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## A VARIATION ON THE THEME OF NICOMACHUS

FLORIAN LUCA, GEREMÍAS POLANCO and WADIM ZUDILIN<sup>✉</sup>

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### Abstract

In this paper, we prove some conjectures of K. Stolarsky concerning the first and third moments of the Beatty sequences with the golden section and its square.

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### 1. Introduction

Nicomachus' theorem asserts that the sum of the first  $m$  cubes is the square of the  $m$ th triangular number,

$$1^3 + 2^3 + \cdots + m^3 = (1 + 2 + \cdots + m)^2. \quad (1.1)$$

(See [2].) With the notation

$$Q(\alpha, m) := \frac{\sum_{n=1}^m \lfloor \alpha n \rfloor^3}{(\sum_{n=1}^m \lfloor \alpha n \rfloor)^2}, \quad (1.2)$$

where  $\alpha \in \mathbb{R} \setminus \{0\}$ , it implies that

$$\lim_{m \rightarrow \infty} Q(\alpha, m) = \alpha. \quad (1.3)$$

Here,  $\lfloor x \rfloor$  is the integer part of the real number  $x$ . The limit in (1.3) follows from  $\lfloor \alpha n \rfloor = \alpha n + O(1)$  and Nicomachus' theorem (1.1).

Recall that the Fibonacci and Lucas sequences,  $\{F_n\}_{n \geq 0}$  and  $\{L_n\}_{n \geq 0}$ , are given by  $F_0 = 0$ ,  $F_1 = 1$  and  $L_0 = 2$ ,  $L_1 = 1$  and the recurrence relations

$$F_{n+2} = F_{n+1} + F_n, \quad L_{n+2} = L_{n+1} + L_n$$

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for  $n \geq 0$ . In a personal communication to the second author, K. Stolarsky observed that the limit relation (1.3) can be ‘quantified’ for  $\alpha = \phi$  and  $\phi^2$ , where  $\phi := \frac{1}{2}(1 + \sqrt{5})$  is the golden mean, and a specific choice of  $m$  along the Fibonacci sequence. The corresponding result is Theorem 2.1 below. We complement it by a general analysis of moments of the Beatty sequences and give a solution to a related arithmetic question in Theorem 2.5.

### 2. Principal results

**THEOREM 2.1.** *For  $k \geq 1$  an integer, define  $m_k := F_k - 1$ . Then*

$$Q(\phi^2, m_{2k}) - Q(\phi, m_{2k}) = \begin{cases} 1 - \frac{1}{(F_{k+1})^2 L_{k+2} L_{k-1}} & \text{if } k \text{ is even,} \\ 1 - \frac{1}{(L_{k+1})^2 F_{k+2} F_{k-1}} & \text{if } k \text{ is odd} \end{cases}$$

and

$$Q(\phi^2, m_{2k-1}) - Q(\phi, m_{2k-1}) = \begin{cases} 1 - \frac{F_{k-2}}{F_{k+1}(F_k)^2(L_{k-1})^2} & \text{if } k \text{ is even,} \\ 1 - \frac{L_{k-2}}{L_{k+1}(L_k)^2(F_{k-1})^2} & \text{if } k \text{ is odd.} \end{cases}$$

The theorem motivates our interest in the numerators and denominators of  $Q(\phi, F_k - 1)$  and  $Q(\phi^2, F_k - 1)$ , which can be thought of as expressions of the form

$$A(k, s) := \sum_{n=1}^{F_k-1} \lfloor \phi n \rfloor^s \quad \text{and} \quad A'(k, s) := \sum_{n=1}^{F_k-1} \lfloor \phi^2 n \rfloor^s$$

for  $k = 1, 2, \dots$  and  $s = 1, 3$ . More generally, our analysis in Section 3 covers the sums

$$A(k, s, j) := \sum_{n=1}^{F_k-1} n^j \lfloor \phi n \rfloor^s \quad \text{where } k = 1, 2, \dots \text{ and } s, j = 0, 1, 2, \dots \quad (2.1)$$

Namely, we find a recurrence relation for  $A(k, j, s)$  and deduce recursions from it for  $A(k, s) = A(k, s, 0)$  and  $A'(k, s)$ . The strategy leads to the following expressions for the numerators and denominators in Theorem 2.1, which are given in Lemmas 2.2–2.4.

