1-5-2023

Modules and Representations up to Homotopy of Lie *n*-Algebroids

M. Jotz
Julius-Maximilians-Universität Würzburg

Rajan Amit Mehta
Smith College, rmehta@smith.edu

T. Papantonis

Journal of Homotopy and Related Structures (2023) 18:23–70
https://doi.org/10.1007/s40062-022-00322-x

Modules and representations up to homotopy of Lie n-algebroids

M. Jotz[1] · R. A. Mehta[2] · T. Papantonis[3]

Received: 23 February 2022 / Accepted: 7 December 2022 / Published online: 5 January 2023

Abstract

This paper studies differential graded modules and representations up to homotopy of Lie n-algebroids, for general $n \in \mathbb{N}$. The adjoint and coadjoint modules are described, and the corresponding split versions of the adjoint and coadjoint representations up to homotopy are explained. In particular, the case of Lie 2-algebroids is analysed in detail. The compatibility of a Poisson bracket with the homological vector field of a Lie n-algebroid is shown to be equivalent to a morphism from the coadjoint module to the adjoint module, leading to an alternative characterisation of non-degeneracy of higher Poisson structures. Moreover, the Weil algebra of a Lie n-algebroid is computed explicitly in terms of splittings, and representations up to homotopy of Lie n-algebroids are used to encode decomposed VB-Lie n-algebroid structures on double vector bundles.

Keywords Lie n-algebroids · Representations up to homotopy · Differential graded modules · Poisson algebras · Adjoint and coadjoint representations

Contents

1 Introduction . . . 24
Outline of the paper . . . 26

Communicated by Jim Stasheff.

✉ M. Jotz
madeleine.jotz-lean@mathematik.uni-wuerzburg.de

R. A. Mehta
rmehta@smith.edu

T. Papantonis
thepapantonis@gmail.com

[1] Institut für Mathematik, Julius-Maximilians-Universität Würzburg, Würzburg, Germany

[2] Department of Mathematics and Statistics, Smith College, Northampton, MA, USA

[3] Düsseldorf, Germany

Relation to other work . . . 27
2 Preliminaries . . . 27
2.1 (Graded) vector bundles and complexes . . . 28
2.2 Dull algebroids vs Lie algebroids . . . 29
2.3 Basic connections and basic curvature . . . 30
3 (Split) Lie n-algebroids and $\mathbb{N}Q$-manifolds . . . 31
3.1 (Split) $\mathbb{N}$-manifolds and homological vector fields . . . 31
3.2 Generalized functions of a Lie n-algebroid . . . 36
3.3 The Weil algebra associated to a Lie n-algebroid . . . 38
4 Differential graded modules . . . 40
4.1 The category of differential graded modules . . . 40
4.2 Adjoint and coadjoint modules . . . 43
4.3 Poisson Lie n-algebroids: coadjoint vs adjoint modules . . . 43
5 Representations up to homotopy . . . 45
5.1 The category of representations up to homotopy . . . 46
5.2 The case of (split) Lie 2-algebroids . . . 47
5.3 Adjoint representation of a Lie 2-algebroid . . . 50
5.4 Adjoint module vs adjoint representation . . . 51
5.5 Coordinate transformation of the adjoint representation . . . 53
5.6 Adjoint representation of a Lie n-algebroid . . . 56
6 Split VB-Lie n-algebroids . . . 56
6.1 Double vector bundles . . . 56
6.2 Linear splittings, horizontal lifts and duals . . . 58
6.3 VB-Lie n-algebroids and $(n+1)$-representations . . . 59
7 Constructions in terms of splittings . . . 63
7.1 The Weil algebra of a split Lie n-algebroid . . . 63
7.2 Poisson Lie algebroids of low degree . . . 65
References . . . 68

1 Introduction

Lie n-algebroids, for $n \in \mathbb{N}$, are graded geometric structures which generalise the notion of Lie algebroids. They have become a field of much interest in mathematical physics, since they form a nice framework for higher analogues of Poisson and symplectic structures.

Courant algebroids [29] give an important example of such higher structures. The work of Courant and Weinstein [12] and of Hitchin and Gualtieri [19, 20, 22] shows that Courant algebroids serve as a convenient framework for Hamiltonian systems with constraints, as well as for generalised geometry. A significant result from Roytenberg [40] and Ševera [42] shows that Courant algebroids are in one-to-one correspondence with Lie 2-algebroids equipped with a compatible symplectic structure.

The standard super-geometric description of a Lie n-algebroid generalises the differential algebraic way of defining a usual Lie algebroid as a vector bundle A over a smooth manifold M together with a degree 1 differential operator on the space $\Omega^\bullet(A) := \Gamma(\wedge^\bullet A^*)$. In the language of graded geometry, this is equivalent to a graded manifold of degree 1 equipped with a homological vector field [45], i.e. a degree 1 derivation on its sheaf of functions which squares to zero and satisfies the graded Leibniz rule. A Lie n-algebroid is then defined as a graded manifold $\mathcal{M}$ of degree n, whose sheaf of functions $C^\infty(\mathcal{M})$ is equipped with a homological vector field $\mathcal{Q}$. In more "classical" geometric terms, a (split) Lie n-algebroid can also be defined as a graded vector bundle $\underline{A} = \bigoplus_{i=1}^{n} A_i[i]$ over a smooth manifold M together with some multi-brackets on its space of sections $\Gamma(\underline{A})$ which satisfy some higher Leibniz and Jacobi identities [43]. A Lie n-algebroid $(\mathcal{M}, \mathcal{Q})$ is called *Poisson* if its underlying graded manifold carries a degree $-n$ Poisson structure $\{\cdot\,,\cdot\}$ on its sheaf of functions $\mathcal{C}^\infty(\mathcal{M})$, such that the homological vector field is a derivation of the Poisson bracket.

A well-behaved representation theory of Lie n-algebroids for $n \geq 2$ has not been developed yet. In the case $n = 1$, i.e. in the case of usual Lie algebroids, Gracia-Saz and Mehta [18], and independently Abad and Crainic [2], showed that the notion of *representation up to homotopy* is a good notion of representation, which includes the adjoint representation. Roughly, the idea is to let the Lie algebroid act via a differential on Lie algebroid forms which take values on a cochain complex of vector bundles instead of just a single vector bundle. This notion is essentially a $\mathbb{Z}$-graded analogue of Quillen's super-representations [39]. After their discovery, representations up to homotopy have been extensively studied in other works, see e.g. [3–5, 8, 10, 15, 17, 25, 35, 36, 44, 48]. In particular, the adjoint representation up to homotopy of a Lie algebroid is proving to be as fundamental in the study of Lie algebroids as the adjoint representation of a Lie algebra is in the study of Lie algebras. As is well known, the adjoint representation controls deformations and symmetries of Lie algebras (see e.g. [13] and references therein), and it is a key to the classification and the algebraic integration of Lie algebras [46, 47]. Similarly, the deformations of a Lie algebroid are controlled by the cohomology with values in its adjoint representation up to homotopy [35, 45], and an ideal in a Lie algebroid is a subrepresentation of the adjoint representation up to homotopy [15]. While a Lie bialgebra is a matched pair of the adjoint and coadjoint representations, a Lie bialgebroid is a matched pair of the adjoint and coadjoint representations up to homotopy [17]. From another point of view, 2-term representations up to homotopy, which are equivalent to decompositions of VB-algebroids [18], have proved to be a powerful tool in the study of multiplicative structures on Lie groupoids (se e.g. [1, 9, 15, 27]), which, at the infinitesimal level, can be described as linear structures on algebroids.

