum_{j=1}^n \alpha_j^2}}{|\alpha_1|} \ R^{N-d_1} \ , \qquad (3.11)$$

where H_{n-1} denotes the $n-1$ dimensional Hausdorff measure in R^n.

Proof of Lemma 3.3. As in the proof of Lemma 3.2, from (3.8) we have that for any $i = 1, ..., n$,

$$H_{n-1}(S_R) \ = \ \frac{\sqrt{\sum_{j=1}^n \alpha_j^2}}{|\alpha_i|} \ H_{n-1}(\pi_i(S_R)) \ , \qquad (3.12)$$

where $\pi_i(S_R)$ represents the projection of $S_R \subset \mathbb{R}^n$ onto the coordinate hyperplane $\{s_i = 0\}$.

We claim that there exists R_o as in the statement of the theorem, such that for $0 < R < R_o$, the projection $\pi_1(S_R)$ is as large as possible, i.e.

$$H_{n-1}(\pi_1(S_R)) \ = \ R^{N-d_1}. \qquad (3.13)$$

To verify this statement we choose any point

$$(s_2, ..., s_n) \in \mathbb{R}^{n-1} \text{ with } |s_i| < R^{d_i}, \text{ for } i = 2, ..., n.$$

Define

$$s_1 = -\sum_{i=2}^{n} \frac{\alpha_i}{\alpha_1} s_i.$$

A simple computation shows that

$$|s_1| \leq \sum_{i=2}^{n} \frac{|\alpha_i|}{|\alpha_1|} R^{d_i} \leq R^{d_1} \sum_{i=2}^{n} \frac{|\alpha_i|}{|\alpha_1|} R^{d_i - d_1}.$$

Hence, for R_o small enough and $0 < R < R_o$ we have $|s_1| < R^{d_1}$, and consequently $(s_1, ..., s_n) \in S_R$. This shows that $(s_2, ..., s_n) \in \pi_1(S_R)$ and proves (3.13). $\square$

Lemma 3.4. *Consider the hyperplane of type k_o,*

$$\Pi = \{g \in \boldsymbol{G} \ \ \textit{such that } x_{k_o,1} = \sum_{i \in \mathcal{N}, i > k_o} a_i x_{i,1}\}.$$

There exist $C = C(\Pi, \boldsymbol{G}) > 0$, and $R_o = R_o(\Pi, \boldsymbol{G}) > 0$, such that for any $0 < R < R_o$, one has

$$\sigma(Box(R) \cap \Pi) \geq C \ R^{Q - k_0} \ . \tag{3.14}$$

Proof. In view of (3.4)-(3.5), the estimate (3.14) will immediately follow from

$$R^{\sum_{j \in \mathcal{N}^C} j m_j} R^{\sum_{l \in \mathcal{N}} l(m_l - 1)} H_{|\mathcal{N}|-1}(S) \geq \ C \ R^{Q - k_0} \ , \tag{3.15}$$

where S is defined as in (3.6). To establish (3.15) we simply apply Lemma 3.3 in which we substitute $n = |\mathcal{N}|$, $\mathcal{N} = I$, $\alpha_i = a_{\mathcal{N}_i}$ and $d_i = \mathcal{N}_i$.

$\square$

The argument in the proof of Lemma 3.4 yields immediately the following.

Corollary 3.5. *In the hypothesis of the previous lemma we have that*

$$\begin{aligned} H_\beta \Big[\pi_{k_o, \dots, r}(Box(R) \cap \Pi) \Big] \qquad & \geq C R^{\sum_{j \in \mathcal{N}^C, j > k_o} j m_j} R^{\sum_{l \in \mathcal{N}} l(m_l - 1)} R^{\sum_{l \in \mathcal{N}, l \neq k_o} l} \\ & = \ C R^{Q - \sum_{j=1}^{k_o - 1} j m_j - k_o}, \end{aligned} \tag{3.16}$$

where $\beta = dim(\mathfrak{g}) - m_1 - ... - m_{k_o - 1} - 1$, and $\pi_{k_o, \dots, r} : \mathfrak{g} \to V_{k_o} \oplus ... \oplus V_r$ denotes the orthogonal projection onto the complement of $V_1 \oplus ... \oplus V_{k_o - 1}$.

Lemma 3.6. *Consider the hyperplane of type k_o,*

$$\Pi = \{g \in \boldsymbol{G} \ \ \textit{such that } x_{k_o,1} = \sum_{i \in \mathcal{N}, i > k_o} a_i x_{i,1}\}.$$

There exist $C = C(\Pi, \boldsymbol{G}) > 0$, and $R_o = R_o(\Pi, \boldsymbol{G}) > 0$, such that for any $0 < R < R_o$, one has

$$\int_{Box(R) \cap \Pi} |X\pi| d\sigma \geq \ C \ R^{Q-1} \ . \tag{3.17}$$

Proof. A simple computation shows that

$$\begin{aligned} |X_i\pi| = & \quad |\sum_{k\in\mathcal{N}} b^1_{k,i} a_k| \\ \geq & \quad |b^1_{k_o,i}| - |\sum_{k\in\mathcal{N},\ k>k_o} b^i_{k,1} a_k| \\ \geq & \quad |b^1_{k_o,i}| - \sum_{k\in\mathcal{N},\ k>k_o} CR^{k-1}. \end{aligned} \tag{3.18}$$

The crucial observation now is that $b^1_{k_o,i}$ is an homogeneous polynomial of weighted degree k_o-1 which depends only on $\xi_1,...,\xi_{k_o-1}$, see Lemma 2.1. Because of the definition of Carnot group, we know that there exists at least one $i=1,...,m_1$, and one $l=1,...,m_{k_o}$ such that $b^l_{k_o,i}\neq 0$. Without loss of generality we can assume that $b^1_{k_o,i}\neq 0$ for some i. In fact, if that is not the case and $b^{l_0}_{k_o,i}\neq 0$ for some $l_0\neq 1$, then we will change the definition of a_{k_o} by rotating the V_{k_o} component of Π onto the direction l_0. The rest of the proof will follow with trivial changes.

Since π does not depend on $\xi_1,...,\xi_{k_o-1}$, we have that

$$\text{Box}(R)\cap\Pi \quad = \quad \Pi_{j=1}^{k_o-1}(-R^j,R^j)^{m_j}\times\pi_{k_o,...,r}(\text{Box}(R)\cap\Pi),$$

where $\pi_{k_o,...,r}$ is as in (3.16).

Estimate (3.18) allows us to infer

$$\begin{aligned} \int_{\text{Box}(R)\cap\Pi}|X\pi|d\sigma \geq & \quad \sum_{i=1}^{m_1}\int_{\text{Box}(R)\cap\Pi}\left(|b^1_{k_o,i}| - \sum_{k\in\mathcal{N},\ k>k_o} CR^{k-1}\right)d\sigma \\ \text{(arguing as in (3.3))} \quad \geq & \quad \left(\sum_{i=1}^{m_1}\int_{\text{Box}(R)\cap\Pi}|b^1_{k_o,i}|d\sigma\right) - \frac{\sqrt{\sum_{j\in\mathcal{N}} a_j^2}}{|a_{k_o}|}\left(\sum_{k\in\mathcal{N},\ k>k_o} CR^{k-1}\right)R^{Q-k_o} \\ \geq & \quad \int_{\Pi_{j=1}^{k_o-1}(-R^j,R^j)^{m_j}}|b^1_{k_o,i}|d\xi_1...d\xi_{k_o-1}\int_{\pi_{k_o,...,r}(\text{Box}(R)\cap\Pi)}d\xi_{k_o}...d\xi_r - CR^Q \\ \text{(by (3.16))} \geq & \quad C_oR^{Q-\sum_{j=1}^{k_o-1}jm_j-k_o}\int_{\Pi_{j=1}^{k_o-1}(-R^j,R^j)^{m_j}}|b^1_{k_o,i}|d\xi_1...d\xi_{k_o-1} - C_1R^Q. \end{aligned}$$

Next, observe that for any non-zero homogeneous polynomial $p(\xi_1,...,\xi_{k_o-1})$, of weighted degree k_o-1, we have that if we set

$$\mathcal{C} = \int_{\mathcal{U}} p(\xi_1,...,\xi_{k_o-1})d\xi_1...d\xi_{k_o-1},$$

where we have let $\mathcal{U}=(-1,1)^{\sum_{j=1}^{k_o-1}jm_j}$, then, through the change of variables $\xi'=\delta_R(\xi)$, one obtains

$$\int_{\Pi_{j=1}^{k_o-1}(-R^j,R^j)^{m_j}} p(\xi'_1,...,\xi'_{k_o-1})d\xi'_1...d\xi'_{k_o-1} = \mathcal{C}R^{\sum_{j=1}^{k_o-1}jm_j+(k_o-1)}.$$

Substituting the latter in (3.19) we complete the proof. ☐

3.3. **Surface area and first order approximation of $C^{1,1}$ domains.** Let $D \subset \mathfrak{g}$ be a $C^{1,1}$ domain such that $0 \in \partial D$. Set k_o to be the type of the origin, and $\Pi = T_0 \partial D$ the tangent space at the origin. We locally parametrize ∂D as the graph $x_{k_o,1} = f(\bar{x})$, with $f \in C^{1,1}$, and Π as the graph $x_{k_0,1} = \sum_{i \in \mathcal{N}, i > k_o} a_i x_{i,1}$, both defined on the hyper-plane $x_{k_o,1} = 0$, with

$$\bar{x} = (x_{1,1}, ..., x_{1,m}, x_{2,1}, ..., x_{k_o-1, m_{k_o-1}}, 0, x_{k_0,2}, ..., x_{r,m_r}).$$

Lemma 3.7. *There exist constants $C_1 = C_1(\mathfrak{g}, D) > 0$, $R_1 = R_1(\mathfrak{g}, D) > 0$, such that for all $0 < R < R_1$ we have*

$$\sigma(B(0,R) \cap \partial D) \ \leq \ C_1 \ \sigma(B(0,R) \cap \Pi) \ . \tag{3.19}$$

Proof. If $\bar{x} + f(\bar{x}) e_{\vec{k_o},1} \in B(0,R)$, then in particular one has $|\bar{x}_{i,j}| \leq R^i$ and

$$|\sum_{i \in \mathcal{N}, i > k_o} a_i x_{i,1}| \leq R^{k_o}.$$

Consequently, we obtain

$$\begin{aligned} \sigma(B(0,R) \cap \partial D) = & \int_{B(0,R) \cap \partial D} d\sigma \\ & = \int_{\bar{x} + f(\bar{x}) \vec{e}_{k_o,1} \in B(0,R)} \sqrt{1 + |\nabla f|^2}(\bar{x}) d\bar{x} \\ & \leq C \int_{|\bar{x}_{i,j}| \leq R^i} d\bar{x} \\ & \leq C_1 \int_{|\bar{x}_{i,j}| \leq R^i} \sqrt{1 + |\nabla(\sum_{i \in \mathcal{N}, i > k_o} a_i x_{i,1})|^2} d\bar{x} \\ & \leq C_1 \int_{\bar{x} + (\sum_{i \in \mathcal{N}, i > k_o} a_i x_{i,1}) \vec{e}_{k_o,1} \in B(0,R) \cap \Pi} \sqrt{1 + |\nabla(\sum_{i \in \mathcal{N}, i > k_o} a_i x_{i,1})|^2} d\bar{x} \\ & = C_1 \sigma(B(0,R) \cap \Pi). \end{aligned}$$

□

The lower bounds require a finer estimate and stronger hypothesis. First, observe that we can write ∂D as a graph over Π in the form $\bar{y} \in \Pi \to \bar{y} + \eta(\bar{y}) \vec{e}_{k_o,1}$, for some choice of function $\eta \in C^{1,1}$.