**LEMMA 2.2.** *Let  $k \geq 1$  be an integer. Then*

$$\begin{aligned} A(k, 1) &= \frac{1}{2}(F_{k+1} - 1)(F_k - 1), \\ A'(k, 1) &= \frac{1}{2}(F_{k+2} - 1)(F_k - 1). \end{aligned} \quad (2.2)$$

**LEMMA 2.3.** *Let  $k \geq 1$  be an integer. Then*

$$\begin{aligned} A(2k, 3) &= \frac{1}{4}(F_{2k-1} - 1)(F_{2k+1} - 1)^2(F_{2k+2} - 1), \\ A(2k - 1, 3) &= \frac{1}{4}(F_{2k-1} - 1)(F_{2k} - 1) \times \frac{1}{5}(L_{4k} - 3L_{2k+1} - L_{2k} + 3). \end{aligned}$$

**LEMMA 2.4.** *Let  $k \geq 1$  be an integer. Then*

$$A'(2k, 3) = \frac{1}{4}(F_{2k} - 1)(F_{2k+2} - 1) \times \frac{1}{5}(L_{4k+4} - 5L_{2k+3} + 13),$$

$$A'(2k - 1, 3) = \frac{1}{4}(F_{2k-1} - 1)(F_{2k+1} - 1) \times \frac{1}{5}(L_{4k+2} - 5L_{2k+2} + 7).$$

Finally, we present an arithmetic formula inspired by Stolarsky’s original question.

**THEOREM 2.5.** *For  $k \geq 1$ ,*

$$\text{LCM}(A(2k, 1), A'(2k, 1)) = \begin{cases} \frac{1}{2}F_{k+1}F_kL_{k+2}L_{k+1}L_{k-1} & \text{if } 2 \mid k, \\ \frac{1}{2}F_{k+2}F_{k+1}F_{k-1}L_{k+1}L_k & \text{if } 2 \nmid k. \end{cases}$$

**REMARK 2.6.** Lemmas 2.2–2.4 indicate that the expression

$$Q(\phi^2, F_k - 1) - Q(\phi, F_k - 1)$$

is expressible as a fraction whose numerator and denominator are polynomials in Fibonacci and Lucas numbers with indices depending linearly on  $k$  according to the parity of  $k$ , yet the statement of Theorem 2.1 presents formulas for these quantities according to the congruence class of  $k$  modulo 4 rather than modulo 2. The discrepancy is related to different factorisations of the factors  $F_n - 1$  that occur in the formulas for  $A(k, j)$  and  $A'(k, j)$  for  $j \in \{1, 3\}$ , since each of the factors  $F_n - 1$  happens to be a product of a Fibonacci and a Lucas number according to the congruence class of  $n$  modulo 4 (see formulas (6.1)).

### 3. Recurrence relations for auxiliary sums

Here, we show how to compute the integer-part sums (2.1). This clearly covers the cases  $A(k, s) = A(k, s, 0)$ . On using

$$\phi^2 = 1 + \phi,$$

which upon multiplication by the integer  $n$  and taking integer parts becomes

$$\lfloor \phi^2 n \rfloor = n + \lfloor \phi n \rfloor,$$

one also gets the explicit formulas

$$A'(k, s) = \sum_{i=0}^s \binom{s}{i} A(k, s - i, i).$$

Using the Binet formula

$$F_k = \frac{\phi^k - (-\phi^{-1})^k}{\sqrt{5}} \quad \text{for all } k \geq 0,$$

one easily proves that

$$\lfloor \phi F_k \rfloor = F_{k+1} - \epsilon_k \quad \text{where } \epsilon_k = \frac{1 + (-1)^k}{2}$$

and

$$[\phi(F_k + n)] = F_{k+1} + [\phi n] \quad \text{for } 1 \leq n \leq F_{k-1} - 1$$