One of the authors proved in [35] that representations up to homotopy of Lie algebroids are equivalent, up to isomorphism, to Lie algebroid modules in the sense of [45]. This paper extends this notion of modules, and consequently of representations up to homotopy, to the context of higher Lie algebroids. The definition is the natural generalisation of the case of usual Lie algebroids explained above, i.e. differential graded modules over the space of smooth functions of the underlying graded manifold. The obtained notion is analysed in detail, including the two most important examples of representations, namely, *the adjoint* and *the coadjoint* representations (up to homotopy). An equivalent geometric point of view of a special class of representations is given by *split VB-Lie n-algebroids*, i.e. double vector bundles with a graded side and a linear split Lie n-algebroid structure over a split Lie n-algebroid.

In addition to the impact of representations up to homotopy in the study of Lie algebroids in the last ten years, our general motivation for studying representations up to homotopy of higher Lie n-algebroids comes from the case $n = 2$, and in particular from Courant algebroids. In light of AKSZ theory, it seems reasonable to expect that the category of representations (up to homotopy) of Courant algebroids might have interesting connections to 3-dimensional topology. The results in this paper should be useful in the study of such representations. The first step is the search for a good notion not only of the adjoint representation of a Courant algebroid, but also of its ideals, similar to the work done in [27]. Since Courant algebroids are equivalent to Lie 2-algebroids with a compatible symplectic structure [40, 42], the following question arises naturally:

Question Is a compatible Poisson or symplectic structure on a Lie n-algebroid encoded in its adjoint representation?

The answer to this question is positive, since it turns out that a Poisson bracket on a Lie n-algebroid gives rise to a natural map from the coadjoint to the adjoint representation which is a morphism of right representations (see Theorem 4.13, Corollary 4.14 and Sect. 7.2), i.e. it anti-commutes with the differentials of their structure as left representations and commutes with the differentials of their structure as right representations. Further, the Poisson structure is symplectic if and only if this map is in fact a right isomorphism. This result is already known in some special cases, including Poisson Lie 0-algebroids, i.e. ordinary Poisson manifolds $(M, \{\cdot\,,\cdot\})$, and Courant algebroids over a point, i.e. quadratic Lie algebras $(\mathfrak{g}, [\cdot\,,\cdot], \langle\cdot\,,\cdot\rangle)$. In the former case the map reduces to the natural map $\sharp\colon T^*M \to TM$ obtained from the Poisson bracket on M, and in the latter case it is the inverse of the map defined by the nondegenerate pairing $\mathfrak{g} \to \mathfrak{g}^*$, $x \mapsto \langle x, \cdot\rangle$.

Let us conclude by explaining why the study of representations up to homotopy of *split* Lie n-algebroids is prominent in this paper. Our approach in this paper emphasises the similarity of the formulas in the split case with the usual formulas for the now well-known representations up to homotopy of Lie algebroids [2, 18]. More precisely, we construct objects evidently generalising this "classical" theory, and we employ techniques and constructions that are similar to those that are well-known. The correspondence between decomposed split VB-Lie n-algebroids and $(n+1)$-representations of Lie n-algebroids is an example of this, since it is a generalisation of the correspondence of decomposed VB-algebroids with 2-representations of Lie 1-algebroids [18].

In addition, some examples naturally have the split form and are easier to work with in this setting. For instance, the symplectic Lie 2-algebroids corresponding to Courant algebroids [40, 42] are often given as split Lie 2-algebroids, after a choice of metric connection on the Courant algebroid.

Outline of the paper

This paper consists of seven sections and is organised as follows. Section 2 sets the notation and conventions, and recalls the definitions and constructions of graded vector bundles and Lie algebroids.

Section 3 offers a quick introduction to graded manifolds, (split) Lie n-algebroids, and Poisson and symplectic structures on Lie n-algebroids. In particular, it discusses the space of generalised functions of a Lie n-algebroid, gives the geometric description of a split Lie 2-algebroid [25] which is used in the rest of the paper, and defines the Weil algebra of a Lie n-algebroid—as it is done in [33] in the case $n = 1$.

Sections 4 and 5 generalise the notions of Lie algebroid modules and representations up to homotopy to the setting of Lie n-algebroids. They offer a detailed explanation of the theory and give some useful examples, including the classes of the adjoint and coadjoint modules, whose properties are discussed thoroughly, especially in the case of Lie 2-algebroids. Section 4 provides the answer to the question expressed above about the connection between higher Poisson or symplectic structures and the adjoint and coadjoint modules.

Section 6 recalls some basic definitions and examples from the theory of double vector bundles and defines VB-Lie n-algebroids together with the prototype example of the tangent prolongation of a Lie n-algebroid. It also shows that there is a 1-1 correspondence between split VB-Lie n-algebroids and representations up to homotopy of degree $n+1$, which relates again the adjoint representation of a Lie algebroid with its tangent prolongation.

Finally, Sect. 7 discusses in the split case the results of this paper. It analyses the Weil algebra of a split Lie n-algebroid using vector bundles and connections, and it gives more details about the map between the coadjoint and adjoint representations for split Poisson Lie algebroids of degree $n \leq 2$.

Relation to other work

During the preparation of this work, the authors learnt that Caseiro and Laurent–Gengoux also consider representations up to homotopy of Lie n-algebroids, in particular the adjoint representation, in their article [11], which was then also in preparation.

In [48], Vitagliano considers representations of strongly homotopy Lie–Rinehart algebras. Strongly homotopy Lie Rinehart algebras are the purely algebraic versions of graded vector bundles, over graded manifolds, equipped with a homological vector field that is tangent to the zero section. If the base manifold has grading concentrated in degree 0 and the vector bundle is negatively graded, the notion recovers the one of split Lie n-algebroids. In that case, Vitagliano's representations correspond to the representations up to homotopy considered in this paper.

In addition, since the DG $\mathcal{M}$-modules considered in this paper are the sheaves of sections of $\mathcal{Q}$-vector bundles, they are themselves also special cases of Vitagliano's strongly homotopy Lie–Rinehart algebras.

2 Preliminaries

This section recalls basic definitions and conventions that are used later on. In what follows, M is a smooth manifold and all the considered objects are supposed to be smooth even if not explicitly mentioned. Moreover, all (graded) vector bundles are assumed to have finite rank.

2.1 (Graded) vector bundles and complexes

Given two ordinary vector bundles $E \to M$ and $F \to N$, there is a bijection between vector bundle morphisms $\phi\colon E \to F$ covering $\phi_0\colon M \to N$ and morphisms of modules $\phi^\star\colon \Gamma(F^*) \to \Gamma(E^*)$ over the pull-back $\phi_0^*\colon C^\infty(N) \to C^\infty(M)$. Explicitly, the map $\phi^\star$ is defined by $\phi^\star(f)(m) = \phi_m^* f_{\phi_0(m)}$, for $f \in \Gamma(F)$, $m \in M$.

Throughout the paper, underlined symbols denote graded objects. For instance, a graded vector bundle is a vector bundle $q\colon \underline{E} \to M$ together with a direct sum decomposition

$$\underline{E} = \bigoplus_{n\in\mathbb{Z}} E_n[n]$$

of vector bundles E_n over M. The finiteness assumption for the rank of $\underline{E}$ implies that $\underline{E}$ is both upper and lower bounded, i.e. there exists a $n_0 \in \mathbb{Z}$ such that $E_n = 0$ for all $|n| > n_0$. Here, an element $e \in E_n$ is (degree-)homogeneous of degree $|e| = -n$. That is, for $k \in \mathbb{Z}$, the degree k component of $\underline{E}$ (denoted with upper index $\underline{E}^k$) equals E_{-k}.

All the usual algebraic constructions from the theory of ordinary vector bundles extend to the graded setting. More precisely, for graded vector bundles $\underline{E}$, $\underline{F}$, the dual

$$\underline{E}^* = \bigoplus_{n\in\mathbb{Z}} E_n^*[-n],$$

the direct sum $\underline{E} \oplus \underline{F}$, the space of graded homomorphisms $\underline{\mathrm{Hom}}(\underline{E}, \underline{F})$, the tensor product $\underline{E} \otimes \underline{F}$, and the symmetric and antisymmetric powers $\underline{S}(\underline{E})$ and $\underline{A}(\underline{E})$ are defined.