Lemma 3.8. *If type(g_o) $= 1$ or 2, then there exist $M = M(\mathfrak{g}, D) >> 1$, $C = C(\mathfrak{g}, D) > 0$, and $R_o = R_o(\mathfrak{g}, D) > 0$, such that for any $0 < R < R_o$, and for any continuous function g defined in a neighborhood of ∂D and Π we have*

$$\int_{\partial D \cap B(0,R)} g(x) d\sigma(x) \geq C \int_{\Pi \cap B(0,R/M)} g(\bar{y} + \eta(\bar{y}) e_{\vec{k_o},1}) d\sigma(\bar{y}) \tag{3.20}$$

Proof. Using the notation introduced earlier in this section we will write ∂D both as a graph over the tangent space Π and over the space $x_{k_o,1} = 0$, i.e.

$$\begin{aligned}\partial\Omega \cap B(0,R) &= \{\bar{y} \in \Pi \text{ such that } \bar{y} + \eta(\bar{y})\vec{e_{k_o,1}} \in B(0,R)\} \\ &= \{\bar{x} \text{ such that } \bar{x} + f(\bar{x})\vec{e_{k_o,1}} \in B(0,R)\}.\end{aligned}$$

The functions η and f are related by

$$f(\bar{x}) = \sum_{i\in\mathcal{N}, i>k_o} a_i x_{i,1} + \eta(\bar{y}),$$

where $\bar{x}$ is a generic point in the hyperplane $x_{k_o,1} = 0$, and $\bar{y}$ is the corresponding point on the tangent space,

$$\bar{y} = \bar{x} + (\sum_{i\in\mathcal{N}, i>k_o} a_i x_{i,1})\vec{e}_{k_o,1}.$$

Since the origin is of type k_0 then $\frac{\partial f}{\partial x_{k,i}}(0) = 0$ for $k = 1, ..., k_o - 1$. Hence

$$|f(\bar{x})| \leq C(|x_{k_o}| + ... + |x_r|) + O(|\bar{x}|^2). \tag{3.21}$$

Consider the portion of ∂D and Π given by

$$G_{\partial D,R} = \{\bar{x} + f(\bar{x})\vec{e}_{k_o,1} \in \partial D \text{ such that } \bar{x} \in B(0,R)\},$$

and

$$G_{\Pi,R} = \bar{y} = \bar{x} + (\sum_{i\in\mathcal{N}, i>k_o} a_i x_{i,1})\vec{e}_{k_o,1} \in \Pi \text{ such that } \bar{x} \in B(0,R)\}$$

First we show that there exists a constant $M >> 1$ such that

$$G_{\Pi,R/M} \subset \Pi \cap B(0,R) \subset G_{\Pi,MR}, \text{ and } G_{\partial D,R/M} \subset \partial\Omega \cap B(0,R) \subset G_{\partial D,MR}. \tag{3.22}$$

To see this we observe that if a point $x = \bar{x} + f(\bar{x})\vec{e}_{k_o,1} \in B(0, R/M)$ then $\bar{x} \in B(0,R)$. Viceversa, if $\bar{x} \in B(0,R)$ then $x = \bar{x} + f(\bar{x})\vec{e}_{k_o,1} \in B(0, MR)$. In fact, $\bar{x} \in B(0,R)$ implies

$$\begin{aligned}|\bar{x} + f(\bar{x})\vec{e_{k_o,1}}|_{\boldsymbol{G}} &\leq C(|\bar{x}|_{\boldsymbol{G}} + |f(\bar{x})|^{\frac{1}{k_o}}) \\ \text{by (3.21)} \quad &\leq C(R + (|x_{k_o}| + ... + |x_r| + O(|\bar{x}|^2)^{\frac{1}{k_o}}) \\ &\leq MR.\end{aligned}$$

We want to stress that in the last inequality we have used the hypothesis $k_o \leq 2$.

On the other hand, if $x = \bar{x} + f(\bar{x})\vec{e}_{k_o,1} \in B(0, R/M)$ then $|\bar{x}_{k,j}| < (R/M)^k$, and hence $\bar{x} \in B(0,R)$. With a similar argument one can see that if $\bar{x} \in B(0,R)$ then $\bar{x} + (\sum_{i\in\mathcal{N}, i>k_o} a_i x_{i,1})\vec{e}_{k_o,1} \in B(0, MR)$, and viceversa, if $\bar{x} + (\sum_{i\in\mathcal{N}, i>k_o} a_i x_{i,1})\vec{e}_{k_o,1} \in B(0,R)$ then $\bar{x} \in B(0, MR)$.

Hence, we have that if $\bar{x} + (\sum_{i\in\mathcal{N}, i>k_o} a_i x_{i,1})\vec{e}_{k_o,1} \in B(0, R/M)$, then $\bar{x} \in B(0,R)$, and consequently $x = \bar{x} + f(\bar{x})\vec{e}_{k_o,1} \in B(0, MR)$.

Let J_1, and J_2 denote the determinants of the Jacobians corresponding respectively to the change of variables $\bar{x} \to \bar{y} \in \Pi$, and $\bar{x} \to x \in \partial D$, so that we have $d\sigma(\bar{y}) = J_1(\bar{x})d\bar{x}$, and $d\sigma(x) = J_2(\bar{x})d\bar{x}$. It is a simple excercise to see that $M^{-1} \leq J_i \leq M$, $i = 1, 2$ for some constant M.

We have

$$\begin{aligned}\int_{\Pi\cap B(0,R/M)} g(\bar y+\eta(\bar y)\vec e_{k_o,1})d\sigma(\bar y) &= \int_{\{\bar x \ \mid\ \bar x+(\sum_{i\in\mathcal N, i>k_o} a_i x_{i,1})\vec e_{k_o,1}\in B(0,R/M)\}} g(\bar x+f(\bar x)\vec{e_{k_o,1}})J_1(\bar x)d\bar x\\ &\le \int_{\bar x\in B(0,R)} g(\bar x+f(\bar x)\vec{e_{k_o,1}})J_1(\bar x)d\bar x\\ &= \int_{\bar x\in B(0,R)} g(\bar x+f(\bar x)\vec{e_{k_o,1}})J_2(\bar x)\frac{J_1(\bar x)}{J_2(\bar x)}d\bar x\\ &\le C\int_{\bar x\in B(0,R)} g(\bar x+f(\bar x)\vec{e_{k_o,1}})J_2(\bar x)d\bar x\\ &\le C\int_{\partial D\cap B(0,MR)} g(x)d\sigma(x),\end{aligned}$$

thus concluding the proof. □

3.4. **Upper bounds for $C^{1,1}$ domains.** Let us consider a Carnot group $\boldsymbol G$ of step r with homogeneous dimension Q. Let $\Omega=\{g\in\boldsymbol G \mid \phi(g)<0\}$, be a bounded open set, where $\phi\in C^{1,1}(\boldsymbol G)$ is a defining function for Ω (i.e., $|\nabla\phi|\neq 0$ in a neighborhood of $\partial\Omega$).

This section is dedicated to the proof of the following,

Theorem 3.9. *For every $g_o\in\partial\Omega$, there exist $M=M(\boldsymbol G\Omega,g_o)>0$ and $R_o=R_o(\boldsymbol G,\Omega,g_o)>0$, depending continuously on g_o, such that, for any $0<R<R_o$, one has*

$$\left(\sup_{B(g_o,R)\cap\partial\Omega}|X\phi|\right)\ \sigma(B(g_o,R)\cap\partial\Omega)\ \le\ M\ R^{Q-\alpha}\ , \tag{3.23}$$

with

$$\alpha=\begin{cases} type(g_o)-1, & \text{if } g_o \text{ is characteristic,}\\ 1, & \text{if } g_o \text{ is not characteristic.}\end{cases}$$

Proof. Let $g_o\in\boldsymbol G$ and consider the group automorphism $L_{g_o^{-1}}:\boldsymbol G\to\boldsymbol G$, see (2.6). Since $L_{g_o^{-1}}$ is an isometry (2.13), we have $L_{g_o^{-1}}(B(g_o,R))=B(e,R)$, where e denotes the group identity. Moreover, $L_{g_o^{-1}}$ is a smooth map, hence in particular it is locally Lipschitz. Consequently, for every $R_o>0$ there exists $C=C(\boldsymbol G,\Omega,g_o,R_o)>0$, depending on the Lipschitz norm of $L_{g_o^{-1}}$ in $B(g_o,R_o)\cap\partial\Omega$, such that for any $0<R<R_o$ one has

$$C^{-1}\sigma(L_{g_o^{-1}}(B(g_o,R)\cap\partial\Omega))\le\sigma(B(g_o,R)\ \cap\ \partial\Omega)\ \le\ C\ \sigma(L_{g_o^{-1}}(B(g_o,R)\cap\partial\Omega))\ . \tag{3.24}$$

Since $\partial\Omega$ is compact, the constant C in (3.24) can be chosen independently of $g_o\in\partial\Omega$, and if we let $R_o\le\min(\mathrm{diam}(\Omega),\frac12)$, we can simply write $C=C(\boldsymbol G,\Omega)$. If we let $\tilde\Omega=L_{g_o^{-1}}(\Omega)$, then (3.24) can be rewritten for some constant $C=C(\boldsymbol G,\Omega)>0$

$$C^{-1}\ \sigma(B(e,R)\cap\partial\tilde\Omega)\ \le\sigma(B(g_o,R)\cap\partial\Omega)\ \le\ C\ \sigma(B(e,R)\cap\partial\tilde\Omega)\ . \tag{3.25}$$

Now, observe that

$$\tilde\Omega\ =\ \{g\in\boldsymbol G\mid\tilde\phi(g)<0\}\ ,$$

where $\tilde{\phi} = \phi \circ L_{g_o^{-1}}$. By the left-invariance of the vector fields $X_1, ..., X_m$ we have $|X\tilde{\phi}| = |X\phi| \circ L_{g_o^{-1}}$. In particular,

$$\sup_{B(g_o,R)\cap\partial\Omega} |X\phi| \;=\; \sup_{B(e,R)\cap\partial\tilde{\Omega}} |X\tilde{\phi}| \;. \tag{3.26}$$

In view of (3.25) and (3.26) we obtain

$$\left(\sup_{B(g_o,R)\cap\partial\Omega} |X\phi|\right)\; \sigma(B(g_o,R)\cap\partial\Omega) \;\le\; C\; \left(\sup_{B(e,R)\cap\partial\tilde{\Omega}} |X\tilde{\phi}|\right)\; \sigma(B(e,R)\cap\partial\tilde{\Omega}) \;. \tag{3.27}$$

Inequality (3.27) allows us to assume that $g_o = e$ in (3.23). Moreover, with a slight abuse of notation we will denote $\tilde{\Omega}$ and $\tilde{\phi}$ by Ω and ϕ, respectively. In addition, thanks to (2.10), it is clear that in (3.27) we can replace the metric balls with gauge pseudo-balls defined in (2.15). Without further mention, we will work with the latter from this moment on.

