

---

5-24-2023

## Decomposition of Beatty and Complementary Sequences

Geremías Polanco

*Smith College*, [gpolanco@smith.edu](mailto:gpolanco@smith.edu)

Follow this and additional works at: [https://scholarworks.smith.edu/mth\\_facpubs](https://scholarworks.smith.edu/mth_facpubs)



Part of the [Discrete Mathematics and Combinatorics Commons](#), and the [Number Theory Commons](#)

---

### Recommended Citation

Polanco, Geremías, "Decomposition of Beatty and Complementary Sequences" (2023). Mathematics and Statistics: Faculty Publications, Smith College, Northampton, MA.

[https://scholarworks.smith.edu/mth\\_facpubs/158](https://scholarworks.smith.edu/mth_facpubs/158)

This Article has been accepted for inclusion in Mathematics and Statistics: Faculty Publications by an authorized administrator of Smith ScholarWorks. For more information, please contact [scholarworks@smith.edu](mailto:scholarworks@smith.edu)

# DECOMPOSITION OF BEATTY AND COMPLEMENTARY SEQUENCES

GEREMIAS POLANCO

**ABSTRACT.** In this paper we express the difference of two complementary Beatty sequences, as the sum of two Beatty sequences closely related to them. In the process we introduce a new Algorithm that generalizes the well known Minimum Excluded algorithm and provides a way to generate combinatorially any pair of complementary Beatty sequences.

## 1. INTRODUCTION AND STATEMENT OF PARTS (1) AND (2) OF THE MAIN THEOREM

Wythoff [30] used

$$(1.1) \quad [n\phi^2] - [n\phi] = n,$$

where  $\phi$  is the golden ratio, in the analysis of a generalized game of Nim (here  $[x]$  denotes the largest integer not exceeding  $x$ ). Later (for instance in [14] and [18]) it was shown that

$$[n\alpha^2] - [n\beta] = tn,$$

for  $\alpha = (2 - t + \sqrt{t^2 + 4})/2$ , and  $\beta$  satisfying  $\alpha^{-1} + \beta^{-1} = 1$ . Other generalizations have been done covering the Beatty sequences parametrized by limited families of irrational numbers. We obtain a generalization of identity (1.1) for all complementary sequences. For example,

$$\left[ n \frac{3 + \sqrt{3}}{2} \right] - [n\sqrt{3}] = \left[ n \frac{\sqrt{3} - 1}{2} \right] + [n(2 - \sqrt{3})] + 1.$$

In particular, we prove Theorem 1.2 below, which has the following as a corollary.

**Corollary 1.1.** *Let  $1 < \alpha < \beta$  be irrational numbers with  $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ . Then*

$$[n\beta] - [n\alpha] = \left[ n \left( \frac{\beta}{\alpha} - 1 \right) \right] + \left[ n \left( 1 - \frac{\alpha}{\beta} \right) \right] + 1.$$

We will give a combinatorial proof of Theorem 1.2. Before stating the theorem, we fix the following notation. We say that  $A$  and  $B$  are complementary sets of natural numbers if  $A \cap B = \emptyset$  and  $A \cup B = \mathbb{N}$ . For any multiset  $X$  of integers that is bounded below, we number the elements (with multiplicity) of  $X$  as  $x_1 \leq x_2 \leq \dots$ , linking in this way indexed lower-case letters (the elements in nondecreasing order) and the upper-case letters (the set). We use set subtraction and addition to denote the termwise operations:

$$X \pm Y = \{x_n \pm y_n : n \in \mathbb{N}\}.$$

A set whose terms are given by  $b_n = [n\beta]$  for some  $\beta > 0$  is called a Beatty sequence. We remind the reader of Beatty's Theorem: The sets  $A, B$  given by  $a_n = [n\alpha]$  and  $b_n = [n\beta]$  are complementary sets of natural numbers if and only if  $\alpha$  is irrational and  $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ .

**Theorem 1.2** (Partial Decomposition of Differences of Complementary Sequences Parts (1) and (2)).  
*Suppose that  $A$  and  $B$  are any pair of complementary sequences. Then*

- (1) *The sequence of difference  $A - B$ , term by term, can always be decomposed in terms of the sum of two other sequences  $C$  and  $R$  in an easy to see way as follows:*

$$(1.2) \quad b_n - a_n = c_n + r_n + 1,$$

where  $c_n$  counts the number of integers strictly between  $a_n$  and  $b_n$  that belong to the sequence  $A$ , and  $r_n$  count the number of integers strictly between  $a_n$  and  $b_n$  that are in the sequence  $B$ .

- (2) *If the two complementary sequences  $A$  and  $B$  are also Beatty sequences given by  $\alpha, \beta$  respectively, with  $1 < \alpha < 2$ , then  $C$  and  $R$  in equation (1.2) are Beatty sequences whose  $n$ th terms are given respectively by the slopes  $\gamma = \frac{2-\alpha}{\alpha-1} = \beta/\alpha - 1$  and  $\rho = 2 - \alpha = 1 - \alpha/\beta$ .*

The paper is outlined as follows. In the next section, we provide a short proof of corollary 1.1 that is independent of the combinatorial proof promised above for theorem 1.2, ending the section by highlighting the importance of the subject. For a deeper understanding of Theorem 1.2 that links to our combinatorial interpretation, in Section 3 we state the MEX algorithm, familiar from the theory of impartial games, and generalize it introducing the novel MES algorithm. In Section 4, we prove the full version of the partial decomposition theorem (which includes Theorem 1.2) and explore further implications of the MES algorithm.

## 2. A SHORT INDEPENDENT PROOF OF COROLLARY 1.1

Before providing an easy proof for Corollary 1.1, we want to highlight two important facts. 1) The proof gives no indication as to where the values  $\gamma = \frac{2-\alpha}{\alpha-1}$  and  $\rho = 2 - \alpha$  used in this theorem are coming from. It simply shows these two values work, but the proof does not tell us the intrinsic reasons why they work. 2) This easy proof leaves us with no way to connect the sequence  $c_n$  and  $r_n$  combinatorially with the sequences  $A$  and  $B$  respectively. In contrast the Partial Decomposition Theorem links  $\gamma$  and  $\rho$  back to  $A$  and  $B$ , and gives us the natural combinatorial interpretation of  $c_n$  and  $r_n$ .

*Proof of Corollary 1.1.* We would like to prove that  $k := [\beta n]$  is equal to the sum of the integers  $t := [\alpha n]$ ,  $r := [\frac{2-\alpha}{\alpha-1}n + 1]$  and  $q := [(2 - \alpha)n]$ . Note that from  $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ , we have

$$(2.1) \quad \frac{2-\alpha}{\alpha-1} = \frac{2-\alpha}{\alpha(1-\frac{1}{\alpha})} = \frac{(2-\alpha)\beta}{\alpha} = (2-\alpha)(\beta-1) = 2\beta - 2 - \alpha\beta + \alpha = 2\beta - 2 - \alpha - \beta + \alpha = \beta - 2.$$

Now, we know that

$$(2.2) \quad [\beta n] = k, \quad \text{if and only if} \quad k < \beta n < k + 1,$$

$$(2.3) \quad \left[ \frac{2-\alpha}{\alpha-1}n + 1 \right] = r \quad \text{if and only if} \quad r < \frac{2-\alpha}{\alpha-1}n + 1 < r + 1,$$

$$(2.4) \quad [\alpha n] = t \quad \text{if and only if} \quad t < \alpha n < t + 1, \quad \text{and}$$

$$(2.5) \quad [(2-\alpha)n] = q \quad \text{if and only if} \quad q < (2-\alpha)n < q + 1.$$