A (cochain) complex of vector bundles is a graded vector bundle $\underline{E}$ over M equipped with a degree one[1] endomorphism over the identity on M

$$\ldots \xrightarrow{\partial} E_{i+1} \xrightarrow{\partial} E_i \xrightarrow{\partial} E_{i-1} \xrightarrow{\partial} \ldots$$

which squares to zero; $\partial^2 = 0$, and is called the *differential*.

Given two complexes $(\underline{E}, \partial)$ and $(\underline{F}, \partial)$, one may construct new complexes by considering all the constructions that were discussed before. Namely, the bundles $\underline{S}(\underline{E})$, $\underline{A}(\underline{E})$, $\underline{E}^*$, $\underline{\mathrm{Hom}}(\underline{E}, \underline{F})$ and $\underline{E} \otimes \underline{F}$ inherit a degree one operator that squares to 0. The basic principle for all the constructions is the graded derivation rule. For example, for $\phi \in \underline{\mathrm{Hom}}(\underline{E}, \underline{F})$ and $e \in \underline{E}$:

$$\partial(\phi(e)) = \partial(\phi)(e) + (-1)^{|\phi|}\phi(\partial(e)).$$

This can also be expressed using the language of (graded) commutators as

$$\partial(\phi) = [\partial, \phi] = \partial \circ \phi - (-1)^{|\phi|}\phi \circ \partial = \partial \circ \phi - (-1)^{|\phi|\cdot|\partial|}\phi \circ \partial.$$

[1] Recall that for $i \in \mathbb{Z}$ the elements of E_i have degree $-i$ by convention.

The shift functor $[k]$, for $k \in \mathbb{Z}$, yields a new complex $(\underline{E}[k], \partial[k])$ whose i-th component is $\underline{E}[k]^i = \underline{E}^{i+k} = E_{-i-k}$ with differential $\partial[k] = \partial$. Formally, $\underline{E}[k]$ is obtained by tensoring with $(\mathbb{R}[k], 0)$ from the right.[2] A degree k morphism between two complexes $(\underline{E}, \partial)$ and $(\underline{F}, \partial)$ over M, or simply k-morphism, is, by definition, a degree preserving morphism $\phi\colon \underline{E} \to \underline{F}[k]$ over the identity on M; that is, a family of vector bundle maps $\phi_i : \underline{E}^i \to \underline{F}[k]^i$ over the identity on M that commutes with the differentials:[3] $\phi \circ \partial = \partial \circ \phi$.

2.2 Dull algebroids vs Lie algebroids

A dull algebroid [23] is a vector bundle $Q \to M$ endowed with an anchor $\rho_Q\colon Q \to TM$ and a bracket (i.e. an $\mathbb{R}$-bilinear map) $[\cdot\,,\cdot]\colon \Gamma(Q) \times \Gamma(Q) \to \Gamma(Q)$ on its space of sections $\Gamma(Q)$, such that

$$\rho_Q\,[q_1, q_2] = \left[\rho_Q\,(q_1)\,, \rho_Q\,(q_2)\right] \tag{1}$$

and the Leibniz identity is satisfied in both entries:

$$[f_1q_1, f_2q_2] = f_1 f_2\,[q_1, q_2] + f_1\rho_Q(q_1) f_2 \cdot q_2 - f_2\rho_Q(q_2) f_1 \cdot q_1,$$

for all $q_1, q_2 \in \Gamma(Q)$ and all $f_1, f_2 \in C^\infty(M)$.

A dull algebroid is a Lie algebroid if its bracket is also skew-symmetric and satisfies the Jacobi identity

$$\mathrm{Jac}_{[\cdot\,,\cdot]}\,(q_1, q_2, q_3) := [q_1, [q_2, q_3]] - [[q_1, q_2]\,, q_3] - [q_2, [q_1, q_3]] = 0,$$

for all $q_1, q_2, q_3 \in \Gamma(Q)$.

Given a skew-symmetric dull algebroid Q, there is an associated operator d_Q of degree 1 on the space of Q-forms $\Omega^\bullet(Q) = \Gamma(\wedge^\bullet Q^*)$, defined by the formula

$$\begin{aligned}\mathrm{d}_Q\tau\,(q_1, \ldots, q_{k+1}) = &\sum_{i<j}(-1)^{i+j}\tau\left(\left[q_i, q_j\right], q_1, \ldots, \hat{q}_i, \ldots, \hat{q}_j, \ldots, q_{k+1}\right)\\ &+\sum_i(-1)^{i+1}\rho_Q(q_i)\left(\tau\left(q_1, \ldots, \hat{q}_i, \ldots, q_{k+1}\right)\right),\end{aligned}$$

for $\tau \in \Omega^k(Q)$ and $q_1, \ldots, q_{k+1} \in \Gamma(Q)$; the notation $\hat{q}$ means that q is omitted. The operator d_Q satisfies as usual

$$\mathrm{d}_Q\,(\tau_1 \wedge \tau_2) = \left(\mathrm{d}_Q\tau_1\right) \wedge \tau_2 + (-1)^{|\tau_1|}\tau_1 \wedge \mathrm{d}_Q\tau_2,$$

for $\tau_1, \tau_2 \in \Omega^\bullet(Q)$. In general, the operator d_Q squares to zero only on 0-forms $f \in \Omega^0(M) = C^\infty(M)$, since $\mathrm{d}_Q^2 f = 0$ for all $f \in C^\infty(M)$ is equivalent to the

[2] If one chose to tensor from the left, the resulting complex would still have i-th component $\underline{E}[k]^i = \underline{E}^{i+k}$, but the Leibniz rule would give the differential $\partial[k] = (-1)^k\partial$.

[3] This becomes $\phi \circ \partial = (-1)^k\partial \circ \phi$ for the other convention.

compatibility of the anchor with the bracket (1). The identity $\mathrm{d}_Q^2 = 0$ on all forms is equivalent to $(Q, \rho_Q, [\cdot\,,\cdot])$ being a Lie algebroid.

2.3 Basic connections and basic curvature

Let $Q \to M$ be a skew-symmetric dull algebroid and $E \to M$ another vector bundle. A Q-connection on E is defined similarly as usual, as a map $\nabla\colon \Gamma(Q) \times \Gamma(E) \to \Gamma(E)$, $(q, e) \mapsto \nabla_q e$ that is $C^\infty(M)$-linear in the first argument and satisfies

$$\nabla_q(fe) = \pounds_{\rho_Q(q)} f \cdot e + f \nabla_q e,$$

for all $q \in \Gamma(Q), e \in \Gamma(E)$ and $f \in C^\infty(M)$. The dual connection ∇^* is the Q-connection on E^* defined by the formula

$$\left\langle \nabla_q^* \varepsilon, e \right\rangle = \pounds_{\rho_Q(q)} \langle \varepsilon, e\rangle - \left\langle \varepsilon, \nabla_q e \right\rangle,$$

for all $\varepsilon \in \Gamma(E^*)$, $e \in \Gamma(E)$ and $q \in \Gamma(Q)$, where $\langle \cdot\,,\cdot\rangle$ is the natural pairing between E and its dual E^*.

A Q-connection on a graded vector bundle $(\underline{E} = \bigoplus_{n\in\mathbb{Z}} E_n[n], \partial)$ is a family of Q-connections ∇^n, $n \in \mathbb{N}$, on each of the bundles E_n. If $\underline{E}$ is a complex with differential ∂, then the Q-connection is *a connection on the complex* $(\underline{E}, \partial)$ if it commutes with ∂, i.e. $\partial(\nabla_q^n e) = \nabla_q^{n-1}(\partial e)$ for $q \in \Gamma(Q)$ and $e \in \Gamma(E_n)$.