(see, for example, [1]). Thus,

$$\begin{aligned} A(k + 1, s, j) &= \sum_{n=1}^{F_k-1} n^j [\phi n]^s + F_k^j [\phi F_k]^s + \sum_{n=F_k+1}^{F_{k+1}-1} n^j [\phi n]^s \\ &= A(k, s, j) + F_k^j (F_{k+1} - \epsilon_k)^s + \sum_{n=1}^{F_{k+1}-F_k-1} (F_k + n)^j [\phi(F_k + n)]^s \\ &= A(k, s, j) + F_k^j (F_{k+1} - \epsilon_k)^s + \sum_{n=1}^{F_{k-1}-1} (F_k + n)^j (F_{k+1} + [\phi n])^s \\ &= A(k, s, j) + F_k^j \sum_{i=0}^s \binom{s}{i} F_{k+1}^i (-\epsilon_k)^{s-i} \\ &\quad + \sum_{n=1}^{F_{k-1}-1} \sum_{\ell=0}^j \binom{j}{\ell} F_k^\ell n^{j-\ell} \sum_{i=0}^s \binom{s}{i} F_{k+1}^i [\phi n]^{s-i} \\ &= A(k, s, j) + \sum_{i=0}^s \binom{s}{i} (-\epsilon_k)^{s-i} F_k^j F_{k+1}^i \\ &\quad + \sum_{\ell=0}^j \sum_{i=0}^s \binom{j}{\ell} \binom{s}{i} F_k^\ell F_{k+1}^i A(k - 1, s - i, j - \ell). \end{aligned}$$

The above reduction, the identity  $A(k, 0, 0) = F_k - 1$  and induction on  $k + j + s$  imply that

$$A(k, s, j) \in \text{span}\{(\phi^i)^k, (-\phi^i)^k : |i| \leq j + s + 1\};$$

in particular, for a fixed choice of  $s$  and  $j$ , the sequence  $\{A(k, s, j)\}_{k \geq 1}$  is linearly recurrent of order at most  $4(s + j) + 6$ . In the following section, we will use this observation about linear recurrency together with the following facts.

- If  $\mathbf{u} = \{u_n\}_{n \geq 0}$  is a linearly recurrent sequence whose roots are all simple in some set  $U$ , then, for fixed integers  $p$  and  $q$ , the sequence  $\{u_{pn+q}\}_{n \geq 0}$  is linearly recurrent with simple roots in  $\{\alpha^p : \alpha \in U\}$ .
- If  $\mathbf{u} = \{u_n\}_{n \geq 0}$  and  $\mathbf{v} = \{v_n\}_{n \geq 0}$  are linearly recurrent and their roots are all simple in some sets  $U$  and  $V$ , respectively, then  $\mathbf{uv} = \{u_n v_n\}_{n \geq 0}$  is linearly recurrent and its roots are all simple in  $UV = \{\alpha\beta : \alpha \in U, \beta \in V\}$ .

In this context, the roots of a linearly recurrent sequence are defined as the zeros of its characteristic polynomial, counted with their multiplicities. It then follows that, for a fixed  $s$ , each of the sequences  $\{A(k, s)\}_{k \geq 1}$  and  $\{A'(k, s)\}_{k \geq 1}$  is linearly recurrent of order at most  $4s + 6$ .

### 4. The proofs of the lemmas

We first establish Lemma 2.2. By the argument in Section 3, both  $A(k, 1)$  and  $A'(k, 1)$  are linearly recurrent with simple roots in the set  $\{\pm\phi^l : |l| \leq 2\}$ . The same is true for the right-hand sides in (2.2). Since the set of roots is contained in a set with 10 elements, it follows that the validity of (2.2) for  $k = 1, \dots, 10$  implies that the relations hold for all  $k \geq 1$ .