It follows from (2.5) that  $2n - q - 1 < \alpha n < 2n - q$  and thus, by (2.4),  $2n - q - 1 = t$ , or

$$(2.6) \quad 2n - 1 = t + q.$$

From (2.1) and (2.3) we find that  $r + 2n - 1 < \beta n < 2n + r$ , hence from (2.2) we deduce that  $2n - 1 + r = k$  and this together with (2.6) gives  $k = t + q + r$ .  $\square$

To prove the results of this paper, we use combinatorial arguments and standard properties of Beatty and Sturmian sequences. For  $\alpha > 1$ , the Sturmian sequence with slope  $1/\alpha$  encodes the Beatty sequence with slope  $\alpha$  [4, Lemma 9.1.3]. There is also a corresponding relationship between non-homogeneous Beatty and Sturmian sequences [10, Lemma 1]. Sturmian sequences in general, and Beatty sequences in particular, are the focus of a growing amount of research, as they play a role in various fields of mathematics [24], biology [16], music [23], computer science [12], and physics (see also [4], [21], [8], [9] and the references therein). The name Sturmian was introduced by Hendlund and Morse in their influential work in the 1940's [20], however, the history of these sequences dates back to 1772 when Bernoulli III worked with what is now known as the inhomogeneous Sturmian sequence (see [4]). In pure mathematics, Sturmian and Beatty sequences have been studied in relation to dynamical systems [20], fix point morphisms [27], logarithm of irrational numbers [25], prime numbers [7], algebraic numbers [15], arithmetic functions [1], primitive roots, divisor functions, character sums, among others (see [2], [3], [6], [17], [24], [26], [28], and the references therein). More recently, the second and third moment of Beatty sequences and its squares have also been studied [22]. We study differences of Beatty sequences in relation to the well known Minimum Excluded Algorithm that is widely used in combinatorial game theory, coloring algorithm, and else where (see for instance [5], [11], [13], [19] and [29]).

### 3. MINIMUM EXCLUDED (MEX) AND MINIMUM EXCLUDED WITH SKIPPING (MES) ALGORITHMS

We present the well known MEX algorithm, as well as definitions and notations necessary to state the more complete version of the main theorem at the end of this section.

**Definition 3.1.** Given a set  $S \subset \mathbb{N}$  the mex of the set  $S$  is defined as  $\text{mex}(S) := \min(S^c \cap \mathbb{N})$ , i.e., the mex( $S$ ) is the minimum natural number that is not in  $S$ .

It is well known that complementary sequences can be generated using the mex rule above (see for instance [15]). We call this process the MEX algorithm, and define it more precisely below. In the rest of the paper, if  $A$  is a sequence we use  $A_n$  to denote the set  $A_n = \{a_k : k \leq n\}$ .

**Definition 3.2** (Minimum Excluded Algorithm MEX). The following set of algorithmic recursions is known as the MEX algorithm, and can be used to generate complementary sequences. The input is a sequence “H”.

- STEP 1: take  $a_1 = 1$ ,
- STEP 2: if  $n \geq 2$ , take  $a_n = \text{mex}\{a_i, b_i \mid i < n\} = \text{mex}(A_{n-1} \cup B_{n-1})$ , and
- STEP 3: if  $n \geq 1$ , take  $b_n = a_n + h_n$ .
- STEP 4: repeat steps 2 and 3.

To our knowledge, the first documented use of the MEX algorithm to generate Beatty sequences combinatorially was by British mathematician Willem A. Wythoff in 1907 [30]. He did so with the purpose of generalizing the combinatorial game Nim. He showed that the winning positions of his game were given by the Beatty pairs  $(a_n, b_n)$ , defined by the golden ratio. In our context, this corresponds to taking  $h_n = n$  in the MEX algorithm above. For the purpose of illustration, let us run the first few rounds of the algorithm with  $h_n = n$ . We apply STEPS 1, 2 and 3 and get  $a_1 = 1$ , and  $b_1 = a_n + n = 1 + 1 = 2$ , so  $A_1 = \{1\}$  and  $B_1 = \{2\}$ . Applying STEP 4, we first obtain  $a_2 = \text{mex}(A_1 \cup B_1) = 3$  and  $b_2 = a_n + n = 3 + 2 = 5$ , so that  $A_2 = \{1, 3\}$  and  $B_2 = \{2, 5\}$ . If we follow this process the first five iterations will give Table 1.

$\text{mex}(A_{n-1} \cup B_{n-1})$	$b_n = a_n + n$	$A_n = \{a_k : k \leq n\}$	$B_n = \{b_k : k \leq n\}$
$a_1 = 1$	$b_1 = 1 + 1 = 2$	$A_1 = \{1\}$	$B_1 = \{2\}$
$a_2 = 3$	$b_2 = 3 + 2 = 5$	$A_2 = \{1, 3\}$	$B_2 = \{2, 5\}$
$a_3 = 4$	$b_3 = 4 + 3 = 7$	$A_3 = \{1, 3, 4\}$	$B_3 = \{2, 5, 7\}$
$a_4 = 6$	$b_4 = 6 + 4 = 10$	$A_4 = \{1, 3, 4, 6\}$	$B_4 = \{2, 5, 7, 10\}$
$a_5 = 8$	$b_5 = 8 + 5 = 13$	$A_5 = \{1, 3, 4, 6, 8\}$	$B_5 = \{2, 5, 7, 10, 13\}$

TABLE 1. first few iterations of the MEX algorithm

Note that the set  $A_n$  agrees with the first  $n$  terms of the sequence  $A$  given by the golden ratio. Similarly,  $B_n$  agrees with the first  $n$  terms of its complementary Beatty sequence.

We now generalize the MEX algorithm by modifying STEP 3, and call this generalization the Minimum Excluded with Skipping (MES) Algorithm. To find  $b_n$ , we choose the minimum excluded element, but after a number of integers are skipped. We base the MES in the following generalization of definition 3.1

**Definition 3.3.** Given a set of integers  $A$  and a nonnegative integer  $k$  we define a function denoted by  $\text{mex}_k(A)$  that consists on selecting the  $(k + 1)$ st minimum excluded positive integer from the set  $A$ . In other words, to find  $\text{mex}_k(A)$  we skip the first  $k$  excluded integers from  $A$  and select the next excluded one.

**Example 1.** For instance if  $A = \{2, 4, 6, 8, \dots\}$ , then  $\text{mex}_3(A) = 7$ , since 1, 3 and 5 are the first 3 excluded integers from  $A$ , and 7 is the next one excluded. Note that with the above definition,  $\text{mex}_0(A) = \text{mex}(A)$ , i.e., if we don't have to skip any excluded integers, we are simply applying the mex rule described above.

**Definition 3.4** (Minimum Excluded with Skipping MES Algorithm). Given a sequence of nonnegative integers  $C = \{c_n\}_{n=1}^{\infty}$ , the MES algorithm is given by the following steps

- STEP 1: set  $a_1 = 1$ ,  $A_1 = \{a_1\} = \{1\}$  and  $B_0 = \emptyset$ .
- STEP 2: for  $n \geq 2$ ,  $a_n = \text{mex}(A_n \cup B_n)$ .

- STEP 3: for  $n \geq 1$ ,  $b_n = \text{mex}_{c_n}(A_n \cup B_{n-1})$ , i.e., skip the first  $c_n$  excluded positive integers and select the next excluded integer from this union.
- STEP 4: iterate steps 2 and 3.