The curvature R_∇ of a Q-connection on a vector bundle E is defined by

$$R_\nabla(q_1, q_2)e = \nabla_{q_1}\nabla_{q_2} e - \nabla_{q_2}\nabla_{q_1} e - \nabla_{[q_1,q_2]} e,$$

for all $q_1, q_2 \in \Gamma(Q)$ and $e \in \Gamma(E)$, and generally, it is an element of $\Gamma(Q^* \otimes Q^* \otimes E^* \otimes E)$. In this situation (where we are assuming Q is skew-symmetric), the curvature is a 2-form with values in the endomorphism bundle $\mathrm{End}(E) = E^* \otimes E$, i.e. $R_\nabla \in \Omega^2(Q, \mathrm{End}(E))$. A connection is called as usual *flat* if its curvature R_∇ vanishes identically.

A Q-connection ∇ on E induces an operator d_∇ on the space of E-valued Q-forms $\Omega^\bullet(Q, E) = \Omega^\bullet(Q) \otimes_{C^\infty(M)} \Gamma(E)$ given by the usual Koszul formula

$$\begin{aligned} \mathrm{d}_\nabla \tau\,(q_1, \ldots, q_{k+1}) = &\sum_{i<j} (-1)^{i+j} \tau\left(\left[q_i, q_j\right], q_1, \ldots, \hat{q_i}, \ldots, \hat{q_j}, \ldots, q_{k+1}\right) \\ &+ \sum_i (-1)^{i+1} \nabla_{q_i}\left(\tau\left(q_1, \ldots, \hat{q_i}, \ldots, q_{k+1}\right)\right), \end{aligned}$$

for all $\tau \in \Omega^k(Q, E)$ and $q_1, \ldots, q_{k+1} \in \Gamma(Q)$. It satisfies

$$\mathrm{d}_\nabla\,(\tau_1 \wedge \tau_2) = \mathrm{d}_Q \tau_1 \wedge \tau_2 + (-1)^k \tau_1 \wedge \mathrm{d}_\nabla \tau_2,$$

for all $\tau_1 \in \Omega^k(Q)$ and $\tau_2 \in \Omega^\bullet(Q, E)$, and squares to zero if and only if Q is a Lie algebroid and ∇ is flat.

Suppose that $\nabla\colon \mathfrak{X}(M)\times\Gamma(Q)\to\Gamma(Q)$ is a TM-connection on the vector bundle Q. The induced *basic connections* on Q and TM are defined similarly as the ones associated to Lie algebroids [2, 18]:

$$\nabla^{\mathrm{bas}}=\nabla^{\mathrm{bas},Q}\colon \Gamma(Q)\times\Gamma(Q)\to\Gamma(Q),\ \nabla^{\mathrm{bas}}_{q_1}q_2=[q_1,q_2]+\nabla_{\rho_Q(q_2)}q_1$$

and

$$\nabla^{\mathrm{bas}}=\nabla^{\mathrm{bas},TM}\colon \Gamma(Q)\times\mathfrak{X}(M)\to\mathfrak{X}(M),\ \nabla^{\mathrm{bas}}_{q}X=\left[\rho_Q(q),X\right]+\rho_Q(\nabla_X q).$$

The basic curvature is the form $R^{\mathrm{bas}}_{\nabla}\in\Omega^2(Q,\mathrm{Hom}(TM,Q))$ defined by

$$R^{\mathrm{bas}}_{\nabla}(q_1,q_2)X=-\nabla_X[q_1,q_2]+[q_1,\nabla_X q_2]+[\nabla_X q_1,q_2]+\nabla_{\nabla^{\mathrm{bas}}_{q_2}X}q_1-\nabla_{\nabla^{\mathrm{bas}}_{q_1}X}q_2.$$

Simple computations show that the basic connections and the basic curvature satisfy

$$\nabla^{\mathrm{bas},TM}\circ\rho_Q=\rho_Q\circ\nabla^{\mathrm{bas},Q}, \tag{2}$$
$$\rho_Q\circ R^{\mathrm{bas}}_{\nabla}=R_{\nabla^{\mathrm{bas},TM}}, \tag{3}$$
$$R^{\mathrm{bas}}_{\nabla}\circ\rho_Q+\mathrm{Jac}_{[\cdot\,,\cdot]}=R_{\nabla^{\mathrm{bas},Q}}. \tag{4}$$

3 (Split) Lie n-algebroids and $\mathbb{N}Q$-manifolds

This section recalls basic results about $\mathbb{N}$-manifolds and Lie n-algebroids (based on [24]), and describes the Weil algebra of a Lie n-algebroid for general n (see [34] for $n=1$). It focuses on the category of *split* $\mathbb{N}$*-manifolds*, which is equivalent to the category of $\mathbb{N}$-manifolds ([7, 40]).

3.1 (Split) $\mathbb{N}$-manifolds and homological vector fields

Graded manifolds of degree $n\in\mathbb{N}$ are defined as follows, in terms of sheaves over ordinary smooth manifolds.

Definition 3.1 An $\mathbb{N}$*-manifold* $\mathcal{M}$ *of degree* n *and dimension* $(m;r_1,\ldots,r_n)$ is a sheaf $C^\infty(\mathcal{M})$ of $\mathbb{N}$-graded, graded commutative, associative, unital $C^\infty(M)$-algebras over a smooth m-dimensional manifold M, which is locally freely generated by $r_1+\ldots+r_n$ elements $\xi_1^1,\ldots,\xi_1^{r_1},\xi_2^1,\ldots,\xi_2^{r_2},\ldots,\xi_n^1,\ldots,\xi_n^{r_n}$ with ξ_i^j of degree i for $i\in\{1,\ldots,n\}$ and $j\in\{1,\ldots,r_i\}$.

A morphism of $\mathbb{N}$-manifolds $\mu\colon\mathcal{N}\to\mathcal{M}$ over a smooth map $\mu_0\colon N\to M$ of the underlying smooth manifolds is a morphism of sheaves of graded algebras $\mu^\star\colon C^\infty(\mathcal{M})\to C^\infty(\mathcal{N})$ over $\mu_0^*\colon C^\infty(M)\to C^\infty(N)$.

For short, "$[n]$-manifold" means "$\mathbb{N}$-manifold of degree n". The degree of a (degree-) homogeneous element $\xi \in \mathcal{C}^\infty(\mathcal{M})$ is written $|\xi|$. Note that the degree 0 elements of $\mathcal{C}^\infty(\mathcal{M})$ are just the smooth functions of the manifold M. By definition, a $[1]$-manifold $\mathcal{M}$ is a locally free and finitely generated sheaf $\mathcal{C}^\infty(\mathcal{M})$ of $C^\infty(M)$-modules. That is, $\mathcal{C}^\infty(\mathcal{M}) = \Gamma(\wedge E^*)$ for a vector bundle $E \to M$. In that case, $\mathcal{M} =: E[1]$. *Recall that this means that the elements of E have degree -1, and so the sections of E^* have degree* 1.

Consider now a (non-graded) vector bundle E of rank r over the smooth manifold M of dimension m. Similarly as before, assigning the degree n to the fibre coordinates of E defines an $[n]$-manifold of dimension $(m; r_1 = 0, \dots, r_{n-1} = 0, r_n = r)$ denoted by $E[n]$, with $\mathcal{C}^\infty(E[n])^n = \Gamma(E^*)$. More generally, let $E_1, \dots, E_n$ be vector bundles of ranks $r_1, \dots, r_n$, respectively, and assign the degree i to the fibre coordinates of E_i, for each $i = 1, \dots, n$. The direct sum $\underline{E} = E_1[1] \oplus \cdots \oplus E_n[n]$ is a graded vector bundle with grading concentrated in degrees $-1, \dots, -n$. When seen as an $[n]$-manifold, $E_1[1] \oplus \cdots \oplus E_n[n]$ has the local basis of sections of E_i^* as local generators of degree i and thus its dimension is $(m; r_1, \dots, r_n)$.