At this point it is convenient to work with the exponential coordinates (2.5) in the Lie algebra $\mathfrak{g}$, rather than dealing directly with the group $\boldsymbol{G}$. We thus set $D = \exp^{-1}(\Omega) \subset \mathfrak{g}$. Observing that the Riemannian Hausdorff measure H_{n-1} in $\boldsymbol{G}$, the Haar measure dg in $\boldsymbol{G}$, and the gauge pseudo-metric, are all obtained by pushing forward via the exponential mapping corresponding measures and pseudo-metric in $\mathfrak{g}$, with another slight abuse of notation we will denote by H_{n-1} the (Euclidean) $(n-1)$-dimensional Hausdorff measure in $\mathfrak{g}$ and set $\sigma = H_{n-1}\lfloor\partial D$. We will continue to indicate with ϕ the pull-back $(\phi\circ\exp)$. In this notation ϕ is a defining function for D. The notation $B(\xi, R) \subset \mathfrak{g}$ will indicate the Lie algebra gauge pseudo-balls of radius R and center $\xi \in \mathfrak{g}$ defined by means of $|\cdot|_{\mathfrak{g}}$ in (2.11).

With these reductions, we have converted the proof of (3.23) into the task of establishing the existence of $C_o = C_o(\mathfrak{g}, D, 0) > 0$ and $R_o = R_o(\mathfrak{g}, D, 0) > 0$, such that for $0 < R < R_o$

$$\left(\sup_{B(0,R)\cap\partial D} |X\phi|\right) \sigma(B(0,R)\cap\partial D) \;\le\; C\; R^{Q-\alpha} \;. \tag{3.28}$$

Our next reduction consists in substituting the quantity $\sigma(B(0,R)\cap\partial D)$ in the left-hand side of (3.28), with $\sigma(B(0,R)\cap\Pi)$, where $\Pi = T_0\partial D \subset \mathfrak{g}$ denotes the tangent plane at the origin $0 \in \partial D$. We observe next that the defining function of Π is given by $\pi(\xi) =< \nabla\phi(0), \xi >$, $\xi \in \mathfrak{g}$. By the hypothesis $\phi \in C^{1,1}$, and by Taylor's theorem, one can write

$$\phi(\xi) = \pi(\xi) + H(\xi) \;, \tag{3.29}$$

with $H = O(|\xi|^2)$ (we recall here that $|\cdot|$ denotes the Euclidean norm on $\mathfrak{g}$), with $\xi \in B(0, R_o)$, and for sufficiently small R_o. Consequently, we obtain

$$\nabla\phi(\xi) = \nabla\phi(0) + \vec{O}(\xi), \tag{3.30}$$

where $\vec{O}(\xi) = \{O_{j,s}(\xi)\}$, $j = 1, ..., r$, $s = 1, ..., m_j$ and $O_{j,s}(\xi) = O(|\xi|)$. In view of (3.30) and Lemma 2.1 we can compute the horizontal gradient of ϕ as follows

$$X_i\phi(\xi) = < X_i,\ \nabla\phi(\xi) > = \quad < X_i, \nabla\phi(0) + \vec{O} >$$
$$= \quad \sum_{j=1}^{r}\sum_{s=1}^{m_j} b^s_{j,i}(\xi_1, ..., \xi_{j-1})\Big(\frac{\partial\pi}{\partial x_{j,s}} + O_{j,s}(\xi)\Big). \tag{3.31}$$

Here, for simplicity, we have let $b^s_{1,i} = \delta_{is}$.

Using the notation introduced in the Section 3.1 we can assume that the equation for Π is given by $\pi(x) = x_{k_o,1} - \sum_{j\in\mathcal{N}, j>k_o} a_j x_{j,1} = 0$.

We choose R_o small enough such that

$$|O_{j,s}(\xi)| \ \le\ \min_{j\in\mathcal{N}} |a_j|\ , \qquad |\xi| \ \le\ R_o\ , \tag{3.32}$$

and note that R_o will depend on the choice of the base point g_o in the statement of the theorem. In view of (3.31) and (3.32) we obtain

$$\begin{aligned} |X_i\phi(\xi)| \ &\le \quad \sum_{j=1}^{r}\sum_{s=1}^{m_j} |b^s_{j,i}(\xi_1, ..., \xi_{j-1})|\Big(a_j\delta_{s1}\ +\ O_{j,s}(\xi)\Big) \\ &\le \quad 2\sum_{j\in\mathcal{N}} |b^1_{j,i}(\xi_1, ..., \xi_{j-1})|\ |a_j|\ +\ \sum_{j\in\mathcal{N}^C}\sum_{s=1}^{m_j} |b^s_{j,i}(\xi_1, ..., \xi_{j-1})|\ |O_{j,s}(\xi)| \\ &\le \quad I(\xi)\ +\ II(\xi)\ . \end{aligned} \tag{3.33}$$

From (2.19), (2.20), and (3.33), we obtain the existence of constants $C_1 = C_1(\mathfrak{g}, D) > 0$, $C_2 = C_2(\mathfrak{g}, D) > 0$, $R_2 = R_2(\mathfrak{g}, D) > 0$ (these constants also depend on the choice of the base point g_o in the statement of the theorem), such that

$$\begin{aligned} \left(\sup_{B(0,R)\cap\partial D} |X\phi|\right)\ \sigma(B(0,R)\cap\partial D) \quad &\le C_1\ \left(\sup_{B(0,R)\cap\partial D}\ [I + II](\xi)\right)\ \sigma(B(0,R)\cap\partial D), \\ &\le\ C_1\ \left(\sup_{\text{Box}(C_2R)\cap\partial D}\ [I + II]\right)\ \sigma(\text{Box}(C_2R)\cap\partial D), \end{aligned} \tag{3.34}$$

for any $0 < R < R(\boldsymbol{G}, \Omega, g_o)$.

We split the proof of (3.34) in two lemmata.

Lemma 3.10. *For R suitably small one has*

$$\left(\sup_{Box(R)}\ I\right)\ \sigma(Box(R)\cap\partial D) \le\ C\ R^{Q-1}.$$

Proof. Using (3.33) and Lemma 2.1 we obtain

$$\sup_{\text{Box}(R)}\ I\ \le\ C\ \left(\sum_{j\in\mathcal{N}} |a_j|\ R^{j-1}\right)\ \le CR^{k_o-1}\ . \tag{3.35}$$

On the other hand, Lemma 3.2 and Lemma 3.7 yield

$$\sigma(\text{Box}(R) \cap \partial D) \leq C R^{Q-k_o}.$$

The desired estimate follows from the latter and from (3.35). □

Lemma 3.11. *For R sufficiently small, one has*

$$\left(\sup_{Box(R)} II \right) \sigma(Box(R) \cap \partial D) \leq C\ R^{Q-1} \begin{cases} R^{-k_o+2} & \text{if } k_o > 1\ ,\ \text{(characteristic point)} \\ R^{j_o} & \text{if } k_o = 1\ ,\ \text{(non-characteristic point)} \end{cases}$$

where $k_o = \min\ \mathcal{N}$ is the type of the origin, and $j_o = \min\ \mathcal{N}^C$.

Proof of Lemma 3.11. Note that for $\xi \in \text{Box}(R)$, the Euclidean norm of ξ is less than R, see (2.19). From Lemma 2.1 we have

$$II(\xi) = \sum_{j \in \mathcal{N}^C} \sum_{s=1}^{m_j} |b^s_{j,i}(\xi_1, ..., \xi_{j-1})|\ |O_{j,s}(\xi)| \ \leq\ C \sum_{j \in \mathcal{N}^C} R^{j-1} R \ \leq\ C R^{j_o}\ . \tag{3.36}$$

In view of the latter and of Lemma 3.2 and Lemma 3.7 one has

$$\left(\sup_{\text{Box}(R)} II \right) \sigma(\text{Box}(R) \cap \partial D) \leq\ C\ R^{j_o} R^{Q-k_o}.$$

The proof follows from observing that if $k_o = 1$, then $j_o > 1$, while if $k_o > 1$ then $j_o = 1$. □

Thus the proof of Theorem 3.9 is concluded. □

Remark 3.12. *If the defining function is "flat" near $0 \in \partial\Omega$, i.e. $\phi \in C^{r,1}(\boldsymbol{G})$ and $\phi(\xi) =< \nabla\phi(0), \xi > +O(|\xi|^r)$, then it is easy to see from the argument in the previous proof that there exist $M = M(\boldsymbol{G}, \Omega) > 0$ and $R_o = R_o(\boldsymbol{G}, \Omega) > 0$ such that, for any $0 < R < R_o$, one has*

$$\left(\sup_{B(0,R) \cap \partial\Omega} |X\phi| \right) \sigma(B(0,R) \cap \partial\Omega)\ \leq\ M\ R^{Q-1}\ . \tag{3.37}$$

3.5. **Upper bounds for real analytic domains.** In this subsection we show that if the defining function ϕ is analytic, then the upper estimates can be dramatically improved.

Theorem 3.13. *Let $\boldsymbol{G}$ be a Carnot group of step r with homogeneous dimension Q. Consider a bounded, open set $\Omega = \{g \in \boldsymbol{G} \mid \phi(g) < 0\}$, where $\phi(0) = 0$, ϕ is analytic near the origin and $\nabla\phi \neq 0$ in a neighborhood of $\partial\Omega$. There exist $M = M(\boldsymbol{G}, \Omega) > 0$ and $R_o = R_o(\boldsymbol{G}, \Omega) > 0$ such that, for any $0 < R < R_o$, one has*

$$\left(\sup_{B(0,R) \cap \partial\Omega} |X\phi| \right) \sigma(B(0,R) \cap \partial\Omega)\ \leq\ M\ R^{Q-1}\ . \tag{3.38}$$

Proof. We will use the notation introduced in the previous theorems, hence we consider the domain $D \subset \mathfrak{g}$ which is the image of Ω under the action of the exponential map. Denote by k_o the type of the origin and let

$$\bar{x} \in \{x_{k_o,1} = 0\} \approx \mathbb{R}^{m_1 + m_2 + \ldots + m_r - 1} \to \bar{x} + f(\bar{x}) e_{\vec{k_o},1},$$

be a parametrization of ∂D near the origin as a graph of the analytic function f. Recall that

$$f(\bar{x}) = \sum_{i \in \mathcal{N}, i > k_o} a_i x_{i,1} + H(\bar{x}),$$

with $H(\bar{x}) = O(|\bar{x}|^2)$, being the same remainder we have in (3.30).

We observe that if $\bar{x} + f(\bar{x}) e_{\vec{k_o},1} \in \partial D \cap B(0,R)$ then

$$|\bar{x}|_{\boldsymbol{G}} \leq R, \text{ and } |f(\bar{x})| \leq R^{k_o}.$$

Consequently, one has

$$|H(\bar{x})| \leq R^{k_0} + \sum_{i \in \mathcal{N}, i > k_o} |a_i x_{i,1}| \leq C R^{k_o}.$$

The crucial point in the argument is that, since f is analytic, then for all $j = 1, ..., r$ and $s = 1, ..., m_r$,

$$|H(\bar{x})| \leq C R^{k_o} \text{ implies } |O_{j,s}| \leq |\nabla H(\bar{x})| \leq C R^{k_o - 1},$$

where $O_{j,s}$ are as in (3.31). At this point we return to the argument in the proof of Theorem 3.9, more precisely in Lemma 3.11, (3.36), and use the previous estimate to deduce

$$|II| \leq \sum_{j \in \mathcal{N}^C} \sum_{s=1}^{m_j} |b^s_{j,i}(\xi_1, ..., \xi_{j-1})| \, |O_{j,s}(\xi)| \leq C \sum_{j \in \mathcal{N}^C} R^{j + k_o - 2} \leq C R^{k_o - 1}. \tag{3.39}$$

Finally, we have that

$$\left(\sup_{\text{Box}(R) \cap D} II \right) \sigma(\text{Box}(R) \cap \Pi) \leq C R^{k_o - 1} R^{Q - k_o} = C R^{Q-1}. \tag{3.40}$$

Substituting (3.40) in place of (3.36) we reach the desired conclusion. □

3.6. **Lower bounds for C^2 domains of type 1 or 2.** The main result of this section is the following.