**Example 2.** We illustrate this algorithm using  $C = \{0, 1, 1, 2, 3, 3, 4, 4, 5, 6, 6, 7, 8, 8, 8, 9, 9, 10, \dots\}$ . We start by setting  $B_0 = \emptyset$ . For the first iteration  $a_1 = 1$ , and  $b_1 = \text{mex}_{c_1}(A_1 \cup B_0) = \text{mex}_0\{1\} = 2$ . So,  $A_1 = \{1\}$  and  $B_1 = \{2\}$ . The second iteration gives  $a_2 = \text{mex}(A_1 \cup B_1) = 3$ , and  $b_2 = \text{mex}_{c_2}(A_2 \cup B_1) = \text{mex}_1\{1, 2, 3\} = 5$ , since 5 is the second minimum excluded integer, i.e., we skip the first excluded integer, 4, and select the next, 5. Next iteration gives  $a_3 = \text{mex}(A_2 \cup B_2) = 4$ , and  $b_3 = \text{mex}_{c_3}(A_3 \cup B_2) = \text{mex}_1\{1, 2, 3, 4, 5\} = 7$ , since we skip the first minimum excluded integer 6, and select the next excluded integer 7. If we continue in this process, we get the rows in Table 2.

$$C = \{0, 1, 1, 2, 3, 3, 4, 4, 5, 6, 6, 7, 8, 8, 8, 9, 9, 10, \dots\}$$

$$a_n = \text{mex}(A_{n-1} \cup B_{n-1})$$

$$b_n = \text{mex}_{c_n}(A_n \cup B_{n-1})$$

$a_n$	$b_n$	$A_n = \{a_k : k \leq n\}$	$B_n = \{b_k : k \leq n\}$
$a_1 = 1$	$b_1 = \text{mex}_0 = 2$	$A_1 = \{1\}$	$B_1 = \{2\}$
$a_2 = 3$	$b_2 = \text{mex}_1 = 5$	$A_2 = \{1, 3\}$	$B_2 = \{2, 5\}$
$a_3 = 4$	$b_3 = \text{mex}_1 = 7$	$A_3 = \{1, 3, 4\}$	$B_3 = \{2, 5, 7\}$
$a_4 = 6$	$b_4 = \text{mex}_2 = 10$	$A_4 = \{1, 3, 4, 6\}$	$B_4 = \{2, 5, 7, 10\}$
$a_5 = 8$	$b_5 = \text{mex}_3 = 13$	$A_5 = \{1, 3, 4, 6, 8\}$	$B_5 = \{2, 5, 7, 10, 13\}$
$a_6 = 9$	$b_6 = \text{mex}_3 = 15$	$A_6 = \{1, 3, 4, 6, 8, 9\}$	$B_6 = \{2, 5, 7, 10, 13, 15\}$
$a_7 = 11$	$b_7 = \text{mex}_4 = 18$	$A_7 = \{1, 3, 4, 6, 8, 9, 11\}$	$B_7 = \{2, 5, 7, 10, 13, 15, 18\}$
$a_8 = 12$	$b_8 = \text{mex}_4 = 20$	$A_8 = \{1, 3, 4, 6, 8, 9, 11, 12\}$	$B_8 = \{2, 5, 7, 10, 13, 15, 18, 20\}$

TABLE 2. first few iterations of the MEX algorithm

**Remark 1.** Because of how it is used in STEP 3 of the MES definition above, we call  $C$  the skipping sequence. After a close look at the resulting Table 5, one can see that the sequence  $C$  used in the MES above can be defined inductively by the rule: a positive integer that has been assigned by the algorithm to  $A$  appears twice in  $C$ , otherwise it appears once. In other words, the algorithm can be generated without explicitly listing the sequence  $C$ , only using a) the initial value  $c_0 = 0$ , and b) the inductive rule just stated. Tables 3, 4 and 5 illustrate how  $C$  is generated inductively in this way. Later we expand on this showing that the resulting sequence from the MES algorithm gives the complementary pair given by the golden ratio. This is Theorem 4.1 (4a).

**Definition 3.5.** Define the sortjoin of two integer sequences  $A$  and  $B$ , denoted by  $A \star B$ , as the union with repetition of the ordered elements of these two sequences.

TABLE 5

$n^{th}$	A	B	C
1)	1	2	0
2)	3	5	1
3)	4	7	1
4)	6	10	2
5)	8	13	3
6)	9	15	3
7)	11	18	4
8)	12	20	4
9)	14	23	5
10)	16	26	6
11)	17	28	6
12)	19	31	7

TABLE 4

$n^{th}$	A	B	C
1)	1	2	0
2)	3	5	1
3)	4	7	1
4)			2
5)			3
6)			3
7)			4
8)			4
9)			5

TABLE 3

$n^{th}$	A	B	C
1)	1	2	0
2)			1
3)			1
4)			2

We use  $A^2$  for the sortjoin of  $A$  with itself i.e.,  $A^2 = A \star A$ , and we represent the sortjoin of  $k$  copies of  $A$  with itself by  $A^k$ . Finally, we use  $\mathbb{Z}_{\{0\}}$  to denote the nonnegative integers. For instance if  $A$  represents the even natural numbers, then  $A \star \mathbb{Z}_{\{0\}}^2 = \{0, 0, 1, 1, 2, 2, 2, 3, 3, 4, 4, 4, 5, 5, \dots\}$ .

We need two more definitions before proving the main theorem. Part (4) of that theorem (below) mentions the case when a sequence  $C$  is given by the sortjoin  $C = D \star \mathbb{Z}_0^{k-1}$  in the MES algorithm. In this case, the sequence  $C$  is completely determined by the sequence  $D$ . Since  $C$  defines the MES algorithm we effectively have that  $D$  defines the MES algorithm. We record this in the following definition.

**Definition 3.6.** If there is an increasing sequence of natural numbers  $D$  such that the sequence  $C$  of the MES algorithm is given by  $C = D \star \mathbb{Z}_0^{k-1}$ , we call  $D$  the defining sequence of the MES algorithm.

We also need the following:

**Definition 3.7.** For  $n \geq 1$ , consider the (possibly empty) half-open integer interval  $(a_n, b_{n-1}]$ , and let  $r_n$  be the cardinality of the intersection of this interval with  $B_{n-1}$ , i.e.,  $r_n = \#\{k \mid k \in (a_n, b_{n-1}] \text{ and } k \in B_{n-1}\}$ . We define the auxiliary sequence  $R$  by  $R := \{r_n\}_{n=1}^\infty$ . Also we define  $R_n := \{r_k : k \leq n\}$ .

#### 4. PARTIAL DECOMPOSITION OF BEATTY SEQUENCES AND THE MES ALGORITHM: PROOF OF THE MAIN THEOREM

We now state the main theorem, and include parts (1) and (2) stated in Section 1 for completion.

**Theorem 4.1** (Partial Decomposition of Differences of Complementary Sequences, Full Version). *Suppose that  $A$  and  $B$  are any pair of complementary sequences. Then*

- (1) The sequence of difference  $A - B$ , term by term, can always be decomposed in terms of the sum of two other sequences  $C$  and  $R$  as follows:

$$(4.1) \quad b_n - a_n = c_n + r_n + 1,$$

where  $c_n$  counts the number of integers strictly between  $a_n$  and  $b_n$  that belong to the sequence  $A$ , and  $r_n$  count the number of integers strictly between  $a_n$  and  $b_n$  that are in the sequence  $B$ .

- (2) If the two complementary sequences  $A$  and  $B$  are also Beatty sequences given by  $\alpha$  and  $\beta$  respectively, with  $1 < \alpha < 2$ , then  $C$  and  $R$  in equation (4.1) are Beatty sequences whose  $n$ th terms are given respectively by the slopes  $\gamma = \frac{2-\alpha}{\alpha-1}$  and  $\rho = 2 - \alpha$ .
- (3) There is a combinatorial interpretation of formula (4.1), that is, an algorithm which we name the *Minimum Excluded with Skipping Algorithm (MES)*, that takes as input a sequence  $C$  and it outputs three sequences  $A, B$  and  $R$  such that if  $a_n, b_n, c_n$  and  $r_n$  are the  $n$ th term of  $A, B, C$  and  $R$  respectively, then equation (4.1) holds, i.e.,

$$b_n - a_n = c_n + r_n + 1.$$

Furthermore, the *MES* characterizes complementary sequences. Specifically,  $A$  and  $B$  are complementary sequences, if and only if, there is a nonnegative sequence  $C$  that generate  $A$  and  $B$  through the *MES* algorithm.