Lemmas 2.3 and 2.4 are similar. The argument in Section 3 shows that the left-hand sides,  $\{A(k, 3)\}_{k \geq 1}$  and  $\{A'(k, 3)\}_{k \geq 1}$ , are linearly recurrent with simple roots in  $\{\pm\phi^l : |l| \leq 4\}$ . Splitting according to the parity of  $k$ , we deduce that  $\{A(2k, 3)\}_{k \geq 1}$ ,  $\{A(2k - 1, 3)\}_{k \geq 1}$ ,  $\{A'(2k, 3)\}_{k \geq 1}$  and  $\{A'(2k - 1, 3)\}_{k \geq 1}$  are linearly recurrent with simple roots in  $\{\phi^{2l} : |l| \leq 4\}$ , a set with nine elements. The same is true about the right-hand sides in the lemmas. Thus, if the relations hold for  $k = 1, \dots, 9$ , then they hold for all  $k \geq 1$ .

A few words about the computation. For the identities presented in Lemmas 2.2–2.4, one can use brute force to compute  $A(k, s)$  and  $A'(k, s)$  for  $s = 1, 3$  and  $k = 2\ell + i$ , where  $i \in \{0, 1\}$  and  $\ell$  is reasonably small (we dealt with  $\ell \leq 9$ ), with any computer algebra system. To check them up to larger values of  $k$  (around 100, say), the brute force strategy no longer works since the summation range up to  $F_k - 1$  becomes too large. Instead one can use the recursion from Section 3 together with  $A(k, 0, 0) = F_k - 1$  to find  $A(k, 1, 0)$ ,  $A(k, 2, 0)$  and  $A(k, 3, 0)$  for all desired  $k$  and, similarly,  $A(k, s, j)$  for small  $j$ , to evaluate  $A'(k, s)$ .

### 5. The proof of Theorem 2.1

Let us now address Theorem 2.1. When  $k = 4\ell$ , this can be rewritten as

$$\begin{aligned} &F_{2\ell+1}^2 L_{2\ell+2} L_{2\ell-1} (A'(4\ell, 3)A(4\ell, 1)^2 - A(4\ell, 3)A'(4\ell, 1)^2) \\ &= A(4\ell, 1)^2 A'(4\ell, 1)^2 (F_{2\ell+1}^2 L_{2\ell+2} L_{2\ell-1} - 1). \end{aligned} \tag{5.1}$$

Since  $A(4\ell, s)$  and  $A'(4\ell, s)$  are linearly recurrent (in  $\ell$ ) with roots contained in  $\{\phi^{4l} : |l| \leq s + 1\}$ , and both the left-most factor in the left-hand side and the right-most factor in the right-hand side each have simple roots in  $\{\phi^{4l} : |l| \leq 2\}$ , it follows that both the left-hand side and the right-hand side are linearly recurrent with simple roots contained in  $\{\phi^{4l} : |l| \leq 10\}$ , a set with 21 elements. Thus, if the above formula holds for  $\ell = 1, \dots, 21$ , then it holds for all  $\ell \geq 1$ . A similar argument applies to the case when  $k = 4\ell + i$  for  $i \in \{1, 2, 3\}$ . Hence, all claimed formulas hold provided they hold for all  $k \leq 100$ , say.

Now we use the lemmas. For  $k = 4\ell$ , Lemmas 2.2–2.4 tell us that (5.1), after eliminating the common factor  $(F_{4\ell} - 1)^2 (F_{4\ell+1} - 1)^2 (F_{4\ell+2} - 1) / 16$ , is equivalent to

$$\begin{aligned} &F_{2\ell+1}^2 L_{2\ell+2} L_{2\ell-1} \times \left(\frac{1}{5}(F_{4\ell} - 1)(L_{8\ell+4} - 5L_{4\ell+3} + 13)\right. \\ &\quad \left. - (F_{4\ell+2} - 1)(F_{4\ell-1} - 1)(F_{4\ell+2} - 1)\right) \\ &= (F_{4\ell} - 1)^2 (F_{2\ell+1}^2 L_{2\ell+2} L_{2\ell-1} - 1) \end{aligned}$$

(and one can perform further reduction using (6.1)). It is sufficient to verify the resulting equality for  $\ell = 1, \dots, 15$  and we have checked it for all  $\ell = 1, \dots, 100$ . The remaining cases for  $k$  modulo 4 are similar. We do not give further details here.