Theorem 3.14. *Let $\boldsymbol{G}$ be a Carnot group of step r with homogeneous dimension Q. Consider a bounded, open set $\Omega = \{g \in \boldsymbol{G} \mid \phi(g) < 0\}$, where ϕ is such that $|\nabla \phi| \neq 0$ in a neighborhood of $\partial \Omega$.*

(i) *If $\phi \in C^{1,1}(\boldsymbol{G})$ and $g_o \in \partial \Omega$ is non-characteristic then there exist $M = M(\boldsymbol{G}, \Omega, g_o) > 0$ and $R_o = R_o(\boldsymbol{G}, \Omega, g_o) > 0$ depending continuously on g_o, such that, for any $0 < R < R_o$, one has*

$$\mu(B(g_o, R)) \;\geq\; M^{-1} \; R^{Q-1} \;. \tag{3.41}$$

(ii) *If $\phi \in C^2(\boldsymbol{G})$ and $g_o \in \partial \Omega$ is of type 2, then there exist $M = M(\boldsymbol{G}, \Omega, g_o) > 0$ and $R_o = R_o(\boldsymbol{G}, \Omega, g_o) > 0$ depending continuously on g_o, such that, for any $0 < R < R_o$, estimate (3.41) still holds.*

The rest of the section is dedicated to the proof of this theorem. We will use the notations and the reductions introduced in the proof of Theorem 3.9 and in the previous sections. In particular, thanks to Lemma 3.8 it is clear that in order to prove (3.41) it suffices to show that if the origin is in ∂D and we denote by Π the tangent space to the boundary at the origin, then there exist $C = C(\mathfrak{g}, \Omega) > 0$, and $R_o = R_o(\mathfrak{g}, \Omega) > 0$ which will depend continuously on the choice of g_o in the statement of the theorem, and such that for any $0 < R < R_o$, one has

$$\int_{\text{Box}(R)\cap\Pi} |X\phi|(\bar{y} + \eta(\bar{y})\vec{e}_{k_o,1})\ d\sigma(\bar{y})\ \geq C\ R^{Q-1}\ . \tag{3.42}$$

Here η is as in (3.21). Next, we distinguish two cases:

Type one: If the origin is not characteristic then $a_1 \neq 0$, and $k_0 = 1$. Since the non-characteristic hypothesis is an open condition, then it holds in a neighborhood of the point in question, giving us local uniform control on $|a_1|$ (from here on a_1 will be part of the constants that we use in the estimates). The key (elementary) observation is that

$$0 < C^{-1} \leq |X\phi| \leq C, \tag{3.43}$$

where $C = C(a_1, \boldsymbol{G}) > 0$.

Estimate (3.42) will immediately follow from

$$\sigma(\text{Box}(R) \cap \Pi) \geq\ C\ R^{Q-1}\ , \tag{3.44}$$

which in turn is an immediate consequence of Lemma 3.4 and Lemma 3.8.

Type two: We need to assume that $\phi \in C^2$ in a neighborhood of $\partial\Omega$. The strategy is to reduce the problem to the case where ∂D is an hyperplane, and then use Lemma 3.6.

Repeating the argument in (3.29)-(3.31) we obtain

$$\phi(\xi) = \pi(\xi) + H, \tag{3.45}$$

with $H = o(|\xi|^2)$, and for $\xi \in B(0, R_o)$, and for sufficiently small R_o. Consequently we obtain

$$\nabla\phi(\xi) = \nabla\phi(0) + \vec{o}(\xi), \tag{3.46}$$

where $\vec{o}(\xi) = \{o_{k,j}(\xi)\}$, $k = 1, ..., r$, $j = 1, ..., m_k$ and $o_{k,j}(\xi) = o(|\xi|)$. In view of (3.46) and Lemma 2.1, we can compute the horizontal gradient of ϕ as follows

$$\begin{aligned} X_i\phi(\xi) =< X_i,\ \nabla\phi(\xi) >= &\quad < X_i, \nabla\phi(0) + \vec{o}(\xi) > \\ = &\quad \sum_{k=1}^{r}\sum_{l=1}^{m_k} b^i_{kl}(\xi_1, ..., \xi_{k-1})\Big(\frac{\partial\pi}{\partial x_{k,l}} + o_{kl}(\xi)\Big). \end{aligned} \tag{3.47}$$

Here, for simplicity, we have let $b^i_{1,l} = \delta_{il}$.

Repeating the argument in Lemma 3.11, with the new regularity hypothesis $\phi \in C^2$, and knowing that the origin is of type two, we obtain

Lemma 3.15. *In the notation established above, for every $\epsilon > 0$ we can choose $R_o = R_o(\epsilon, \mathfrak{g}, \Omega) > 0$ such that if $0 < R < R_o$, one has*

$$\left(\sup_{Box(R)} |\sum_{k=1}^{r} \sum_{l=1}^{m_k} b_{ki}^{l}(\xi_1, ..., \xi_{k-1}) o_{kl}(\xi)| \right) \ \sigma(Box(R) \cap \Pi) \leq \ \epsilon \ R^{Q-1} \tag{3.48}$$

Consequently, the estimate (3.42) is reduced to the proof of

$$\int_{\text{Box}(R)\cap\Pi} |X\pi|(\bar{y} + \eta(\bar{y})\vec{e}_{k_o,1}) d\sigma(\bar{y}) \geq C \ R^{Q-1} \tag{3.49}$$

In fact, if (3.49) holds, then from (3.47), and (3.48) we would obtain

$$\begin{aligned} \int_{\text{Box}(R)\cap\Pi} &|X\phi|(\bar{y} + \eta(\bar{y})\vec{e}_{k_o,1}) \ d\sigma(\bar{y}) \\ &\geq C \int_{\text{Box}(R)\cap\Pi} |X\pi|(\bar{y} + \eta(\bar{y})\vec{e}_{k_o,1}) d\sigma(\bar{y}) - \epsilon R^{Q-1} \geq \ CR^{Q-1}. \end{aligned} \tag{3.50}$$

We now proceed with the proof of (3.49): Since the origin is of type two, we have

$$\pi(\xi) = \pi(\xi_2, ..., \xi_r) = \sum_{k \in \mathcal{N}} a_k \xi_{k,1},$$

and for $\xi \in \text{Box}(R) \cap \Pi$, it follows that

$$\begin{aligned} |X_i \pi(\xi)| = & \quad |\sum_{k\in\mathcal{N}} b_{k,i}^1 a_k| \\ \geq & \quad |b_{2,i}^1 a_2| - |\sum_{k\in\mathcal{N}, \ k>2} b_{k,i}^1 a_k| \\ \geq & \quad |b_{2,i}^1 a_2| - \sum_{k\in\mathcal{N}, \ k>2} CR^{k-1}. \end{aligned} \tag{3.51}$$

As we observed in Lemma 3.6, $b_{2,i}^1$ depends only on ξ_1, and not on the higher layers coordinates. Hence, since $k_0 = 2$, then $b_{2,i}^1(\bar{y} + \eta(\bar{y})\vec{e}_{k_o,1}) = b_{2,i}^1(\bar{y})$.

At this point, we have that

$$\begin{aligned} \int_{\text{Box}(R)\cap\Pi} |X\pi|(\bar{y} + \eta(\bar{y})\vec{e}_{k_o,1}) d\sigma(\bar{y}) \quad &= \int_{\text{Box}(R)\cap\Pi} |X\pi|(\bar{y}) d\sigma(\bar{y}) \\ \text{(in view of Lemma 3.6)} \quad &\geq CR^{Q-1}, \end{aligned}$$

and the proof is concluded.

4. Some examples.

In Theorem 1.2 we have established the 1-Ahlfors regularity of the X-perimeter measure under the assumption that Ω be a C^2 domain of type ≤ 2. We recall that such hypothesis is automatically fulfilled when the step of the group is $r = 2$. In this section we show that the type assumption is optimal, in the sense that we prove the existence of a group $\boldsymbol{G}$ of step 3, and of a domain $\Omega \subset \boldsymbol{G}$ of type 3 for which the X-perimeter measure fails to be 1−Ahlfors regular. We remark that additional smoothness does not suffice since in our example the defining function ϕ of Ω is of class C^∞.

We consider the cycle group $\boldsymbol{G} = K_3$, see ex. 1.1.3 in [CGr], whose Lie algebra is given by the stratification,

$$\boldsymbol{G} \ = \ V_1 \oplus V_2 \oplus V_3 \ ,$$

where $V_1 = span\{X_1, X_2\}$, $V_2 = span\{X_3\}$, and $V_3 = span\{X_4\}$, so that $m_1 = 2$ and $m_2 = m_3 = 1$. We assign the commutators

$$[X_1, X_2] \ = \ X_3 \qquad\qquad [X_1, X_3] \ = \ X_4 \ , \tag{4.1}$$

all other commutators being assumed trivial. We observe that the homogeneous dimension of $\boldsymbol{G}$ is

$$Q \ = \ m_1 \ + \ 2\, m_2 \ + \ 3\, m_3 \ = \ 7 \ .$$

The group law in $\boldsymbol{G}$ is given by the Baker-Campbell-Hausdorff formula (2.8). In exponential coordinates, if $g = \exp(X)$, $g' = \exp(X')$, where $X = \sum_{i=1}^4 x_i X_i$, $X' = \sum_{i=1}^4 y_i X_i$, we have

$$g \ \circ \ g' \ = \ X \ + \ X' \ + \frac{1}{2}\, [X, X'] \ + \ \frac{1}{12}\, \{[X,[X,X']] \ - \ [X',[X,X']]\}.$$

A computation based on (4.1) gives (see also ex. 1.2.5 in [CGr])

$$g \ \circ \ g' \ = \ \Big(x_1 + y_1, x_2 + y_2, x_3 + y_3 + P_3, x_4 + y_4 + P_4 \Big) \ ,$$

where

$$P_3 \ = \ \frac{1}{2}\, (x_1 y_2 - x_2 y_1) \ ,$$

$$P_4 \ = \ \frac{1}{2}(x_1 y_3 - x_3 y_1) + \frac{1}{12}\Big(x_1^2 y_2 - x_1 y_1 (x_2 + y_2) + x_2 y_1^2 \Big) \ .$$

Using (2.7), (2.8) we find that a left invariant basis of the Lie algebra $\mathfrak{g}$ is given by the vector fields

$$\begin{aligned} X_1 \ &= \ \frac{\partial}{\partial x_1} \ - \ \frac{x_2}{2}\, \frac{\partial}{\partial x_3} \ - \ \left(\frac{x_3}{2} + \frac{x_1 x_2}{12} \right) \frac{\partial}{\partial x_4} \ , \\ X_2 \ &= \ \frac{\partial}{\partial x_2} \ + \ \frac{x_1}{2}\, \frac{\partial}{\partial x_3} \ + \ \frac{x_1^2}{12}\, \frac{\partial}{\partial x_4} \ , \\ X_3 \ &= \ \frac{\partial}{\partial x_3} \ + \ \frac{x_1}{2}\, \frac{\partial}{\partial x_4} \ , \\ X_4 \ &= \ \frac{\partial}{\partial x_4} \ . \end{aligned} \tag{4.2}$$