- (4) The *MES* generalizes the *MEX* algorithm as follows: suppose there is an increasing sequence of positive integers  $D$  such that  $C = D \star \mathbb{Z}_0^{k-1}$  for some  $k \in \mathbb{N}$ . Then we have the following
- (a) If  $k = 2$  and  $D$  is the Beatty sequence defined by the golden ratio, then the *MES* and the *MEX* algorithm coincide, i.e., they both produce  $A$ . In other words, the *MES* produces the sequence  $A$  if and only if the skipping sequence  $C$  can be defined as follows: any nonnegative integer  $c$  appears once or twice in  $C$ , and  $c$  appears twice in  $C$  if and only if  $c \in A$ .
- (b) If  $D$  is the sequence given by  $\alpha$  where

$$\alpha = \frac{k-1 + \sqrt{k^2+1}}{k},$$

then the *MES* algorithm produces the Beatty sequence  $D$  and its complement. In other words, the *MES* produces the sequence  $D$  if and only if the skipping sequence  $C$  can be defined as follows: any nonnegative integer  $c$  appears  $k$  or  $k-1$  times in  $C$ , and  $c$  appears  $k$  times in  $C$  if and only if  $c \in D$ . This coincides with the *MEX* and the golden ratio when  $k = 2$ .

In this section, we are going to prove the Partial Decomposition Theorem. Parts (1) and (3) are the simplest to prove, and we do so immediately. We complete the proof of Part (2) in Theorems 4.5 and 4.6. Finally, we prove Part (4) of the main theorem in Corollary 4.10.

*Proof of Parts (1) and (3) of Theorem 4.1.* Part 1 of Theorem 4.1 is evident, by virtue of the fact that  $A$  and  $B$  are complementary, and thus, the integers in the interval  $(a_n, b_n)$  will belong either to  $A$  or  $B$ , hence the stated formula in part 1 must follow. Part 3 of the theorem follows by construction



of the MES algorithm. On the one hand, given a nonnegative sequence  $C$  the MES can be implemented, skipping  $c_n$  excluded integers in the  $n$ th step. By construction it will produce complementary sequences. Conversely, if  $C$  is negative, the skipping cannot be performed in the natural sense that is done in the algorithm. On the other hand, if a pair of complementary sequences is given, one can define  $r_n$  as in Definition 3.7. Then  $c_n$  can be defined as the following complement:  $[(a_n, b_n) \cap B_n]^c$ . With this set up, we can implement the algorithm with  $C = \{c_n\}_{n=1}^\infty$ . From which it will clearly follow that if  $R = \{r_n\}_{n=1}^\infty$ , then  $b_n - a_n = c_n + r_n + 1$ .  $\square$

We spend the rest of this section proving parts 2) and 4) of Theorem (4.1). We do so through a sequence of lemmas regarding Sturmian and Beatty sequences. Given a Beatty sequence with slope  $0 < \alpha < 1$ , each positive integer will occur in the sequence  $k$  times or  $k + 1$  times, for some number  $k \geq 1$ . This is simply a consequence of the size of  $\alpha$ . We have the following:

**Lemma 4.1.** *If  $\alpha$  belongs to the interval  $\frac{1}{k+1} < \alpha < \frac{1}{k}$  for  $k$  a positive integer, then all nonnegative integers appear in the Beatty sequence  $C$  with slope  $\alpha$ , each of them repeating  $k$  or  $k + 1$  times. Both occur infinitely often. In the language of sortjoin, there exists, in this case, a sequence  $B$  of nonnegative integers such that  $C = B \star \mathbb{Z}_0^{k-1}$ .*

*Proof.* It's not hard to see that the inequality  $\frac{1}{k+1} < \alpha < \frac{1}{k}$  implies each integer repeats  $k$  or  $k + 1$  times in  $\{[n\alpha]\}_{n=1}^\infty$ . Indeed,  $k\alpha < 1$  implies each integer  $k$  occurs at least  $k$  times, and an integer can not occur  $k + 2$  times nor more, for if  $[n\alpha] = [(n + 1)\alpha] = \dots = [(n + k + 1)\alpha] = k$ , then  $0 = [(n + k + 1)\alpha] - [n\alpha] \geq [n\alpha] + [(k + 1)\alpha] - [n\alpha] = [(k + 1)\alpha] > 1$ . Now some integers  $n$ , will indeed need to repeat  $k + 1$  times, for if all large integers repeat  $k$  times, then the sequence would be ultimately periodic and then the following equality would need to hold:

$$\lim_{n \rightarrow \infty} \frac{[n\alpha]}{n} = \frac{1}{k},$$

i.e.,  $\alpha = 1/k$ , a contradiction to the irrationality of  $\alpha$ . For the same reasons it cannot repeat  $k + 1$  times. Hence both  $k$  and  $k + 1$  occur infinitely often.  $\square$

**Definition 4.1.** For a real number  $\alpha$ , such that  $0 < \alpha < 1$ , we define the characteristic function of  $\alpha$  as

$$(4.2) \quad f_\alpha(n) = [\alpha(n + 1)] - [n\alpha].$$

Clearly  $f_\alpha(n) = 1$  or  $f_\alpha(n) = 0$ . Since the sum telescopes, we have

$$(4.3) \quad \sum_{n=1}^m f_\alpha(n) = [\alpha(m + 1)].$$

**Remark 2.** Notice that (4.3) gives the number of integers  $n \leq m$  for which  $f_\alpha(n) = 1$ .

**Definition 4.2.** For  $1 < \beta \in \mathbf{R} \setminus \mathbf{Q}$ , define

$$(4.4) \quad g'_\beta(n) = \begin{cases} 1 & \text{if } n = [k\beta], \text{ for } k \in \mathbf{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

It is a well known fact (see [25] and [4, Lemma 9.1.3]) that for all integers  $n$

$$(4.5) \quad g'_\beta(n) = f_{1/\beta}(n).$$

**Lemma 4.2.** *For each irrational number  $\alpha$  such that  $\frac{1}{k+1} < \alpha < \frac{1}{k}$ , there exists a number  $\beta := \frac{\alpha}{1-\alpha}$ , such that  $[n\alpha] = t$  for  $k+1$  different numbers  $n = m, m+1, \dots, m+k$ , if and only if  $[n\beta] = t$  for  $k$  numbers  $n = q, q+1, \dots, q+k-1$ . In other words, if  $C = [n\alpha]_{n=1}^\infty$  and  $B = \left[ \frac{\alpha}{1-\alpha} \cdot n \right]_{n=1}^\infty$  then  $C = D \star \mathbb{Z}_0^k$  and  $B = D \star \mathbb{Z}_0^{k-1}$  for some increasing sequence  $D$  of nonnegative integers.*