### 6. The proof of Theorem 2.5

This follows from Lemma 2.2, the classical formulas

$$\begin{aligned} F_{4\ell} - 1 &= F_{2\ell+1}L_{2\ell-1}, & F_{4\ell+1} - 1 &= F_{2\ell}L_{2\ell+1}, \\ F_{4\ell+2} - 1 &= F_{2\ell}L_{2\ell+2}, & F_{4\ell+3} - 1 &= F_{2\ell+2}L_{2\ell+1} \end{aligned} \tag{6.1}$$

as well as known facts about the greatest common divisor of Fibonacci and Lucas numbers with close arguments. For example, for  $k = 2\ell$ ,

$$\begin{aligned} \text{LCM}(2A(4\ell, 1), 2A'(4\ell, 1)) &= \text{LCM}((F_{4\ell+1} - 1)(F_{4\ell} - 1), (F_{4\ell+2} - 1)(F_{4\ell} - 1)) \\ &= \text{LCM}(F_{2\ell}L_{2\ell+1}, F_{2\ell}L_{2\ell+2})F_{2\ell+1}L_{2\ell-1} \\ &= F_{2\ell}L_{2\ell+1}L_{2\ell+2}F_{2\ell+1}L_{2\ell-1} \\ &= F_{k+1}F_kL_{k+2}L_{k+1}L_{k-1}, \end{aligned}$$

where we used the fact that  $\text{gcd}(L_{2\ell+1}, L_{2\ell+2}) = 1$ . The case  $k = 2\ell + 1$  is similar.

### 7. Further variations

First, we give an informal account of a more general result lurking, perhaps, behind the formulas in Theorem 2.1. Consider a homogeneous (rational) function  $r(\mathbf{x}) = r(x_1, \dots, x_m)$  of degree 1, that is, satisfying

$$r(t\mathbf{x}) = tr(\mathbf{x}) \quad \text{for } t \in \mathbb{Q},$$

and an algebraic number  $\alpha$  solving the equation

$$\sum_{k=0}^m c_k \alpha^k = 0, \tag{7.1}$$

where the  $c_k$  are integers. If  $r(\mathbf{x})$  vanishes at a vector  $\mathbf{x}^* = (x_1^*, \dots, x_m^*)$ , then automatically

$$\sum_{k=0}^m c_k r(\alpha^k \mathbf{x}^*) = 0 \tag{7.2}$$

in view of the homogeneity of the function. We can then enquire whether equation (7.2) is ‘approximately’ true if  $r(\mathbf{x}^*) = 0$  is ‘approximately’ true. In this note, we merely examined the golden ratio case in which (7.1) is  $\alpha^2 - \alpha - 1 = 0$ , while the choice

$$r(x_1, \dots, x_m) = \frac{\sum_{n=1}^m x_n^3}{(\sum_{n=1}^m x_n)^2}$$

for the rational function and  $\mathbf{x}^* = (1, 2, \dots, m)$  for its exact solution originated from the Nicomachus identity.

Notice that Nicomachus' theorem (1.1) is the first entry in the chain of identities

$$1^{2r-1} + 2^{2r-1} + \cdots + m^{2r-1} = P_r(1 + 2 + \cdots + m) \quad \text{for } r = 2, 3, \dots,$$

where  $P_r(x)$  are known as the Faulhaber polynomials. Our approach in this note gives a clear strategy to deal with the quantities that replace (1.2) in these settings.

Some further variations on the topic can be investigated in the  $q$ -direction, based on  $q$ -analogues of (1.1) (see [3]).

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