Definition 3.2 An $[n]$-manifold of the form $E_1[1] \oplus \cdots \oplus E_n[n]$ as above is called a *split* $[n]$*-manifold.*

The relation between $[n]$-manifolds and split $[n]$-manifolds is explained by the following theorem, which is implicit in [40] and explicitly proved in [7].

Theorem 3.3 *Any* $[n]$*-manifold is non-canonically diffeomorphic to a split* $[n]$*-manifold.*

Note that under the above correspondence, the structure sheaf of an $[n]$-manifold $\mathcal{M} \simeq \underline{E} = E_1[1] \oplus \cdots \oplus E_n[n]$ becomes

$$\mathcal{C}^\infty(\mathcal{M}) \simeq \Gamma\left(\underline{S}\left(\underline{E}^*\right)\right),$$

and a different choice of splitting leaves the bundles unchanged, up to isomorphism. In particular, for the case of a split $[2]$-manifold $\mathcal{M} = \underline{E} = E_1[1] \oplus E_2[2]$ the graded functions are

$$\mathcal{C}^\infty(\mathcal{M}) = \Gamma\left(\underline{S}\left(\underline{E}^*\right)\right) = \Gamma\left(\wedge E_1^* \otimes S E_2^*\right),$$

where the grading is defined such that

$$\mathcal{C}^\infty(\mathcal{M})^i = \bigoplus_{k+2\ell=i} \Gamma\left(\wedge^k E_1^* \otimes S^\ell E_2^*\right).$$

Using the language of graded derivations, the usual notion of vector field can be generalized to a notion of vector field on an $[n]$-manifold $\mathcal{M}$.

Definition 3.4 A vector field of degree j on $\mathcal{M}$ is a degree j (graded) derivation of $\mathcal{C}^\infty(\mathcal{M})$, i.e. a map $\mathcal{X}\colon \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})$ such that $|\mathcal{X}(\xi)| = j + |\xi|$ and $\mathcal{X}(\xi\zeta) = \mathcal{X}(\xi)\zeta + (-1)^{j|\xi|}\xi\mathcal{X}(\zeta)$, for homogeneous elements $\xi, \zeta \in \mathcal{C}^\infty(\mathcal{M})$.

As usual, $|\mathcal{X}|$ denotes the degree of a homogeneous vector field $\mathcal{X}$. The Lie bracket of two vector fields $\mathcal{X}, \mathcal{Y}$ on $\mathcal{M}$ is the graded commutator

$$[\mathcal{X}, \mathcal{Y}] = \mathcal{X}\mathcal{Y} - (-1)^{|\mathcal{X}||\mathcal{Y}|}\mathcal{Y}\mathcal{X}.$$

The following relations hold:

(i) $[\mathcal{X}, \mathcal{Y}] = -(-1)^{|\mathcal{X}||\mathcal{Y}|}[\mathcal{Y}, \mathcal{X}]$,
(ii) $[\mathcal{X}, \xi\mathcal{Y}] = \mathcal{X}(\xi)\mathcal{Y} + (-1)^{|\mathcal{X}||\xi|}\xi[\mathcal{X}, \mathcal{Y}]$,
(iii) $(-1)^{|\mathcal{X}||\mathcal{Z}|}[\mathcal{X}, [\mathcal{Y}, \mathcal{Z}]] + (-1)^{|\mathcal{Y}||\mathcal{X}|}[\mathcal{Y}, [\mathcal{Z}, \mathcal{X}]] + (-1)^{|\mathcal{Z}||\mathcal{Y}|}[\mathcal{Z}, [\mathcal{X}, Y]] = 0$,

for $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ homogeneous vector fields on $\mathcal{M}$, and ξ a homogeneous element of $\mathcal{C}^\infty(\mathcal{M})$.

Local generators ξ_i^j of $\mathcal{C}^\infty(\mathcal{M})$ over an open set $U \subseteq M$ given by the definition of $\mathcal{M}$ define the (local) vector fields $\partial_{\xi_i^j}$ of degree $-j$, which sends ξ_i^j to 1 and the other local generators to 0. The sheaf $\underline{\mathrm{Der}}_U(\mathcal{C}^\infty(\mathcal{M}))$ of graded derivations of $\mathcal{C}_U^\infty(\mathcal{M})$ is freely generated as a $\mathcal{C}_U^\infty(\mathcal{M})$-module by ∂_{x_k} and $\partial_{\xi_i^j}$, where $x_1, \ldots, x_m$ are coordinates for M defined on U.

Note that in the case of a split $[n]$-manifold $E_1[1] \oplus \cdots \oplus E_n[n]$, each section $e \in \Gamma(E_j)$ defines a derivation $\hat{e}$ of degree $-j$ on $\mathcal{M}$ by the relations: $\hat{e}(f) = 0$ for $f \in \mathcal{C}^\infty(M)$, $\hat{e}(\varepsilon) = \langle \varepsilon, e\rangle$ for $\varepsilon \in \Gamma(E_j^*)$ and $\hat{e}(\varepsilon) = 0$ for $\varepsilon \in \Gamma(\underline{E}^*)$ with $|\varepsilon| \neq j$. In particular, $\widehat{e_j^i} = \partial_{\varepsilon_j^i}$ for $\{e_j^i\}$ a local basis of E_j and $\{\varepsilon_j^i\}$ the dual basis of E_j^*.

Given TM-connections $\nabla^i : \mathfrak{X}(M) \to \mathrm{Der}(E_i)$ for all i, the space of vector fields over a split $[n]$-manifold $\mathcal{M}$ is generated as a $\mathcal{C}^\infty(\mathcal{M})$-module by

$$\left\{\nabla_X^1 \oplus \cdots \oplus \nabla_X^n \mid X \in \mathfrak{X}(M)\right\} \cup \left\{\hat{e} \mid e \in \Gamma\left(E_i\right) \text{ for some } i\right\}.$$

The vector fields of the form $\nabla_X^1 \oplus \cdots \oplus \nabla_X^n$ are of degree 0 and are understood to send $f \in C^\infty(M)$ to $X(f) \in C^\infty(M)$, and $\varepsilon \in \Gamma(E_i^*)$ to $\nabla_X^{i,*}\varepsilon \in \Gamma(E_i^*)$. The negative degree vector fields are generated by those of the form $\hat{e}$.

Definition 3.5 A homological vector field $\mathcal{Q}$ on an $[n]$-manifold $\mathcal{M}$ is a degree 1 derivation of $\mathcal{C}^\infty(\mathcal{M})$ such that $\mathcal{Q}^2 = \frac{1}{2}[\mathcal{Q}, \mathcal{Q}] = 0$.

A homological vector field on a [1]-manifold $\mathcal{M} = E[1]$ is a differential d_E associated to a Lie algebroid structure on the vector bundle E over M [45]. The following definition generalizes this to arbitrary degrees.

Definition 3.6 A *Lie n-algebroid* is an $[n]$-manifold $\mathcal{M}$ endowed with a homological vector field $\mathcal{Q}$—the pair $(\mathcal{M}, \mathcal{Q})$ is also called *$\mathbb{N}\mathcal{Q}$-manifold of degree n*. A *split Lie n-algebroid* is a split $[n]$-manifold $\mathcal{M}$ endowed with a homological vector field $\mathcal{Q}$. A *morphism of (split) Lie n-algebroids* is a morphism μ of the underlying $[n]$-manifolds such that $\mu^\star$ commutes with the homological vector fields.