*Proof.* Notice that  $\frac{1}{k+1} < \alpha < \frac{1}{k}$  if and only if  $k < \frac{1}{\alpha} < k+1$ , if and only if  $k-1 < \frac{1}{\alpha} - 1 < k$  and  $\frac{1}{\alpha} - 1 = \frac{1-\alpha}{\alpha} = \frac{1}{\beta}$ . Thus we claim that it is enough to prove that the sequence  $[n\alpha] = t$  for  $k+1$  values of  $n$ , if and only if  $[(t+1)\frac{1}{\alpha}] - [t\frac{1}{\alpha}] = k+1$ . Indeed, if this is true, since  $[(t+1)\frac{1}{\beta}] - [t\frac{1}{\beta}] = [(t+1)(\frac{1}{\alpha} - 1)] - [t(\frac{1}{\alpha} - 1)] = [(t+1)\frac{1}{\alpha}] - [t\frac{1}{\alpha}] - 1 = k$ , it would follow from this claim that  $[n\beta]$  repeats  $k$  times. So we just need to prove the double implication in the claim. For that purpose, from (4.2), we write  $f(n) := f_\alpha(n) = [(n+1)\alpha] - [n\alpha]$ . And from (4.5), it follows that  $f(n) = 1$  if and only if there exists a  $t$  such that  $n = [t\frac{1}{\alpha}]$ . Thus, from this last sentence we find that  $[\alpha(n+m)] = r$  for the first  $k+1$  values of  $m$  exactly, if and only if the following three things happen: First,  $f(n-1) = 0$ ; second,  $f(n) = f(n+k+1) = 1$  and third,  $f(n+m) = 0$  for all numbers in between  $n$  and  $n+k+1$ , i.e., for all  $m$  such that  $1 \leq m < k+1$ . For such an  $n$ , it follows from (4.5) that  $f(n+m) = 0$  for  $1 \leq m < k+1$  if and only if there is no  $j$  such that  $n+m = [j\frac{1}{\alpha}]$ , for  $1 \leq m \leq k$ . Thus we see that  $[(t+1)\frac{1}{\alpha}] - [t\frac{1}{\alpha}] > k$ . Since the difference between two consecutive numbers in the Beatty sequence  $[t\frac{1}{\alpha}]$  is either  $k$  or  $k+1$ , we see that  $[(t+1)\frac{1}{\alpha}] - [t\frac{1}{\alpha}] = k+1$ . So, we have proved that each time the sequence  $[n\alpha]$  repeats  $k+1$  times, then the difference  $[(t+1)\frac{1}{\alpha}] - [t\frac{1}{\alpha}] = k+1$ , and thus, the Theorem holds.  $\square$

Combining Lemma 4.1 and Lemma 4.2, we see that if a Beatty sequence is generated by an  $\alpha$  such that each nonnegative integer repeats  $k$  or  $k+1$  times, then in the Beatty sequence generated by  $\frac{\alpha}{1-\alpha} = -1 + \frac{1}{1-\alpha}$  integers repeat  $k-1$  or  $k$  times. If we iterate this process we have the following Theorem.

**Lemma 4.3.** *For each number  $\alpha$  with  $\frac{1}{k+1} < \alpha < \frac{1}{k}$  for some integer  $k$ , there exists a number  $\beta$  with*

$1 < \beta < 2$  given by  $\beta - 1 + \frac{1}{2 - \frac{1}{2 - \frac{1}{2 - \dots - \frac{1}{2 - \frac{1}{1-\alpha}}}}}}$ , and  $k-2$  two's in this expansion, such that

$$\beta - 1 + \frac{1}{2 - \frac{1}{2 - \frac{1}{2 - \dots - \frac{1}{2 - \frac{1}{1-\alpha}}}}}}$$

$[n\alpha] = k$  for  $k+1$  different numbers if and only if  $k$  is in the Beatty sequence given by  $[n\beta]$ .

*Proof.* In Lemma 4.2, an irrational  $\alpha$  giving a maximum of  $k+1$  repetitions, implies that the number

$$(4.6) \quad \alpha_1 : -1 + \frac{1}{1-\alpha}$$

produces a maximum of  $k$  repetitions. We iterate this process a second time and find that

$$(4.7) \quad \alpha_2 := -1 + \frac{1}{1 - \alpha_1} = -1 + \frac{1}{1 - \left(-1 + \frac{1}{1 - \alpha_1}\right)} = -1 + \frac{1}{2 - \frac{1}{1 - \alpha}}$$

produces a maximum of  $k - 1$  repetitions. If we iterate this process a total of  $k$  times, we will get a sequence with no repetition, i.e., a Beatty sequence  $[n\beta]$ , for some  $\beta > 1$ . Note that in each iteration after the second one, a new digit 2 will continue to appear as in the right hand side of (4.7). Thus the slope  $\beta$  of the Beatty sequence is necessarily of the form

$$-1 + \frac{1}{2 - \frac{1}{2 - \frac{1}{2 - \frac{\dots}{2 - \frac{1}{1 - \alpha}}}}}}$$

The number of times the digit 2 appears in this expression is  $k - 2$ , for the following reasons. We start the process with the number  $\alpha$  giving  $k$  and  $k + 1$  repetition. Next step gives  $\alpha_1$  which does not have the digit 2 in it by (4.6). The third number in this process,  $\alpha_2$ , is the first one having the digit 2, and thus this digit will appear  $k - 2$  times in  $\beta$ .  $\square$

**Corollary 4.2.** *Consider a Beatty sequence with slope  $\alpha < 1$ , and let  $k$  be the integer such that  $\frac{1}{k+1} < \alpha < \frac{1}{k}$ . Then the sequence of numbers that repeat  $k$  times and the sequence of numbers that repeat  $k + 1$  times form a partition of the integers given by two complementary Beatty sequences.*

*Proof.* Since  $0 < \alpha < 1$ , each integer  $n$  is in the Beatty sequence with slope  $\alpha$ , and it is repeated  $k$  or  $k + 1$  times. By Lemma 4.3 those numbers that repeat  $k + 1$  times form a Beatty sequence with slope  $\beta$ , hence those that only repeat  $k$  times are the complementary Beatty sequence with slope  $\frac{1}{1 - \frac{1}{\beta}}$ .  $\square$

We reinterpret the previous result in terms of sortjoin of sequences, to get

**Corollary 4.3.** *Let  $\alpha \in [0, 1]$  be an irrational number, and let  $k$  be the integer such that  $\frac{1}{k+1} < \alpha < \frac{1}{k}$ . Then the sequence  $C = \{[\alpha n]\}_{n=1}^\infty$  is given by the  $k$  fold sortjoin of  $\mathbb{Z}_0$  with a beatty sequence  $D$ . Specifically,  $C = D \star \mathbb{Z}_0^k$ .*

**Lemma 4.4.** *Let  $A$  and  $B$  be the complementary Beatty sequences generated by  $\alpha$  and  $\beta$  respectively, with  $(1 < \alpha < 2)$ , and let  $a_n$  and  $b_n$  be the  $n^{\text{th}}$  elements of  $A$  and  $B$ , respectively. Define  $r_n$  as in Definition (3.7). Then the difference,  $r_{n+1} - r_n$  equals 0 or 1, and  $r_{n+1} - r_n = 1$  if and only if  $a_{n+1} - a_n = 1$ , hence the sequence  $r_n - r_{n-1}$  is Sturmian.*