We now consider the smooth function

$$\phi(g) \ = x_4 - (x_1^2 + x_2^2 + x_3^2),$$

for which we obviously have

$$\nabla\phi(g) \ = \ \Big(- 2x_1, -2x_2, -2x_3, 1\Big) \ \neq \ 0 \ ,$$

and the C^∞ domain $\Omega = \{g \in \boldsymbol{G} \mid \phi(g) < 0\}$. Using this formula and (4.2) we easily obtain

$$X\phi(g) \ = \ (X_1\phi(g), X_2\phi(g)) \ = \ \Big(- 2x_1 + x_2x_3 - p_2, -2x_2 - x_1x_3 + p_1\Big) \ , \tag{4.3}$$

with $p_2 = \left(\frac{x_3}{2} + \frac{x_1x_2}{12}\right)$, $p_1 = \frac{x_1^2}{12}$. We note explicitly that the characteristic set Σ of Ω is non-empty, therefore Ω is at least of type 2. In particular, one easily computes

$$[X_1, X_2]\phi(g) \ = \ X_3\phi(g) \ = \ -2x_3 + \frac{x_1}{2} \ ,$$

and the latter function vanishes along a line passing through the origin. This shows that the origin is of type 3, and therefore so is Ω. Since the tangent hyperplane at the origin Π is given by $\{x_4 = 0\}$, we easily find

$$\sigma(\text{Box}(R) \ \cap \ \Pi) \ = \ 2\pi R^4 \ . \tag{4.4}$$

Observe that if

$$|x_1|, |x_2| \le \left(\frac{R}{10}\right)^{3/2}, \text{ and } |x_3| \le \left(\frac{R}{10}\right)^{2}, \text{ then } x = (x_1, x_2, x_3, x_1^2 + x_2^2 + x_3^2) \in \partial\Omega \cap \text{Box}(R),$$

and if

$$x = (x_1, x_2, x_3, x_1^2 + x_2^2 + x_3^2) \in \partial\Omega \cap \text{Box}(R), \text{ then } |x_1|, |x_2| \le R^{3/2}, \text{ and } |x_3| \le R^2.$$

The latter yields immediately that

$$\begin{aligned} \sigma(\partial\Omega \cap B(0,R)) \quad &= \int_{(x_1,x_2,x_3,x_1^2+x_2^2+x_3^2)\in\partial\Omega\cap B(0,R)} \sqrt{1 + 4(x_1^2 + x_2^2 + x_3^2)}\, dx_1dx_2dx_3 \\ &\approx CR^3R^2 = CR^5. \end{aligned} \tag{4.5}$$

Together with (4.4), estimate (4.5) shows that the size of $\partial\Omega \cap B(0,R)$ and $\text{Box}(R) \cap \Pi$ are not comparable in general (we have shown that they are comparable if the type is at most two).

Next, we show that the estimate

$$P_X(\Omega, B(0,R)) \ \cong \ \int_{\text{Box}(R) \ \cap \ \{\phi=0\}} |X\phi|(g)d\sigma(g) \ \ge \ C \ R^{Q-1} \tag{4.6}$$

fails for our choice of ϕ. Observe that

$$|X\phi|^2 = 4(x_1^2 + x_2^2) + (x_1^2 + x_2^2)x_3^2 + p_1^2 + p_2^2 - 2x_2x_3p_2 + 4x_1p_2 - 4x_2p_1 - 4x_1x_3p_1.$$

Consequently we have the simple estimates

$$\sup_{\text{Box}(R) \ \cap \ \{\phi=0\}} |X\phi|^2 \le CR^3, \tag{4.7}$$

and

$$(4.8)\quad P_X(\Omega; B(0,R)) \cong \int_{\mathrm{Box}(R)\,\cap\,\{\phi=0\}} |X\phi|(g)d\sigma(g)$$
$$\leq \Big(\sup_{\mathrm{Box}(R)\,\cap\,\{\phi=0\}} |X\phi|(g)\Big)\sigma(\partial\Omega\cap B(0,R)) \leq CR^5R^{\frac{3}{2}}.$$

Since $Q = 7$ then it is obvious that the lower bound on $P_X(\Omega; B(0,R))$ cannot be comparable to R^6.

5. Geometric estimates for a system of free vector fields of Hörmander type

Our next goal consists in extending the results in the previous section and prove area estimates for surface Carnot-Caratheodory balls associated to a free system of Hörmander vector fields.

We will use the notation introduced in Theorem 2.7, and in the preceding paragraphs. Moreover, we will denote by $B(x,R)$ the solid Carnot-Caratheodory balls in $\mathbb{R}^n$, and with $B_R(y)$ the Carnot-Caratheodory balls in Lie groups or in their Lie algebras.

Let $\phi \in C^{1,1}(M^n)$, with $|\nabla\phi| \neq 0$ near the level set $\Omega = \{x \in M^n \mid \phi(x) < 0\} \subset M^n$ and such that Ω is a bounded $C^{1,1}$ domain. Set $X_1,...,X_m$ smooth vector fields, free up to step r in a neighborhood of $\partial\Omega$, and satisfying the Hörmander condition (1.2). Choose a positive $R_1 = R_1(X,\Omega,x_o) < 1$, small enough such that $B(x_o, 2R_1) \subset V$, where V is the neighborhood of $x_o \in \partial\Omega$, which is the domain for the coordinate chart $\theta_{x_o}(\cdot) : V \to U \subset \boldsymbol{G}_{m,r}$, as in Theorem 2.7 (**A**). For $0 < R < R_1$, set $\Delta = \Delta(x_o,R) = B(x_o,R)\cap\partial\Omega$. Let us adopt the following notation

$$(5.1)\qquad \begin{aligned} \Omega' &= \theta_{x_o}(\Omega) \subset \boldsymbol{G}_{m,r} \\ \phi'(x) &= \phi(\theta_{x_o}^{-1}(x)),\ \text{so that}\ \Omega' = \{y \in \boldsymbol{G}_{m,r} \mid \phi'(y) < 0\} \end{aligned}$$

Definition 5.1. *Let $X_1,...,X_m$ smooth vector fields in M^n, free up to step r and satisfying the Hörmander condition (1.2). Let $\Omega = \{x \in M^n \mid \phi(x) < 0\} \subset M^n$ be a bounded $C^{1,1}$ domain. Choose any collection $\{X_{ik}\}$, of commutators of length k with $X_{i1} = X_i$ such that the system $\{X_{ik}\}$, $k = 1,...,r$ evaluated at $x_o \in \partial\Omega$ is a basis of $\mathbb{R}^n$. We define the "type" of x_o to be the smallest $k = 1,...,r$ such that there exists $l = 1,...,m_k$ for which $X_{k,l}\phi(g_0) \neq 0$. We will denote by k_o = type(x_o) the type of x_o, and if for every $x_o \in \partial\Omega$ we have that type$(x_o) \leq s \in \mathbb{N}$ then we will say that Ω has type less or equal than s.*

Remark 5.2. *Note that the definition of "type" of x_o is independent of the choice of the collection $\{X_{ik}\}$, in view of the definition of free vector fields. An equivalent definition is the following: We define the "type" of x_o to be the smallest $k = 1,...,r$ such that there exists a commutator $Z = [X_{i_1},[X_{i_2},...,[X_{i_{k-1}},X_{i_k}]...]$ of order k such that $Z\phi(x_o) \neq 0$.*

We want to rephrase the notion of type in terms of the osculating group $\boldsymbol{G}_{m,r}$.

Lemma 5.3. *Using the notation introduced above, one has that if the point $x_o \in \partial\Omega$ has type k_o in Ω, then the origin is of type k_o in Ω' (according to Definition 2.2).*

Proof. We will prove the statement for $k_o = 1$ and $k_o = 2$. The general case follows similarly. Notice that for any $N > r$, and $i = 1, ..., m$, we have

$$\mathcal{R}_i\phi'(0) = \sum_{l=1}^{r}\sum_{|\alpha|=l} p_{\alpha,i,N}(0)\partial_{y_\alpha}\phi'(0) \tag{5.2}$$

where $p_{\alpha,i,N}$ are homogeneous group polynomials of order greater or equal than $|\alpha| \geq 1$. In particular, $p_{\alpha,i,N}(0) = 0$, and we obtain

$$\mathcal{R}_i\phi'(0) = 0, \quad i = 1, ..., m. \tag{5.3}$$

In case, type$(x_o) = 1$ then (2.22) implies

$$X\phi(x_o) = Y\phi'(0). \tag{5.4}$$

Hence the origin is of type one in $\Omega' \subset \boldsymbol{G}_{m,r}$. Next, we assume type$(x_o) = 2$ and consider any collection $\{X_{k,i}\}$ as in Definition 5.1. Recall from Theorem 2.7 that for any multi-index $\beta = (2, i)$, $i = 1, ..., m_2$ we have $X_\beta\phi(x_o) = Y_\beta\phi'(0) + \mathcal{R}_\beta\phi'(0)$, where $\mathcal{R}_\beta$ is a vector field of order less or equal than 1. At this point we observe that modulo higher order terms (which will vanish at the origin) we must have

$$\mathcal{R}_{(2,i)}\phi'(0) = \sum_{l=1}^{r}\sum_{|\alpha|=l} p_{\alpha,\beta,N}(0)\partial_{y_\alpha}\phi'(0) \tag{5.5}$$

with the degree of $p_{\alpha,\beta,N}$ greater or equal than $|\alpha| - 1$. The only term which will not vanish in this expression are those corresponding to $|\alpha| = 1$, which lead us to

$$\mathcal{R}_{2,i}\phi'(0) = \sum_{|\alpha|=1} p_{\alpha,\beta,N}(0)\partial_{y_\alpha}\phi'(0) = \sum_{j=1}^{m} c_j Y_j\phi'(0), \tag{5.6}$$

for the choice of the coefficients $c_j = p_{(1,j),\beta,N}(0)$. Since we are assuming that $X\phi(x_o) = 0$ then in view of (5.4), we also have $Y\phi'(0) = 0$, and consequently $\mathcal{R}_\beta\phi'(0) = 0$. At this point Theorem 2.7 gives us the equality $X_\beta\phi(x_o) = Y_\beta\phi'(0)$, for any $\beta = (2, i)$, $i = 1, ..., m_2$. This implies that the type of the origin in Ω' is two. $\square$

Theorem 5.4. *Let $X_1, ..., X_m$ be smooth vector fields in M^n, free up to step r and satisfying the Hörmander condition (1.2). Let $\Omega = \{x \in M^n \mid \phi(x) < 0\} \subset M^n$ be a bounded $C^{1,1}$ domain. For each point x_o in $\partial\Omega$, denote by k_o its type. There exists constants $C = C(\Omega, x_o) > 0$ and $R = R(\Omega, x_o) > 0$ depending continuously on x_o, such that if $0 < R < R_o$ then*

$$\left(\sup_{x\in\Delta(x_o,R)} |X\phi(x)|\right) \sigma(\Delta(x_o, R)) \;\leq\; C(\Omega, X, x_o)\frac{|B(x_o, R)|}{R^{s(x_o)}}. \tag{5.7}$$

with

$$s(x_o) = \begin{cases} k_o - 1, & \textit{if } k_o \geq 3, \\ 1, & \textit{if } \partial\Omega, \textit{ and } X_1, ..., X_m \textit{ are real analytic near } x_o \textit{ or if } k_0 \leq 2. \end{cases} \tag{5.8}$$

Proof. The idea of the proof is very simple: We rephrase the estimate (5.7) in terms of the "tangent" free algebra $\mathfrak{g}_{m,r}$ via the map $\theta(\cdot,\cdot)$ defined in Theorem 2.7 (B) and the exponential coordinates. At this point, formula (2.22) allows us to divide the problem in two steps. First we estimate the part corresponding to Y_i, using the results from the previous section, and then we deal with the error term $\mathcal{R}_i$ in (2.23). This error term is an operator of order less or equal than zero, hence it does not contribute (modulo higher order perturbations) to the final estimate. In the following we describe in detail this general idea.