The homological vector field associated to a split Lie n-algebroid $\underline{A} = A_1[1] \oplus \cdots \oplus A_n[n] \to M$ can be equivalently described by a family of brackets which satisfy some Leibniz and higher Jacobi identities [43]. More precisely, a homological vector field on $\underline{A}$ is equivalent to an L_∞-algebra structure[4] on $\Gamma(\underline{A})$ that is anchored by a vector bundle morphism $\rho\colon A_1 \to TM$. Such a structure is given by multibrackets $[\![\cdot, \ldots, \cdot]\!]_i \colon \Gamma(\underline{A})^i \to \Gamma(\underline{A})$ of degree 1 for $1 \le i \le n+1$ such that

(1) $[\![\cdot\,,\cdot]\!]_2$ satisfies the Leibniz identity with respect to ρ,
(2) $[\![\cdot, \ldots, \cdot]\!]_i$ is $C^\infty(M)$-linear in each entry for all $i \neq 2$,
(3) *(graded skew symmetry)* each $[\![\cdot, \ldots, \cdot]\!]_i$ is graded alternating: for a permutation $\sigma \in S_i$ and for all $a_1, \ldots, a_i \in \Gamma(\underline{A})$ degree-homogeneous sections

$$\left[\!\!\left[a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(k)}\right]\!\!\right]_i = \mathrm{Ksgn}\,(\sigma, a_1, \ldots, a_k) \cdot [\![a_1, a_2, \ldots, a_k]\!]_i\,,$$

and
(4) *(strong homotopy Jacobi identity)* for $k \in \mathbb{N}$ and $a_1, \ldots, a_k \in \Gamma(\underline{A})$ sections of homogeneous degree:

$$\sum_{i+j=k+1} (-1)^{i(j-1)} \sum_{\sigma \in \mathrm{Sh}_{i,k-i}} \mathrm{Ksgn}(\sigma, a_1, \ldots, a_k)[\![[\![a_{\sigma(1)}, \ldots, a_{\sigma(i)}]\!]_i, a_{\sigma(i+1)}, \ldots, a_{\sigma(k)}]\!]_j = 0.$$

Here, $\mathrm{Sh}_{i,k-i}$ is the set of all $(i, k-i)$-shuffles[5] and $\mathrm{Ksgn}(\sigma, a_1, \ldots, a_k)$ is the $(a_1, \ldots, a_k)$-graded signature of the permutation $\sigma \in S_k$, i.e.

$$a_1 \wedge \ldots \wedge a_k = \mathrm{Ksgn}(\sigma, a_1, \ldots, a_k) a_{\sigma(1)} \wedge \ldots \wedge a_{\sigma(k)}.$$

This gives the following alternative geometric description of a split Lie 2-algebroid $(\mathcal{M} = A_1[1] \oplus A_2[2], \mathcal{Q})$, see [25]. For consistency with the notation in [25], set $A_1 := Q$ and $A_2^* =: B$.

Definition 3.7 A split Lie 2-algebroid $Q[1] \oplus B^*[2]$ is given by a pair of an anchored vector bundle $(Q \to M, \rho_Q)$ and a vector bundle $B \to M$, together with a vector bundle map $\ell\colon B^* \to Q$, a skew-symmetric dull bracket $[\cdot\,,\cdot]\colon \Gamma(Q) \times \Gamma(Q) \to \Gamma(Q)$, a linear Q-connection ∇ on B, and a vector valued 3-form $\omega \in \Omega^3(Q, B^*)$ such that

(i) $\nabla^*_{\ell(\beta_1)}\beta_2 + \nabla^*_{\ell(\beta_2)}\beta_1 = 0$, for all $\beta_1, \beta_2 \in \Gamma(B^*)$,
(ii) $[q, \ell(\beta)] = \ell(\nabla^*_q \beta)$ for all $q \in \Gamma(Q)$ and $\beta \in \Gamma(B^*)$,
(iii) $\mathrm{Jac}_{[\cdot\,,\cdot]} = \ell \circ \omega \in \Omega^3(Q, Q)$,
(iv) $R_{\nabla^*}(q_1, q_2)\beta = -\omega(q_1, q_2, \ell(\beta))$ for $q_1, q_2 \in \Gamma(Q)$ and $\beta \in \Gamma(B^*)$,
(v) $\mathrm{d}_{\nabla^*}\omega = 0$.

[4] We note that the sign convention agrees with, e.g. [28, 49]. In [41], the term "$L_\infty[1]$-algebra" was used for brackets with this sign convention.

[5] A $(i, k-i)$-shuffle is an element $\sigma \in S_k$ such that $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(k)$.

To pass from the definition above to the homological vector field $\mathcal{Q}$, set $\mathcal{Q}(f) = \rho^* \mathrm{d}f \in \Gamma(Q^*)$, $\mathcal{Q}(\tau) = \mathrm{d}_Q\tau + \partial_B\tau \in \Omega^2(Q) \oplus \Gamma(B)$, and $\mathcal{Q}(b) = \mathrm{d}_\nabla b - \langle \omega, b\rangle \in \Omega^1(Q, B) \oplus \Omega^3(Q)$ for $f \in C^\infty(M)$, $\tau \in \Omega(Q)$ and $b \in \Gamma(B)$, where $\partial_B := \ell^*$.

On the other hand we may obtain the data of Definition 3.7 from a given homological vector field $\mathcal{Q}$ as follows. Define the vector bundle map ℓ to be the 1-bracket and ρ to be the anchor. The 2-bracket induces the skew-symmetric dull bracket on Q and the Q-connection on B^* via the formula

$$[\![q_1 \oplus \beta_1, q_2 \oplus \beta_2]\!]_2 = [q_1, q_2]_Q \oplus \left(\nabla^*_{q_1}\beta_2 - \nabla^*_{q_2}\beta_1\right).$$

Finally, the 3-bracket induces the 3-form ω via the formula

$$[\![q_1 \oplus 0, q_2 \oplus 0, q_3 \oplus 0]\!]_3 = 0 \oplus \omega\,(q_1, q_2, q_3)\,.$$

Example 3.8 (Lie 2-algebras) If we consider a Lie 2-algebroid over a point, then we recover the notion of *Lie 2-algebra* [6]. Specifically, a Lie 2-algebroid over a point consists of a pair of vector spaces $\mathfrak{g}_0, \mathfrak{g}_1$, a linear map $\ell\colon \mathfrak{g}_0 \to \mathfrak{g}_1$, a skew-symmetric bilinear bracket $[\cdot\,,\cdot]\colon \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_1$, a bilinear *action bracket* $[\cdot\,,\cdot]\colon \mathfrak{g}_1 \times \mathfrak{g}_0 \to \mathfrak{g}_0$, and an alternating trilinear bracket $[\cdot\,,\cdot\,,\cdot]\colon \mathfrak{g}_1 \times \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$ such that

(1) $[\ell(x), y] + [\ell(y), x] = 0$ for $x, y \in \mathfrak{g}_0$,
(2) $[x, \ell(y)] = \ell([x, y])$ for $x \in \mathfrak{g}_1$ and $y \in \mathfrak{g}_0$,
(3) $\mathrm{Jac}_{[\cdot,\cdot]}(x, y, z) = \ell([x, y, z])$ for $x, y, z \in \mathfrak{g}_1$,
(4) $[[x, y], z] + [y, [x, z]] - [x, [y, z]] = [x, y, \ell(z)]$ for $x, y \in \mathfrak{g}_1$ and $z \in \mathfrak{g}_0$,
(5) and the higher Jacobi identity

$$\begin{aligned} 0 = {} & [x, [y, z, w]] - [y, [x, z, w]] + [z, [x, y, w]] - [w, [x, y, z]] - [[x, y], z, w] \\ & + [[x, z], y, w] - [[x, w], y, z] - [[y, z], x, w] + [[y, w], x, z] - [[z, w], x, y]\,. \end{aligned}$$

holds for $x, y, z, w \in \mathfrak{g}_1$.