*Proof.* To prove the first part of the theorem, notice that by Definition 3.7, we are given that, for an integer  $n$ ,  $R_n$  has  $r_n$  elements, all of which belong to  $B$ . Let us list all the elements of  $R_n$ , in descending order:

$$(4.8) \quad R_n = \{b_{n-1}, b_{n-2}, \dots, b_{n-r_n}\}.$$

Similarly,

$$(4.9) \quad R_{n+1} = \{b_n, b_{n-1}, \dots, b_{n-r_{n+1}}\}.$$

We can see from these two sets that  $R_{n+1}$  always contains an element that is not in  $R_n$ , namely  $b_n$ . Also note that the smaller elements of  $R_n$  and  $R_{n+1}$ , namely  $b_{n-r_n}$  and  $b_{n-r_{n+1}}$  respectively, could be the same element. Indeed, one can see that if  $a_n$  and  $a_{n+1}$  are consecutive elements, then from Definition 3.7 we see that  $b_{n-r_n} = b_{n-r_{n+1}}$ . On the other hand, if  $a_n$  and  $a_{n+1}$  are not consecutive, then since  $A$  and  $B$  are complementary Beatty sequences, there have to be some element of  $B$  between  $a_n$  and  $a_{n+1}$ . In fact there can only be one element, since  $\frac{1}{\alpha} + \frac{1}{\beta} = 1$  and  $1 < \alpha < \beta$  implies  $\beta > 2$  and thus  $b_{n+1} - b_n \geq 2$ . Hence, we see that  $r_{n+1} - r_n = 1$  if and only if  $a_{n+1} - a_n = 1$ , and if the difference  $a_{n+1} - a_n \neq 1$ , then  $r_{n+1} - r_n = 0$ . Finally, the above argument together with the fact that  $a_{n+1} - a_n = 1$  or  $2$ , implies that  $r_{n+1} - r_n = a_{n+1} - a_n - 1$ . Since the sequence given by the right hand side of this equality is Sturmian, it follows that the sequence given by the left hand side is also Sturmian. □

In a similar manner we get the following lemma.

**Lemma 4.5.** *Let  $A$  and  $B$  be complementary Beatty sequences and suppose that the slope  $\beta > 2$  and let  $q = [\beta]$ . Suppose  $a_n$  and  $b_n$  are the  $n$ th term of  $A$  and  $B$  respectively, and let  $c_n = \{a \in A : a_n < a < b_n\}$ . Then, the difference  $c_{n+1} - c_n$  equals  $q - 2$  or  $q - 1$ , and  $c_{n+1} - c_n = q - 2$  if and only if  $b_{n+1} - b_n = q$ .*

The following is a corollary to Lemma 4.4.

**Corollary 4.4.** *If  $A$  and  $B$  are complementary Beatty sequences, then the sequence  $R = \{r_n\}_{n=1}^{\infty}$  is Beatty with slope  $r$  such that  $0 < r < 1$ .*

*Proof.* The difference between two elements of  $R$  is either 0 or 1. Thus if  $R$  is Beatty, the slope should be a number  $r$  with  $0 < r < 1$ . Since  $A$  is a Beatty sequence, Lemma 4.4 together with the fact that  $r_{n+1} - r_n$  can take only two possible values, implies that  $R$  is also Beatty. Hence the corollary holds. □

We now get the following proposition that is directly related to part 2 of the main theorem. Indeed, Theorem 4.5 and 4.6 complete the proof of the second part of the main theorem.

**Theorem 4.5.** *If  $A$  and  $B$  are complementary Beatty sequences, then the sequence  $R = \{r_n\}_{n=1}^{\infty}$  is given by the slope  $r := 2 - \alpha$ , where  $\alpha$  is the slope of the sequence  $A$ .*

*Proof.* Since  $0 < 2 - \alpha < 1$ , in light of Theorem 4.4 and its corollary, it is enough to prove that  $[(2 - \alpha)(n + 1)] - [(2 - \alpha)n] = 1$  if and only if  $a_{n-1} - a_n = 1$ . For this purpose, note that  $[(2 - \alpha)(n + 1)] - [(2 - \alpha)n] = 2(n + 1) - 2n + [-\alpha(n + 1)] - [-\alpha n] = 2 + [-\alpha(n + 1)] - [-\alpha n]$ . It is clear that, for  $x$  non integral,  $[-x] = -1 - [x]$ , thus we see that  $2 + [-\alpha(n + 1)] - [-\alpha n] = 2 - (a_{n+1} - a_n)$ . This is equal to 1 if and only if  $a_{n+1} - a_n = 1$ . □

In a similar manner we get the following theorem, using the definition of  $C$  given in Lemma 4.4

**Theorem 4.6.** *If  $A$  and  $B$  are complementary Beatty sequences, then, the sequence  $C = \{c_n\}_{n=1}^\infty$  is given by the slope  $c := \frac{2-\alpha}{\alpha-1}$ , where  $\alpha$  is the slope of the sequence  $A$ .*

We get the following theorem, that we find it to be a striking formula involving homogeneous Beatty sequences. It also has a very natural and straightforward interpretation that we highlight in the remark that follows the theorem. This interpretation is highlighted in part 2) of the main theorem.

**Theorem 4.7.** *Let  $\alpha$  and  $\beta$  be two irrational numbers generating two complementary Beatty sequences,  $A$  and  $B$ , respectively, and let  $1 < \alpha < 2$ . Set  $a_n = [\alpha n]$  and  $b_n = [\beta n]$ . Then for any integer  $n$  we have*

$$(4.10) \quad [\beta^{-1}(b_{n-1} + 1)] - [\beta^{-1}(a_n)] = [(2 - \alpha)n].$$

Similarly, we have

$$(4.11) \quad [\alpha^{-1}b_n] - [\alpha^{-1}(a_n + 1)] = \left\lfloor \frac{2 - \alpha}{\alpha - 1} n \right\rfloor.$$

*Proof.* We will prove identity (4.10), since (4.11) follows by a similar argument using Theorem 4.6. To prove equation (4.10) in light of Theorem 4.5, we just need to prove that  $r_n = [\frac{1}{\beta}(b_{n-1} + 1)] - [\frac{1}{\beta}(a_n)]$ . For that purpose, recall that  $r_n$  can be written as:

$$(4.12) \quad r_n = \#\{b \in B : a_n < b \leq b_{n-1}\} = \sum_{\substack{b \in B \\ a_n < b \leq b_{n-1}}} 1.$$

Now define  $f(k) := f_{\frac{1}{\beta}}(k)$  as in (4.2). By (4.5),  $f(a_n) = 0$ , and thus by the same equation we find that (4.12) becomes:

$$(4.13) \quad \sum_{a_n \leq k \leq b_{n-1}} f(k) = \sum_{1 \leq k \leq b_{n-1}} f(k) - \sum_{1 \leq k < a_n} f(k) = [\beta^{-1}(b_{n-1} + 1)] - [\beta^{-1}(a_n)].$$

This last equality follows from (4.3). □

**Remark 3.** We will interpret equation (4.10), and a similar interpretation follows for equation (4.11). Note that in Corollary 4.7 the Sturmian sequence with slope  $1/\beta$  is the indicator function of the Beatty sequence with slope  $\beta$ . Hence, in light of Lemmas 4.3 and 4.5, the left hand side of equation 4.10, gives the number of integers strictly between  $a_n$  and  $b_n$  which are in  $B$ . As a passing comment, we mention that the right hand side of (4.10) gives us another way to write this quantity. Since these  $r_n$  integers are in  $B$ , if we want to generate a complementary pair by the MES we simply need to find a sequence  $c'_n$  that consistently gives the number of integers between  $a_n$  and  $b_n$  that are in  $A$ . We

would then take these numbers  $c'_n$  as the skipping sequence in the algorithm. This is what we show in the following Theorem. As mentioned before, the last part of Lemma 4.6 finishes the proof of the Partial Decomposition Theorem.