Choose a positive $R_1 = R_1(X,\Omega,x_o) < 1$, small enough such that $B(x_o, 2R_1) \subset V$, where V is the neighborhood of x_o, which is the domain for the coordinate chart $\theta_{x_o}(\cdot) : V \to U \subset \boldsymbol{G}_{m,r}$, as in Theorem 2.7 (**A**). For $0 < R < R_1$, set $\Delta = \Delta(x_o, R) = B(x_o, R) \cap \partial\Omega$, and let Ω' and ϕ' be as in (5.1). Notice that if Ω and $X_1, ..., X_m$ are real analytic in a neighborhood of x_o, then $\theta_{x_o}(\cdot)$ is real analytic, and consequently Ω' will be real analytic in a neighborhood of the identity. Since $\theta_{x_o} : V \to U$ is a smooth map, then there exists a constant $C = C(X,\Omega,x_o) > 0$ depending also on the Lipschitz norm of θ_{x_o} in $B(0, 2R_1)$ such that

$$\begin{aligned} \sigma(\Delta) \leq &\quad C\sigma(\theta_{x_o}(\Delta)) \\ \leq &\quad C\sigma(\theta_{x_o}(B(x_o,R)) \cap \partial\Omega'). \end{aligned} \tag{5.9}$$

Here we are introducing a slight ambiguity in the notation, in fact in the previous formula, we have used the same symbol σ to denote the surface measure in M^n (on the left hand side) and the surface measure of $\boldsymbol{G}_{m,r}$ (on the right hand side). Since it is clear which is which, and the two measures are bi-lipschitz equivalent, we will continue to use this notation in the interest of clarity.

Let us observe that since θ is a diffeomorphism we can choose a constant $C > 0$ such that we also have the estimate

$$\sigma(\Delta) \geq C^{-1}\sigma(\theta_{x_o}(B(x_o,R)) \cap \partial\Omega'). \tag{5.10}$$

Since $X_1, ..., X_m$ are free up to step $r \in \mathbb{N}$ in a neighborhood of x_o, then the argument in Remark 2.6 allows us to write that for some constants $C_1, C_2 > 0$ depending only on Ω we have

$$\text{Box}_{\,C_1R} \subset B(x_o,R) \subset \text{Box}_{\,C_2R}, \tag{5.11}$$

where Box_R denotes the box-like sets defined in Remark 2.6, and $0 < R < R_1$, with a smaller R_1, if needed. Consequently, $\theta_{x_o}(B(x_o,R)) \subset \boldsymbol{G}_{m,r}$ will be contained in one of the group "boxes" Box_{CR} of size comparable to R defined in (2.19). By virtue of the box-ball theorem (2.20) (see Theorem 3, [NSW]) we have

$$\theta_{x_o}(B(x_o,R)) \subset B_{C_3R}(0) \subset \boldsymbol{G}_{m,r}, \tag{5.12}$$

for some positive constant C_3 depending only on X, and Ω. From (5.9)–(5.12) we obtain

$$\sigma(\Delta) \leq C\ \sigma(\text{Box}_R(0) \cap \partial\Omega) \leq C\ \sigma(B_{C_3R}(0) \cap \partial\Omega') = C\ \sigma(\Delta'), \tag{5.13}$$

where we have let $\Delta' = B_{C_3R}(0) \cap \partial\Omega'$. Once again, we also have the reverse inequality

$$\sigma(\Delta') \leq C\sigma(\Delta).$$

In view of Lemma 5.3, we know that $\Omega' \subset \boldsymbol{G}_{m,r}$ is a $C^{1,1}$ domain of type less or equal than two. This observation allows us to invoke Theorem 3.9, proved in the previous section, and infer that there exists $R_2 = R_2(\Omega', \boldsymbol{G}_{m,r}) > 0$, such that for any $0 < R < R_2$ one has

$$\sigma(\Delta') \sup_{B_R(0)} |Y\phi'| \leq C \begin{cases} R^{Q-k_o+1}, & \text{if } k_o \geq 3 \\ R^{Q-1}, & \text{if } \partial\Omega, \text{ and } X_1, ..., X_m \text{ are real analytic, or if } k_o = 1, 2 \end{cases} \tag{5.14}$$

for some positive constant $C = C(\Omega', \boldsymbol{G}_{m,r})$. Choose $R_0 = \min\{R_1, R_2\}$. In order to prove Theorem 5.4, we need to estimate the quantity

$$\sigma(\Delta) \sup_{B(x_o,R)} |X\phi| \leq C\sigma(\Delta') \left(\sup_{B_R(0)} |Y\phi'| + \sup_{B_R(0)} \sum_{i=1}^{m} |\mathcal{R}_i \phi'| \right) \tag{5.15}$$

in the range $0 < R < R_0$. We will prove the following

Lemma 5.5. *In the notation established above, there exists $R(\Omega', \boldsymbol{G}_{m,r}) > 0$ such that for any $0 < R < R(\Omega', \boldsymbol{G}_{m,r})$ and for every $i = 1, ..., m$*

$$|\Delta'| \sup_{B_R(0)} |\mathcal{R}_i \phi'| \leq C \begin{cases} R^{Q-k_o+2}, & \textit{if } k_o \geq 3 \\ R^{Q}, & \textit{if } \partial\Omega, \textit{ and } X_1, ..., X_m \textit{ are real analytic, or if } k_o = 1, 2 \end{cases}, \tag{5.16}$$

for some positive $C = C(\Omega, \boldsymbol{G}_{m,r})$.

The proof of the Theorem follows immediately from Lemma 5.5, (5.14) and (5.15). □

Proof of Lemma 5.5. Following the arguments in (3.24)-(3.31), we make a number of reductions on the problem. In particular we will assume without loss of generality that the surface portion Δ' is a portion of the tangent hyper-plane

$$\Pi = \{(y_{ik}) \in \mathfrak{g}_{m,r} \mid \pi(y) \ = 0 \ \},$$

with $\pi(y) = \sum_{j \in \mathcal{N}} a_j y_{1,j}$, and we will substitute the gauge ball $B_R(0)$ with the box-like set $\text{Box}_R(0)$. As in (3.29) we write

$$\phi(y) = \pi(y) + H, \tag{5.17}$$

with $H = O(|y|^2)$. After such reductions we have that for some positive $R(\Omega', \boldsymbol{G}_{m,r}) < 1$, and $C = C(\Omega', \boldsymbol{G}_{m,r}) > 0$, if we choose $0 < R < R(\Omega', \boldsymbol{G}_{m,r})$ then

$$\begin{aligned} \sigma(\Delta') \sup_{B_R(0)} |\mathcal{R}_i \phi'| \leq & \quad C\sigma(\text{Box}_R(0) \cap \Pi) \left\{ \sup_{\text{Box}_R(0)} |\mathcal{R}_i \left(\sum_{j \in \mathcal{N}} a_j y_{1,j} \right)| \ + \ \left| \mathcal{R}_i(\ H \ ) \right| \right\}, \\ \leq & \quad C \left\{ I \ + \ II \right\}, \end{aligned} \tag{5.18}$$

where we have let

$$I \ = \ \sigma(\text{Box}_R(0) \cap \Pi) \sup_{\text{Box}_R(0)} |\mathcal{R}_i \left(\sum_{j \in \mathcal{N}} a_j y_{1,j} \right)|, \tag{5.19}$$

and

$$II \ = \ \sigma(\text{Box}_R(0) \cap \Pi) \sup_{\text{Box}_R(0)} \left| \mathcal{R}_i(\ H \ ) \right|. \tag{5.20}$$

Now, we choose any integer N larger than the homogeneous dimension $Q = \sum_{i=1}^r im_i$ of the group $\boldsymbol{G}_{m,r}$, and let $g_{\alpha,i,N}$ denote the higher order terms in the $N-$th order Taylor expansion of R_i (see 2.23).

Estimate of I: A direct computation yields

$$
\begin{aligned}
\mathcal{R}_i \quad \Big(\sum_{j\in\mathcal{N}} a_j y_{1,j}\Big) &= \sum_{l=1}^r \sum_{|\alpha|=l} [p_{\alpha,i,N}(y) + g_{\alpha,i,N}(y)] \partial_{y_\alpha} \Big(\sum_{j\in\mathcal{N}} a_j y_{1,j}\Big) \\
&= \sum_{l=k_o}^r \Big(\sum_{|\alpha|=l} [p_{\alpha,i,N} + g_{\alpha,i,N}]\Big) a_l. \qquad (5.21)
\end{aligned}
$$

Because of the homogeneity of the polynomials $p_{\alpha,i,N}$, and the growth condition $g_{\alpha,i,N}(y) = O(|y|_{\mathfrak{g}}^N)$, we have for any $R > 0$ suitably small

$$
\sup_{\text{Box}_R(0)} \sum_{|\alpha|=l} \Big[|p_{\alpha,i,N}| + |g_{\alpha,i,N}|\Big] \le C_{\mathfrak{g}_{m,r}} R^l, \ l = 1, ..., r. \qquad (5.22)
$$

From Section 3.1 we recall that

$$
\sigma(\text{Box}_R(0) \cap \Pi) \le C R^{Q-k_o}. \qquad (5.23)
$$

From (5.21)-(5.23) it is easy to deduce the following estimate

$$
\begin{aligned}
|\text{Box}_r(0) \cap \Pi| \sup_{\text{Box}_r(0)} |R_i\Big(\sum_{j=1}^r a_j y_{1,j}\Big)| \quad &\le C_{m,r} R^{Q-k_o} \sum_{l=k_o}^r R^l \\
&\le C_{m,r} R^{Q-k_o} R^{k_o} = C_{m,r} R^Q.
\end{aligned}
$$

Estimate of II: Let $N > r$ and recall that $g_{\alpha,i,N}(y) = O(|y|_{\mathfrak{g}}^N)$. For every multi-index $\alpha = (k, j)$, denote by $O_\alpha = \partial_{y_\alpha} H$, and observe that since $\phi \in C^{1,1}$ then $O_\alpha = O(|y|)$. In view of the upper estimates in the Carnot group setting and (1.11) we have

$$
\begin{aligned}
II \ \le \quad & C R^{Q-k_0} \Big(|\mathcal{R}_i(H)| \Big) \\
\le \quad & C R^{Q-k_0} \Big(|\sum_{l=1}^r \sum_{|\alpha|=l} p_{\alpha,i,N} \partial_{y_\alpha} H| + R^N \Big) \\
\le \quad & C R^{Q-k_0} \Big(\sum_{l=1}^r R^l \sum_{|\alpha|=l} |O_\alpha| + R^N \Big) \qquad (5.24) \\
\le \quad & C R^{Q-k_0} \Big(\sum_{l=1}^r R^{l+1} \Big) \\
\le \quad & C R^{Q-k_0+2}
\end{aligned}
$$

On the other hand, if Ω' is real analytic in a neighborhood of the identity, then the argument in Section 3.5 yields the estimate $|\nabla H| \le C R^{k_o-1}$. Substituting the latter in (5.24) we obtain $II \le C R^Q$. □

We now turn to the proof of the lower Ahlfors estimates.