Example 3.9 (Derivation Lie 2-algebr(oid)) For any Lie algebra $(\mathfrak{g}, [\cdot\,,\cdot]_\mathfrak{g})$, the derivation Lie 2-algebra is defined as the complex

$$\mathrm{ad}\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$$

with brackets given by $[\delta_1, \delta_2] = \delta_1\delta_2 - \delta_2\delta_1$, $[\delta, x] = \delta x$, $[\delta_1, \delta_2, \delta_3] = 0$ for all $\delta, \delta_i \in \mathrm{Der}(\mathfrak{g})$, $i = 1, 2, 3$, and $x \in \mathfrak{g}$.

A global analogue of this construction can be achieved only under strong assumptions on the Lie algebroid $A \to M$. Precisely, let $A \to M$ be a Lie algebra bundle. Then the space of all derivations D of the vector bundle A which preserve the bracket

$$D[a_1, a_2] = [Da_1, a_2] + [a_1, Da_2]$$

is the module of sections of a vector bundle over M, denoted $\mathrm{Der}_{[\cdot,\cdot]}(A) \to M$. Together with the usual commutator bracket and the anchor $\rho'(D) = X$, where D

is a derivation of $\Gamma(A)$ covering $X \in \mathfrak{X}(M)$, the vector bundle $\mathrm{Der}_{[\cdot,\cdot]}(A)$ is a Lie algebroid over M [30]. Since the anchor of A is trivial, the complex

$$A \xrightarrow{\mathrm{ad}} \mathrm{Der}_{[\cdot,\cdot]}(A) \xrightarrow{\rho'} TM$$

becomes a Lie 2-algebroid with $\mathrm{Der}_{[\cdot,\cdot]}(A)$-connection on A given by $\nabla_D a = Da$ and $\omega = 0$.

Example 3.10 (Courant algebroids) Let $E \to M$ be a Courant algebroid with pairing $\langle\cdot,\cdot\rangle\colon E \times_M E \to E$, anchor ρ and Dorfman bracket $[\![\cdot,\cdot]\!]$, and choose a metric linear connection $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$. Then $E[1] \oplus T^*M[2]$ becomes as follows a split Lie 2-algebroid. The skew-symmetric dull bracket is given by $[e, e'] = [\![e, e']\!] - \rho^*\langle \nabla_{\cdot} e, e'\rangle$ for all $e, e' \in \Gamma(E)$. The *basic connection* is $\nabla^{\mathrm{bas}}\colon \Gamma(E) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, $\nabla^{\mathrm{bas}}_e X = [\rho(e), X] + \rho(\nabla_X e)$, and the *basic curvature* is given by $\omega_\nabla \in \Omega^2(E, \mathrm{Hom}(TM, E))$

$$\omega_\nabla\left(e, e'\right) X = -\nabla_X \left[\!\left[e, e'\right]\!\right] + \left[\!\left[\nabla_X e, e'\right]\!\right] + \left[\!\left[e, \nabla_X e'\right]\!\right] + \nabla_{\nabla^{\mathrm{bas}}_{e'} X} e - \nabla_{\nabla^{\mathrm{bas}}_{e} X} e' - P^{-1}\left\langle \nabla_{\nabla^{\mathrm{bas}}_{\cdot} X} e, e'\right\rangle,$$

where $P\colon E \to E^*$ is the isomorphism defined by the pairing, for all $e, e' \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$. The map ℓ is $\rho^*\colon T^*M \to E$, the E-connection on T^*M is $\nabla^{\mathrm{bas},*}$ and the form $\omega \in \Omega^3(E, T^*M)$ is given by $\omega(e_1, e_2, e_3) = \langle \omega_\nabla(e_1, e_2)(\cdot), e_3\rangle$. The kind of split Lie 2-algebroids that arise in this way are the split symplectic Lie 2-algebroids [40]. They are splittings of the symplectic Lie 2-algebroid which is equivalent to the tangent prolongation of E, which is an LA-Courant algebroid [25, 26].

3.2 Generalized functions of a Lie *n*-algebroid

In the following, $(\mathcal{M}, \mathcal{Q})$ is a Lie n-algebroid with underlying manifold M. Consider the space $C^\infty(\mathcal{M}) \otimes_{C^\infty(M)} \Gamma(\underline{E})$ for a graded vector bundle $\underline{E} \to M$ of finite rank. For simplicity, $C^\infty(\mathcal{M}) \otimes_{C^\infty(M)} \Gamma(\underline{E})$ is sometimes written $C^\infty(\mathcal{M}) \otimes \Gamma(\underline{E})$. That is, these tensor products in the rest of the paper are always of $C^\infty(M)$-modules.

First suppose that $(\mathcal{M}, \mathcal{Q}) = (A[1], \mathrm{d}_A)$ is a Lie algebroid. The space of $\underline{E}$-valued differential forms $\Omega(A; \underline{E}) := \Omega(A) \otimes_{C^\infty(M)} \Gamma(\underline{E}) = C^\infty(A[1]) \otimes_{C^\infty(M)} \Gamma(\underline{E})$ has a natural grading given by

$$\Omega\left(A; \underline{E}\right)_p = \bigoplus_{i-j=p} \Omega^i\left(A; E_j\right).$$

It is well-known (see [2]) that any degree preserving vector bundle map $h\colon \underline{E} \otimes \underline{F} \to \underline{G}$ induces a wedge product operation

$$(\cdot \wedge_h \cdot)\colon \Omega(A; \underline{E}) \times \Omega(A; \underline{F}) \to \Omega(A; \underline{G})$$

which is defined on $\omega \in \Omega^p(A; E_i)$ and $\eta \in \Omega^q(A; F_j)$ by

$$(\omega \wedge_h \eta)\left(a_1, \ldots, a_{p+q}\right) = \sum_{\sigma \in \mathrm{Sh}_{p,q}} (-1)^{qi} \operatorname{sgn}(\sigma) h\left(\omega\left(a_{\sigma(1)}, \ldots, a_{\sigma(p)}\right), \eta\left(a_{\sigma(p+1)}, \ldots, a_{\sigma(p+q)}\right)\right),$$

for all $a_1, \ldots, a_{p+q} \in \Gamma(A)$.

In particular, the above rule reads

$$\theta \wedge_h \zeta = (-1)^{qi} (\omega \wedge \eta) \otimes h(e, f),$$

for all $\theta = \omega \otimes e$ and $\zeta = \eta \otimes f$ where ω is a p-form, η is a q-form, and e and f are homogeneous sections of $\underline{E}$ and $\underline{F}$ of degree i and j, respectively.

Some notable cases for special choices of the map h are given by the identity, the composition of endomorphisms, the evaluation and the 'twisted' evaluation maps, the graded commutator of endomorphisms and the natural pairing of a graded vector bundle with its dual. In particular, the evaluation $(\Phi, e) \mapsto \Phi(e)$ and the twisted evaluation $(e, \Phi) \mapsto (-1)^{|\Phi||e|}\Phi(e)$ make $\Omega(A; \underline{E})$ a graded $\Omega(A; \underline{\mathrm{End}}(\underline{E}))$-bimodule.