**Lemma 4.6.** *Let  $\alpha$  be an irrational number such that  $1 < \alpha < 2$ , and let  $k \geq 1$  be the unique positive integer for which  $\frac{2k-1}{k} < \alpha < \frac{2k+1}{k+1}$ . The skipping sequence  $C$  that generates the MES algorithm with output the Beatty sequence given by  $\alpha$ , can be generated by a Beatty sequence  $D$  with slope  $\delta$  given as follows:*

- (1) *if  $k > 1$ , (i.e., if  $\frac{3}{2} < \alpha < 2$ ), then the slope  $\delta$  is given by:*

$$\delta := -1 + \frac{1}{2 - \frac{1}{2 - \frac{1}{2 - \frac{\dots}{2 - \frac{1}{\dots - \frac{1}{2 - \frac{1}{\alpha - 1}}}}}}}}$$

*with  $k - 1$  two's in this expansion, and the frequency for the skipping sequence is given as follows: Each number  $n$  will appear  $k$  times in  $C$  if  $n \in D$ , and  $k - 1$  times otherwise.*

- (2) *If  $k = 1$  (the case  $1 < \alpha < \frac{3}{2}$ ), then  $\delta = \frac{2-\alpha}{\alpha-1}$ , and the frequency is given as follows: each number  $n$  will appear once in the skipping sequence if  $n \in D$  and will not appear in the skipping sequence if  $n \notin D$ .*
- (3) *Furthermore,  $C$  is a Beatty sequence and is given by the slope  $c = \frac{2-\alpha}{\alpha-1}$ .*

**Remark 4.** Note here that if  $\alpha = \phi$ , where  $\phi$  is the golden ratio, then  $c = \frac{2-\phi}{\phi-1} = (2-\phi)\phi = 2\phi - \phi^2 = \phi - 1 = \frac{1}{\phi}$ . Also, observe here that the number  $\delta$  can be written as

$$\delta = \frac{2 - \alpha}{k\alpha - 2k + 1}.$$

This reduces to

$$\delta = \frac{2 - \alpha}{\alpha - 1}$$

when  $k = 1$ .

*Proof.* We already proved in Theorem 4.6 that  $C$  is generated by  $c := \frac{2-\alpha}{\alpha-1}$ . Since  $1 < \alpha < 2$ , there exists a unique positive integer  $k$  such that

$$(4.14) \quad \frac{2k-1}{k} < \alpha < \frac{2k+1}{k+1}.$$

Case I:  $k > 1$ . We will apply Theorem 4.3 with the  $\alpha$  of that Theorem equal to  $c = \frac{2-\alpha}{\alpha-1} = -1 + \frac{1}{\alpha-1}$  (i.e., the slope of the sequence  $C$ ). We first notice that the last portion of the continued fraction of  $\beta$

in Theorem 4.3 is given by

$$(4.15) \quad 2 - \frac{1}{1-c} = 2 - \frac{1}{2 - \frac{1}{\alpha-1}}.$$

Thus, we gain an extra 2 here, compared to Lemma 4.3, and hence, when applying that lemma we will need to adjust for that extra 2. With that in mind, we see that (4.14) implies that  $\frac{1}{k} < \frac{2-\alpha}{\alpha-1} < \frac{1}{k-1}$ . Since, there are  $k-1$  two's in the expansion of  $\delta$ , it is equivalent to having  $k-2$  two's in Lemma 4.3, because we need to adjust for the extra 2, shown in (4.15). Thus each positive integer will appear in  $C$   $k$  or  $k-1$  times. Also, Lemma 4.3 tells us that those numbers that will appear  $k$  times are, precisely, the integers belonging to the Beatty sequence  $D$  with the slope that we here called  $\delta$ .

Case II:  $k = 1$ , i.e.,  $1 < \alpha < \frac{3}{2}$ . We see that  $c = \frac{2-\alpha}{\alpha-1} > 1$ . In this case, each positive integer either will appear once in  $C$  or will not appear at all. Thus  $D$  and  $C$  turn out to be the same sequence, and thus  $C$  can be described as follows. Each integer  $n$  will appear once in  $C$  if  $n \in D$ , and will not appear in  $C$  otherwise.  $\square$

If we compare parts (1) and (3) in Lemma 4.6 with Corollary 4.3 we get:

**Corollary 4.8.** *To generate a Beatty sequence with the MES algorithm as in Lemma 4.6 the skipping sequence is given by the sortjoin of  $D$  and  $k$  copies of  $\mathbb{Z}_0$ . Specifically,  $C = D \star \mathbb{Z}_0^k$ , and the slope of  $C$  is given by  $\frac{2-\alpha}{\alpha-1}$*

**Theorem 4.9.** *The MES algorithm generates a Beatty sequence if and only if the defining sequence is also a Beatty sequence  $D$  given by the slope  $\delta$  from Lemma 4.6, and the frequency is given by: if a number belongs to  $D$ , it will appear  $k$  times in the skipping sequence, otherwise, it will appear  $k-1$  times.*

*Proof.* The number  $\delta$  given in Lemma 4.6 can be also written as

$$(4.16) \quad \delta = \frac{2-\alpha}{k\alpha - 2k + 1},$$

or equivalently

$$(4.17) \quad \alpha = \frac{2-\delta + 2k\delta}{1+k\delta}.$$

Equation (4.16) tell us that to produce the Beatty sequence with slope  $\alpha$ , we just need to start with the skipping sequence generated by the number  $\delta$  given in the right hand side of the equality, with frequency  $k$ . We know that if  $A$  is a Beatty sequence, so is  $B$ . So we may assume both  $A$  and  $B$  are Beatty sequences. In this case, Lemma 4.6 tells us that the defining sequence  $D$  is also a Beatty sequence given by (4.16). For the converse, (4.17) tells us that given a defining sequence with slope  $\delta$ , we will generate a different Beatty sequence with slope  $\alpha$  for each positive integer  $k$  we take, as given in (4.17).  $\square$

We now state and prove the following theorem, which is part (4), of the Partial Decomposition Theorem.

**Corollary 4.10.** *The MES generalizes the MEX algorithm as follows: Suppose there is an increasing sequence of positive integers  $D$  such that  $C = D \star \mathbb{Z}_0^{k-1}$  for some  $k \in \mathbb{N}$ . Then we have the following*

- (1) *If  $k = 2$  and  $D$  is the Beatty sequence defined by the golden ratio, then the MES and the MEX algorithm coincide, i.e., they both produce  $A$ . In other words, the MES produces the sequence  $A$  if and only if the skipping sequence  $C$  can be defined as follows: any nonnegative integer  $c$  appears once or twice in  $C$ , and  $c$  appears twice in  $C$  if and only if  $c \in A$ .*
- (2) *Let  $D$  be the sequence given by*

$$\alpha = \frac{k - 1 + \sqrt{k^2 + 1}}{k}.$$

*Then the MES algorithm produces the Beatty sequence  $D$  and its complement. In other words, the MES produces the sequence  $D$  if and only if the skipping sequence  $C$  can be defined as follows: any nonnegative integer  $c$  appears  $k$  or  $k - 1$  times in  $C$ , and  $c$  appears  $k$  times in  $C$  if and only if  $c \in D$ . This coincides with the MEX and the golden ratio when  $k = 2$ .*

*Proof.* Substitute  $\alpha = \delta$  in Equation 4.17, and solve for  $\delta$  in the resulting rational expression to obtain a quadratic irrational with parameter  $k$ . The case  $k = 2$  gives the first part of the theorem.  $\square$

**Remark 5.** As seen in (4.16) and (4.17), a given  $\delta$  can generate infinitely many  $\alpha$ 's. However each given  $\alpha$  is generated by a unique choice of  $\delta$  and  $k$ .

**Example.** As an illustration of Theorem 4.9 consider the number  $\alpha = \frac{187 + 2\sqrt{13}}{113}$ . The Beatty sequence generated by  $\alpha$  is  $\{1, 3, 5, 6, 8, 10, 12, 13, 15, 17, 18, 20, 22, 24, 25, \dots\}$ . If we run the MES algorithm with frequency  $k = 3$  and defining sequence given by  $\delta = \sqrt{13}/2$ , we obtain Table 4.