Theorem 5.6. *Let $X_1, ..., X_m$ be smooth vector fields in M^n, free up to step r and satisfying the Hörmander condition (1.2). Let $\Omega = \{x \in M^n \mid \phi(x) < 0\} \subset M^n$ be a bounded C^2 domain. For each point x_o in*

$\partial\Omega$ of type less or equal than two, there exists constants $C = C(\Omega, X, x_o) > 0$ and $R = R(\Omega, X, x_o) > 0$ depending continuously on x_o, such that if $0 < R < R_o$ then

$$\int_{\Delta(x_o,R)} |X\ \phi(x)| d\sigma(x)\ \geq\ C(\Omega, X, x_o)\frac{|B(x_o,R)|}{R}. \tag{5.25}$$

Here we recall that $\Delta(x_o, R) = B(x_o, R) \cap \partial\Omega$, and $|\cdot|$ denotes the Lebesgue-Hausdorff measure of the set.

Proof of Theorem 5.6. We adopt the same notation as in the previous theorem. Using the same arguments as those in (5.9)–(5.13), we find that there exist constants $C = C(X, \Omega, x_o) > 0$ and $R_0 = R_0(X, \Omega, x_o) > 0$ such that

$$\int_{\Delta(x_o,R)} |X\ \phi(x)| d\sigma(x) \geq C \int_{\Delta'} \sum_{i=1}^{m} \Big(|Y_i \phi'| + |\mathcal{R}_i \phi'| \Big) d\sigma. \tag{5.26}$$

In view of Theorem 3.14 and Lemma 5.5 we obtain

$$\begin{aligned} \int_{\Delta(x_o,R)} |X\ \phi(x)| d\sigma(x) \qquad & \geq C \int_{\Delta'} \sum_{i=1}^{m} \Big(|Y_i \phi'| + |\mathcal{R}_i \phi'| \Big) d\sigma \\ & \geq C \Big(\int_{\Delta'} \sum_{i=1}^{m} |Y_i \phi'| d\sigma - |\Delta'| \sup_{B_R(0)} |\mathcal{R}_i \phi'| \Big) \\ & \geq C \Big(R^{Q-1} - R^{Q} \Big) \\ & \geq C R^{Q-1}, \end{aligned} \tag{5.27}$$

concluding the proof.

□

6. The case of a general CC space : Proof of Theorem 1.3

In this section we establish the Ahlfors regularity for C^2 domains of type less or equal than two, in a general CC space (M^n, d). We will assume that $X_1, ..., X_m$ are smooth vector fields which satisfy (1.2) with step $r \in \mathbb{N}$ at every point of an open subset $U \subset M^n$.

A routine modification of the proof also shows that if $X_1, ..., X_m$ are analytic and $\partial\Omega$ is analytic then the Ahlfors regularity follows with no restriction on the type.

The results in this section rest on the Rothschild-Stein Lifting Theorem 2.8 and on Lemma 2.10.

Consider an open, bounded C^2 set $\Omega \subset U \subset M^n$, and assume that there exists a (C^2) function $\phi : U \to \mathbb{R}$ such that $\Omega = \{x \in M^n \mid \phi(x) < 0\}$. Let $x_o \in \partial\Omega$. The condition that the type of Ω is less or equal than two means that either x_o is not characteristic or there exist indices $i_0, j_0 = 1, ..., m$ such that

$$[X_{i_o}, X_{j_o}]\phi(x_o) \neq 0 \tag{6.1}$$

Proof of Theorem 1.3. The strategy behind the proof of this result is straightforward. We lift the vector fields, the domain and the metric. Then we use the results from Theorems 5.4, 5.6 and Lemma 2.10 to establish the desired estimates. Let $E \subset U$ be a compact set which contains Ω. We will use E as in Lemma 2.10. At this point we refer to Theorem 2.8, which yields a neighborhood V of the origin in $\mathbb{R}^{\tilde{n}-n}$, and the "lifted" vector fields $\tilde{X}_1, ..., \tilde{X}_m$ in $U \times V$. Denote by $\tilde{\Omega} = \Omega \times V$, and $\tilde{x_o} = (x_o, 0) \in \partial\tilde{\Omega}$. Choose $R_1 = R_1(X, \Omega, x_o) > 0$ small enough so that $\tilde{B}((x_o,0),R) \subset E \times V$. For $0 < R < R_1$, set $\tilde{\Delta} = \tilde{\Delta}((x_o,0),R) = \tilde{B}((x_o,0),R) \cap \tilde{\Omega}$. Let $\tilde{\phi}(x,t) = \phi(x)$, so that $\tilde{\Omega} = \{\tilde{\phi} < 0\}$, and $|\tilde{X}\tilde{\phi}| = |X\phi|$.

Notice that since $\tilde{X}_i\tilde{X}_j\tilde{\phi} = X_iX_j\phi$, if Ω is of type less or equal than two according to Definition 5.1, then so is $\tilde{\Omega}$. Also[1], if Ω and $X_1, ..., X_m$ are real analytic, then so is $\tilde{\Omega}$.

Let us start by proving the upper Ahlfors bound. For this part of the proof we only require $\phi \in C^{1,1}$. Notice that $|\tilde{X}\tilde{\phi}|$ is only a function of x, hence

$$\left(\sup_{P \in \Delta(x_o,R)} |X\phi(P)| \right) = \left(\sup_{(x,s) \in \tilde{\Delta}((x_o,0),R)} |\tilde{X}\tilde{\phi}(x,s)| \right). \tag{6.2}$$

It is convenient to rewrite $\tilde{\Delta}$ and $\tilde{B}((x_o,0),R)$ in the following (completely obvious) way

$$\begin{aligned} \tilde{B}((x_o,0),R) = & \Big(B(x_o,R) \times V \Big) \cap \tilde{B}((x_o,0),R), \\ \tilde{\Delta} = & \Big(\Delta(x_o,R) \times V \Big) \cap \tilde{\Delta} = \Big(\Delta(x_o,R) \times V \Big) \cap \tilde{B}((x_o,0),R). \end{aligned} \tag{6.3}$$

At this point we recall from Theorem 5.4 that there exist $C(\Omega, X) > 0$ and $R_2 = R_2(X, \Omega) > 0$ such that

$$\left(\sup_{(x,s) \in \tilde{\Delta}((x_o,0),R)} |\tilde{X}\tilde{\phi}(x,s)| \right) \sigma(\tilde{\Delta}) \leq C(\Omega, X) \frac{|\tilde{B}((x_o,0),R)|}{R}, \tag{6.4}$$

for any $0 < R < R_2$. Set $R_0 = \min\{R_1, R_2\}$. In view of (6.3), and (2.25) one has for $0 < R < R_0$, and $v \in C_0^\infty(V)$,

$$\begin{aligned} \sigma(B(x_o,R) \cap \partial\Omega) \left| \int_V \chi_{\tilde{B}(x_o,0),R)}(y,s)v(s)ds \right| & \leq C \left| \int_{\Delta(x_o,R)} \int_V \chi_{\tilde{B}(x_o,0),R)}(y,s)v(s)ds\, d\sigma(y) \right| \\ & = C \left| \int_{(\Delta(x_o,R) \times V)} \chi_{\tilde{B}((x_o,0),R)}(y,s)v(s)ds\, d\sigma(y) \right| \\ & = C \left| \int_{(\Delta(x_o,R) \times V) \cap \tilde{B}((x_o,0),R)} v(s)ds\, d\sigma(y) \right| \\ \text{from (6.3)} \quad & \leq C(v,X)\sigma(\tilde{\Delta}). \end{aligned} \tag{6.5}$$

The conclusion now follows from (6.2), (6.4) and (6.5). In fact, for R suitably small one has

[1] Although not explicitly stated in [RS], one can see that if the original system of vector fields is analytic then so are the lifted vector fields. See for instance [Go], page 79.

$$\begin{aligned}
\left(\sup_{P\in\Delta(x_o,R)}|X\phi(P)|\right) \quad & \sigma(\Delta(x_o,R)) \\
&= \left(\sup_{(x,s)\in\tilde\Delta((x_o,0),R)}|\tilde X\tilde\phi(x,s)|\right)\sigma(\Delta(x_o,R)) \\
\text{by (6.5)} \quad & \le C(v,X)\frac{\sigma(\tilde\Delta)}{\left|\int_V \chi_{\tilde B(x_o,0),R)}(y,s)\ v(s)ds\right|}\left(\sup_{(x,s)\in\tilde\Delta((x_o,0),R)}|\tilde X\tilde\phi(x,s)|\right) \\
\text{by (6.4)} \quad & \le C(v,X)\frac{1}{\left|\int_V \chi_{\tilde B(x_o,0),R)}(y,s)\ v(s)ds\right|}\frac{|\tilde B((x_o,0),R)|}{R} \qquad (6.6)\\
\text{(from Lemma 2.10)} \quad & \le C(v,X,\Omega)\frac{|B(x_o,R)|}{|\tilde B(x_o,0),R)|}\frac{|\tilde B((x_o,0),R)|}{R} \\
&= C(v,X,\Omega)\frac{|B(x_o,R)|}{R}.
\end{aligned}$$

We now turn our attention to the lower bounds of Ahlfors type. Let $\tilde B = \tilde B((x_o,0),R)$, and $B = B(x_o,R)$. Since $\tilde B \subset \tilde{\tilde B} \subset U\times V$, then for every $x\in\partial\Omega$ and suitably small $R_0 = R_0(X,\Omega) > 0$, we can find $v\in C_0^\infty(V)$ such that

$$v(s)\chi_{\tilde B}(x,s) = \chi_{\tilde B}(x,s), \tag{6.7}$$

for any $0<R<R_0$ and $s\in V$.

Recalling that $\tilde X\tilde\phi(x,s) = X\phi(x)$, and in view of (6.3), and Lemma 2.10 we have

$$\begin{aligned}
\int_{\tilde\Delta}|\tilde X\tilde\phi(x,s)|d\sigma(s,t) \quad &= \int_{\tilde\Delta}|\tilde X\tilde\phi(x,s)|\ v(s)\chi_{\tilde B}(x,s)d\sigma(s,t) \\
&= \int_{\Delta\times V}|\tilde X\tilde\phi(x,s)|\ v(s)\chi_{\tilde B}(x,s)d\sigma(s,t) \\
&= \int_{\Delta}|X\phi|(x)\int_V v(s)\chi_{\tilde B}(x,s)ds\ d\sigma(x) \\
&\le C(X,\Omega)\int_\Delta |X\phi|(x)d\sigma(x)\frac{|\tilde B|}{|B|}. \qquad (6.8)
\end{aligned}$$

The proof now follows immediately from (6.8) and from Theorem 5.6 □

7. 1-Ahlfors regularity of the X-perimeter and the Dirichlet problem

In this section we bring up an interesting connection between 1-Ahlfors regularity of the X-perimeter $P_X(\Omega;\cdot)$ and the Dirichlet problem for the sub-Laplacian $\mathcal{L} = \sum_{i=1}^m X_i^* X_i$ associated with the system X. We recall that the latter consists in finding, for a given $\phi\in C(\partial\Omega)$, a $\mathcal{L}$-harmonic function u in Ω, i.e., a solution to $\mathcal{L}u = \sum_{i=1}^m X_i^* X_i u = 0$, such that $u=\phi$ on $\partial\Omega$ continuously. For any bounded open set

$\Omega \subset M^n$ there exists a unique (generalized) solution H^Ω_ϕ to the Dirichlet problem, see [CG]. A point $x_o \in \partial\Omega$ is called *regular* if for any $\phi \in C(\partial\Omega)$ one has

$$\lim_{x \to x_o} H^\Omega_\phi(x) \; = \; \phi(x_o) \; .$$

We will prove the following result.