$n^{th}$	A	B	C	R	D
1)	1	2	0	0	1
2)	3	4	0	0	3
3)	5	7	1	0	5
4)	6	9	1	1	7
5)	8	11	1	1	9
6)	10	14	2	1	10
7)	12	16	2	1	12
8)	13	19	3	2	14
9)	15	21	3	2	16
10)	17	23	3	2	18
11)	18	26	4	3	19
12)	20	28	4	3	21

This Table shows the MES algorithm when the frequency  $k$  is given by  $k = 3$  and the defining sequence  $D$  is given by  $\delta = \sqrt{13}/2$ . Note that the output sequences  $A$  is given by

$$\alpha = \frac{2 - \delta + 2l\delta}{1 + k\delta} = \frac{2 - \delta + 6\delta}{1 + 6\delta} = \frac{187 + 2\sqrt{13}}{113}$$



**Example 3.** *Example 2 can be rephrased recursively as follows: The skipping sequence  $C$  of the MES algorithm that generates the sequence  $A$  in Example 2 is given by the condition “if  $n \in A$ , then it will repeat twice in  $C$ . Otherwise,  $n$  will appear only once in  $C$ ”. Note that if  $\delta = \frac{1+\sqrt{5}}{2}$ , and  $k = 2$  then we get sequence in Example 2, i.e., we get the sequence with the golden ratio, and this slope coincides with the defining sequence.*

## 5. CONCLUDING REMARKS

What does the MES algorithm offer that the MEX does not? Notice that both algorithms allow us to generate any pair of Beatty (and indeed any pair of complementary) sequences. The MEX uses function  $b_n - a_n = h_n$  to generate these sequences, and thus any interesting application requires that we are able to easily describe the function  $h_n$  independent of  $a_n$  and  $b_n$ . To obtain the Beatty sequences generated  $\alpha$  and  $\beta$ , a few cases are easy to describe: for instance [30] shows that  $h_n = n$  gives the case  $\alpha = \frac{1+\sqrt{5}}{2}$ , and more generally, [14] and [18] show that  $h_n = tn$  gives the family  $\frac{2-t+\sqrt{t^2+1}}{2}$ . The MES is more advantageous because it gives the general formula  $h_n = c_n + r_n + 1$  with explicit formulas for  $c_n$  and  $r_n$  given in the main theorem. It also gives the added benefit of the combinatorial interpretation for  $c_n$  and  $r_n$  linking it back to the complementary sequences  $A$  and  $B$ . We conclude with an open-ended question: Given the many applications of the mex function or the MEX algorithm as highlighted in section 2, Does the MES algorithm reveal interesting connections that illuminate some aspects of these applications?

**Acknowledgements.** This research has been partially supported by the Ministerio de Educación Superior, Ciencia y Tecnología (MESCyT) of the Dominican Republic, grant No. 2018-2019-1D2-261

## REFERENCES

- [1] Abercrombie, A., Banks, W. & Shparlinski, I. Arithmetic functions on Beatty sequences. *Acta Arith.* **136**, 81-89 (2009)
- [2] Abercrombie, A. Beatty sequences and multiplicative number theory. *Acta Arithmetica.* **70**, 195-207 (1995)
- [3] Allouche, J. & Dekking, F. Generalized Beatty sequences and complementary triples. *Moscow Journal Of Combinatorics And Number Theory.* **8**, 325-341 (2019)
- [4] Allouche, J. & Shallit, J. Automatic sequences: theory, applications, generalizations. (Cambridge university press,2003)
- [5] Andrews, G. & Newman, D. Partitions and the minimal excludant. *Annals Of Combinatorics.* **23**, 249-254 (2019)
- [6] Ballot, C. On Functions Expressible as Words on a Pair of Beatty Sequences.. *J. Integer Seq.* **20**, 17-4 (2017)
- [7] Banks, W. & Shparlinski, I. Prime numbers with Beatty sequences. *ArXiv Preprint arXiv:0708.1015.* (2007)
- [8] Berstel, J. Recent results in Sturmian words Developments in Language Theory ed J Dassow and A Salomna. (Singapore: World Scientific,1996)
- [9] Blanchet-Sadri, F. & Simmons, S. Counting minimal semi-Sturmian words. *Discrete Applied Mathematics.* **161**, 2851-2861 (2013)
- [10] Bosma, W., Dekking, M. & Steiner, W. A Remarkable Integer Sequence Related to  $\pi$  and  $\sqrt{2}$  *Integers.* **18**, 1-9 (2018), <http://math.colgate.edu/~integers/sjs4/sjs4.pdf>
- [11] Brettell, N., Oxley, J., Semple, C. & Whittle, G. The excluded minors for 2-and 3-regular matroids. *ArXiv Preprint arXiv:2206.15188.* (2022)
- [12] Bruckstein, A. Self-similarity properties of digitized straight lines. *Contemp. Math.* **119** pp. 1-20 (1991)

- [13] Conway, J. On numbers and games. (AK Peters/CRC Press,2000)
- [14] Fraenkel, A. How to Beat Your Wythoff Games' Opponent on Three Fronts. *The American Mathematical Monthly*. **89**, 353-361 (1982), <http://www.jstor.org/stable/2321643>
- [15] Fraenkel, A. Iterated floor function, algebraic numbers, discrete chaos, Beatty subsequences, semigroups. *Transactions Of The American Mathematical Society*. **341**, 639-664 (1994)
- [16] Hata, M. Neurons: A Mathematical Ignition. (World Scientific,2014)
- [17] Hildebrand, A., Li, J., Li, X. & Xie, Y. Almost Beatty Partitions. *ArXiv Preprint arXiv:1809.08690*. (2018)
- [18] Holladay, J. Some generalizations of Wythoff's game and other related games. *Mathematics Magazine*. **41**, 7-13 (1968)
- [19] Illingworth, F., Scott, A. & Wood, D. Product structure of graphs with an excluded minor. *ArXiv Preprint arXiv:2104.06627*. (2022)
- [20] Morse, M. & Hedlund, G. Symbolic dynamics II. Sturmian trajectories. *American Journal Of Mathematics*. **62**, 1-42 (1940)
- [21] Lothaire, M. & Lothaire, M. Algebraic combinatorics on words. (Cambridge university press,2002)
- [22] Luca, F., Polanco, G. & Zudilin, W. A variation on the theme of Nicomachus. *Bulletin Of The Australian Mathematical Society*. **97**, 367-373 (2018)
- [23] Noll, T. Sturmian sequences and morphisms: a music-theoretical application. *Mathématique Et Musique, Journée Annuelle De La Société Mathématique De France*. pp. 79-102 (2008)
- [24] O'Bryant, K. A generating function technique for Beatty sequences and other step sequences. *Journal Of Number Theory*. **94**, 299-319 (2002)
- [25] Polanco, G. The logarithm of irrational numbers and Beatty sequences. *Acta Arithmetica*. **179** pp. 101-123 (2017)
- [26] Porta, H. & Stolarsky, K. Half-silvered mirrors and Wythoff's game. *Canadian Mathematical Bulletin*. **33**, 119-125 (1990)
- [27] Stolarsky, K. Beatty sequences, continued fractions, and certain shift operators. *Canadian Mathematical Bulletin*. **19**, 473-482 (1976)
- [28] Tang, H. Prime divisors in special Beatty sequences. *Chinese Journal Of Contemporary Mathematics*., 221-230 (2010)
- [29] Welsh, D. & Powell, M. An upper bound for the chromatic number of a graph and its application to timetabling problems. *The Computer Journal*. **10**, 85-86 (1967)
- [30] Wythoff, W. A modification of the game of Nim. *Nieuw Arch. Wisk.* **7**, 199-202 (1907)