Theorem 7.1. *Let Ω be a bounded domain in a Carnot group $\boldsymbol{G}$. If the perimeter measure $P_X(\Omega;\cdot)$ is 1-Ahlfors regular, then every $g_o \in \Omega$ is regular for the Dirichlet problem.*

The full proof of this result will be accomplished in several steps, and it is ultimately based on an important generalization to sub-Laplacians of the classical criterion of Wiener. We begin by introducing the relevant definitions. A couple (K,Ω), $K \subset \Omega \subset M^n$, with K compact and Ω open, is called a condenser. For a given condenser (K,Ω), we let

$$\mathcal{F}(K,\Omega) \; = \; \{\phi \in C^\infty_o(\Omega) \mid \phi \geq 1 \text{ on } K\} \, .$$

The X-*capacity* of the condenser (K,Ω) is defined as follows, see [CDG2],

$$cap_X \; (K,\Omega) \; = \; \inf_{\phi \in \mathcal{F}(K,\Omega)} \int_\Omega |X\phi(g)|^2 \; dg.$$

When $\Omega = \boldsymbol{G}$, then we simply write $cap_X \; K$, instead of $cap_X \; (K,\boldsymbol{G})$. The following properties of the capacity are simple consequences of its definition, and we list them without proof.

Proposition 7.2. *Let $K \subset \Omega_1 \subset \Omega_2$, then*

$$cap_X \; (K,\Omega_2) \; \leq \; cap_X \; (K,\Omega_1).$$

If, instead, one has $K_1 \subset K_2 \subset \Omega$, then

$$cap_X \; (K_1,\Omega) \; \leq \; cap_X(K_2,\Omega).$$

According to the capacitary estimates in [D], [CDG2], when M^n is a Carnot group $\boldsymbol{G}$ with homogeneous dimension Q, then there exists $C = C(\boldsymbol{G}) > 0$ such that for every $g_o \in \boldsymbol{G}$ and $r > 0$

$$cap_X \; (\overline{B}(g_o,r), B(g_o,2r)) \; = \; C \; r^{Q-2} \; . \tag{7.1}$$

The following basic criterion of Wiener type was proved in [NS], see also [D] for a generalization to quasilinear equations.

Theorem 7.3. *Given a bounded open set $\Omega \subset M^n$, a point $x_o \in \partial\Omega$ is regular if and only if for some small $\delta > 0$*

$$\int_0^\delta \frac{cap_X \; (\Omega^c \cap \overline{B}(g_o,t), B(g_o,2t))}{cap_X \; (\overline{B}(g_o,t), B(g_o,2t))} \; \frac{dt}{t} \; = \; \infty.$$

We will also need the following result.

Let $\mathcal{H}^{Q-1}(\cdot)$ denote the $(Q-1)-$Hausdorff measure in $\boldsymbol{G}$ with respect to the Carnot-Carathéodory metric. The next simple theorem is a particular instance of a potential theoretic result which holds in any metric space with controlled geometry, See Theorem 5.9 in [HK].

Theorem 7.4. *In a Carnot group $\boldsymbol{G}$ consider a compact subset F of a ball $B = B(g_0, R)$. If for some $0 < \lambda \le 1$ we have*

$$\mathcal{H}^{Q-1}(F) \ \ge\ \lambda\ \frac{|B(g_o,R)|}{R}\ , \tag{7.2}$$

then there exists $C = C(Q) \ge 1$ such that for every $u \in C_o^\infty(B(g_o, CR))$ satisfying $u \ge 1$ on F, one has

$$\int_{B(g_o,CR)} |Xu(g)|^2\ dg\ \ge\ \frac{\lambda}{C}\ \frac{|B(g_o,R)|}{R^2}\ . \tag{7.3}$$

Corollary 7.5. *In a Carnot group $\boldsymbol{G}$ consider a bounded open set $\Omega \subset \boldsymbol{G}$. If the Hausdorff measure $\mathcal{H}^{Q-1}$ is lower $1-$Ahlfors regular, then there exists $C = C(\boldsymbol{G}) > 0$, such that for any $g_o \in \partial\Omega$ and any $r > 0$*

$$cap_X\ (\overline{B}(g_o,R) \cap \partial\Omega, B(g_o, 2R))\ \ge\ C\ R^{Q-2}\ .$$

Proof. Let $F = \overline{B}(g_o, R)$ be the compact subset of $B(g_o, 2R)$ in the statement of Theorem 7.4. Since we are assuming the lower $1-$Ahlfors regularity, then hypothesis (7.2) is automatically satisfied. By virtue of Theorem 7.4 and of the definition of capacity we obtain that there exists $C_1 \ge 1$ and $C_2 > 0$ such that

$$cap_X\ (\overline{B}(g_o,R) \cap \partial\Omega, B(g_o, 2C_1R))\ \ge\ C_2\ R^{Q-2}\ .$$

The conclusion now follows from the latter inequality and from Proposition 7.2. □

We next establish, in the special setting of Carnot groups, a simple and general property of the Hausdorff measure in a space of homogeneous type.

Proposition 7.6. *Let μ be a Borel measure in a Carnot group $\boldsymbol{G}$ with homogeneous dimension Q, and suppose that for a bounded open set $\Omega \subset \boldsymbol{G}$ the measure $P_X(\Omega;\cdot)$ is upper 1-Ahlfors regular, i.e., for some $M > 0$ one has for every $g \in \partial\Omega$ and $R > 0$*

$$P_X(\Omega; B(g,R))\ \le\ M\ R^{Q-1}\ . \tag{7.4}$$

There exists a constant $C = C(M) > 0$ such that for every

$$\mathcal{H}^{Q-1}(\partial\Omega \cap B(g,R))\ \ge\ C\ P_X(\Omega; B(g,R))\ . \tag{7.5}$$

Proof. Consider the compact set

$$K\ =\ \partial\Omega\ \cap \overline{B}(g_o,R)\ .$$

For each $\epsilon > 0$ we can find a covering $\{B(x_i,r_i)\}_{i\in N}$ of K such that $0 < r_i < \epsilon$. Using the hypothesis (7.4) we obtain

$$P_X(\Omega; \overline{B}(g,R))\ \le\ \sum_{i=1}^{\infty} P_X(\Omega; B(g_i,r_i))\ \le\ M\ \sum_{i=1}^{\infty} r_i^{Q-1}.$$

Since

$$\mathcal{H}^{Q-1}(\partial\Omega \cap \overline{B}(g,R)) = \liminf_{\epsilon\to 0}\ \{\sum_{i=1}^{\infty} {r_i}^{Q-1} \mid \partial\Omega \cap \overline{B}(g,R) \subset \bigcup_{i=1}^{\infty} B(g_i,r_i), r_i < \epsilon\},$$

we reach the conclusion.

□

This a particular case of Lemma 1.6 in [DS1]. Notice that there is nothing special about the role of the perimeter measure in the last proposition. The same result still holds when $P_X(\Omega;\cdot)$ is substituted by any Borel measure on $\partial\Omega$. We are now ready to give the

Proof of Theorem 7.1. Assume that $P_X(\Omega;\cdot)$ is 1-Ahlfors regular, i.e., we have for some constant $M>0$

$$M^{-1}\ R^{Q-1}\ \leq\ P_X(\Omega;B(g,R))\ \leq\ M\ R^{Q-1}\ , \tag{7.6}$$

for every $g\in\partial\Omega$ and every $R>0$. Using the upper bound in (7.6) in view of Proposition 7.6 we conclude that (7.5) holds. The lower bound in (7.6) yields the lower $1-$Ahlfors regularity of the Hausdorff measure $\mathcal{H}^{Q-1}$. Thanks to Theorem 7.4 and Corollary 7.5, this estimate implies

$$cap_X\ (\partial\Omega\cap\overline{B}(g_o,R),B(g_o,2R))\ \geq\ C''\ R^{Q-2}\ .$$

Due to (7.1) the latter estimate implies

$$\frac{cap_X\ (\partial\Omega\cap\overline{B}(g_o,R),B(g_o,2R))}{cap_X\ (\overline{B}(g_o,R),B(g_o,2R))}\ \geq\ C'''\ >\ 0\ . \tag{7.7}$$

We now apply the second part of Proposition 7.2 with $K_2=\Omega^c\cap\overline{B}(g_o,R)$, $K_1=\partial\Omega\cap\overline{B}(g_o,R)$, $\Omega=\boldsymbol{G}$, obtaining

$$cap_X\ (\partial\Omega\cap\overline{B}(g_o,R),B(g_o,2R))\ \leq\ cap_X(\Omega^c\cap\overline{B}(g_o,R),B(g_o,2R))\ .$$

This estimate, combined with (7.7), and with Theorem 7.3, finally implies the regularity of the point $g\in\partial\Omega$. By the arbitrariness of $g\in\partial\Omega$ we reach the conclusion.

□

Interestingly, using Theorem 7.1 in combination with some examples due to Hansen and Hueber [HH], it is possible to provide a proof of the optimal character of the type assumption in Theorem 1.3, which is alternative to that in section 4. Let us start by recalling Theorem 3.6, [HH].

Theorem 7.7. *Let $\boldsymbol{G}$ be a Carnot group of step $r\in\mathbb{N}$, and denote by m_1 the dimension of the first layer of the stratification V_1. If $r\leq 2$, or if $m_1=2$ and $r\leq 4$, then every bounded domain $\Omega\subset\boldsymbol{G}$ is regular for the Dirichlet problem, provided it satisfies a pointwise exterior ball condition (with respect to the underlying Euclidean metric). In all other cases there exist bounded domains with smooth boundary which are not regular.*

Let us describe more explicitly the smooth domains with irregular boundary points mentioned in the theorem. If $r\geq 3$ and $m_1\geq 3$, or $m_1=2$ and $r\geq 4$, then for every $\gamma>0$ we set $y=\{y_{k,j}\}$ the be the point on the $y_{r,m_r}-$axis at distance γ from the origin, i.e. $y_{j,k}=0$ if $k=1,...,r$, $j\neq m_r$, and $y_{r,m_r}=\gamma$. Consider the Euclidean ball

$$B_E=\left\{x=\{x_{k,j}\}\in\boldsymbol{G}\ \text{ such that }\sum_{k=1}^{r}\sum_{j=1}^{m_j}(x_{k,j}-y_{k,j})^2<\gamma^2\right\}\subset\boldsymbol{G}.$$

In Theorem 3.4 and 3.5, in [HH], it is proved that B_E is thin at the origin, and consequently the origin is an irregular boundary point for B_E^C. This proves that there exist bounded smooth domains with irregular points in Carnot groups of step $r \geq 3$ (for instance consider $B(0, 100\gamma) \setminus B_E$). Consequently, in view of Theorem 7.1, the perimeter measure of such domains cannot be $1-$Ahlfors regular. Note that the origin is always of type r in these examples.

Corollary 7.8. *If $r \geq 3$ and $m_1 \geq 3$, or $m_1 = 2$ and $r \geq 4$, then there exist Carnot groups $\boldsymbol{G}$ of step $r \in \mathbb{N}$, with dim $V_1 = m_1$, and bounded, smooth domains $\Omega \subset \boldsymbol{G}$, whose perimeter measure $P_X(\Omega; \cdot)$ is not $1-$Ahlfors regular.*

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