In the general case of a Lie n-algebroid $(\mathcal{M}, \mathcal{Q})$, the space $\Omega(A)$ is replaced by the generalized smooth functions $C^\infty(\mathcal{M})$ of $\mathcal{M}$. The space $C^\infty(\mathcal{M}) \otimes_{C^\infty(M)} \Gamma(\underline{E})$ has a natural grading, where the homogeneous elements of degree p are given by

$$\bigoplus_{i-j=p} C^\infty(\mathcal{M})^i \otimes \Gamma\left(E_j\right).$$

Similarly as in the case of a Lie algebroid, given a degree preserving map

$$h \colon \underline{E} \otimes \underline{F} \to \underline{G},$$

one obtains the multiplication

$$\begin{aligned}\left(C^\infty(\mathcal{M}) \otimes \Gamma(\underline{E})\right) \times \left(C^\infty(\mathcal{M}) \otimes \Gamma(\underline{F})\right) &\to C^\infty(\mathcal{M}) \otimes \Gamma(\underline{G}) \\ (\omega, \eta) &\mapsto \omega \wedge_h \eta.\end{aligned}$$

In particular, for elements of the form $\xi \otimes e \in C^\infty(\mathcal{M})^i \otimes \Gamma(E_j)$, $\zeta \otimes f \in C^\infty(\mathcal{M})^k \otimes \Gamma(F_\ell)$ the above rule reads

$$(\xi \otimes e) \wedge_h (\zeta \otimes f) = (-1)^{(-j)k} \xi\zeta \otimes h(e, f),$$

where on the right hand side the multiplication $\xi\zeta$ is the one in $C^\infty(\mathcal{M})$. The special cases above are defined similarly for the n-algebroid case. Moreover, $C^\infty(\mathcal{M}) \otimes_{C^\infty(M)} \Gamma(\underline{E})$ is endowed with the structure of a graded $C^\infty(\mathcal{M}) \otimes_{C^\infty(M)} \Gamma(\underline{\mathrm{End}}(\underline{E}))$-bimodule.

Finally, the following fact will be useful later as it is a generalisation of [2, Lemma A.1], and gives the connection between the space $C^\infty(\mathcal{M}) \otimes \Gamma(\underline{\mathrm{Hom}}(\underline{E}, \underline{F}))$ and the homomorphisms from $C^\infty(\mathcal{M}) \otimes \Gamma(\underline{E})$ to $C^\infty(\mathcal{M}) \otimes \Gamma(\underline{F})$.

There is a 1-1 correspondence between the degree n elements of $\mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{\mathrm{Hom}}(\underline{E}, \underline{F}))$ and the operators $\Psi\colon \mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{E}) \to \mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{F})$ of degree n which are $\mathcal{C}^\infty(\mathcal{M})$-linear in the graded sense:

$$\Psi(\xi \wedge \eta) = (-1)^{nk}\xi \wedge \Psi(\eta),$$

for all $\xi \in \mathcal{C}^\infty(\mathcal{M})^k$, and all $\eta \in \mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{E})$. The element $\Phi \in \mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{\mathrm{End}}(\underline{E}))$ induces the operator $\widehat{\Phi}$ given by left multiplication by Φ:

$$\widehat{\Phi}(\eta) = \Phi \wedge \eta.$$

This clearly satisfies $\widehat{\Phi}(\xi \wedge \eta) = (-1)^{nk}\xi \wedge \widehat{\Phi}(\eta)$, for all $\xi \in \mathcal{C}^\infty(\mathcal{M})^k$, $\eta \in \mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{E})$. Conversely, an operator Ψ of degree n must send a section $e \in \Gamma(E_k)$ into the sum

$$\Gamma(F_{k-n}) \oplus \left(\mathcal{C}^\infty(\mathcal{M})^1 \otimes \Gamma\left(F_{k-n+1}\right)\right) \oplus \left(\mathcal{C}^\infty(\mathcal{M})^2 \otimes \Gamma\left(F_{k-n+2}\right)\right) \oplus \ldots,$$

defining the elements

$$\Psi_i \in \mathcal{C}^\infty(\mathcal{M})^i \otimes \Gamma\left(\underline{\mathrm{Hom}}^{n-i}(\underline{E}, \underline{F})\right).$$

Thus, this yields the (finite) sum $\widetilde{\Psi} = \sum \Psi_i \in \left(\mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma\left(\underline{\mathrm{Hom}}\left(\underline{E}, \underline{F}\right)\right)\right)^n$. Clearly,

$$\widetilde{\widehat{\Phi}} = \Phi \text{ and } \widehat{\widetilde{\Psi}} = \Psi.$$

□

Schematically, for a Lie n-algebroid $\mathcal{M}$, the above discussion gives the following diagram:

$$\left(\mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{\mathrm{Hom}}(\underline{E}, \underline{F}))\right)^n \overset{1\text{-}1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Degree } n \text{ operators } \Psi \\ \mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{E}) \to \mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{F}) \\ \text{which are } \mathcal{C}^\infty(\mathcal{M})\text{-linear in the graded sense} \end{array} \right\}$$

In particular, if $\underline{E} = \underline{F}$, then

$$\left(\mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{\mathrm{End}}(\underline{E}))\right)^n \overset{1\text{-}1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Degree } n \text{ operators } \Psi \text{ on } \mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{E}) \text{ which} \\ \text{are } \mathcal{C}^\infty(\mathcal{M})\text{-linear in the graded sense} \end{array} \right\}$$

3.3 The Weil algebra associated to a Lie *n*-algebroid

Let $\mathcal{M}$ be an $[n]$-manifold over a smooth manifold M and $\xi_1^1, \ldots, \xi_1^{r_1}, \xi_2^1, \ldots, \xi_2^{r_2}, \ldots, \xi_n^1, \ldots, \xi_n^{r_n}$ be its local generators over some open $U \subset M$ with degrees $1, 2, \ldots, n$, respectively. By definition, the tangent bundle $T\mathcal{M}$ of $\mathcal{M}$ is an $[n]$-manifold over TM [33, 34], whose local generators over $TU \subset TM$ are given by

$$\mathcal{C}^{\infty}_{TU}(T\mathcal{M})^0 = C^{\infty}(TU) \text{ and } \xi_i^1, \ldots, \xi_i^{r_i}, \mathrm{d}\xi_i^1, \ldots, \mathrm{d}\xi_i^{r_i} \in \mathcal{C}^{\infty}_{TU}(T\mathcal{M})^i.$$

The shifted tangent prolongation[6] $T[1]\mathcal{M}$ is an $[n+1]$-manifold over M, with local generators over U given by

degree 0	$C^{\infty}(U)$
degree 1	$\xi_1^1, \ldots, \xi_1^{r_1}, \Omega^1(U)$
degree 2	$\xi_2^1, \ldots, \xi_2^{r_2}, \mathrm{d}\xi_1^1, \ldots, \mathrm{d}\xi_1^{r_1}$
$\vdots$	$\vdots$
degree n	$\xi_n^1, \ldots, \xi_n^{r_n}, \mathrm{d}\xi_{n-1}^1, \ldots, \mathrm{d}\xi_{n-1}^{r_{n-1}}$
degree $n+1$	$\mathrm{d}\xi_n^1, \ldots, \mathrm{d}\xi_n^{r_n}$

It carries a bigrading (p, q), where p comes from the grading of $\mathcal{M}$ and q is the grading of "differential forms". In other words, the structure sheaf of $T[1]\mathcal{M}$ assigns to every coordinate domain $(U, x^1, \ldots, x^m)$ of M that trivialises $\mathcal{M}$, the space

$$\mathcal{C}^{\infty}_U(T[1]\mathcal{M}) = \bigoplus_i \underbrace{\mathcal{C}^{\infty}_U(\mathcal{M})^i}_{(i,0)} \left\langle \underbrace{\left(\mathrm{d}x^k\right)_{k=1}^m}_{(0,1)}, \underbrace{\left(\mathrm{d}\xi_1^k\right)_{k=1}^{r_1}}_{(1,1)}, \ldots, \underbrace{\left(\mathrm{d}\xi_n^k\right)_{k=1}^{r_n}}_{(n,1)} \right\rangle.$$

Suppose now that $(\mathcal{M}, \mathcal{Q})$ is a Lie n-algebroid over M. Then $T[1]\mathcal{M}$ is an $[n+1]$-manifold, which inherits the two commuting differentials $\pounds_Q$ and $\mathbf{d}$ defined as follows:

- the de Rham differential $\mathbf{d}\colon \mathcal{C}^{\infty}(T[1]\mathcal{M})^{\bullet} \to \mathcal{C}^{\infty}(T[1]\mathcal{M})^{\bullet+1}$ is defined on generators by $C^{\infty}(M) \ni f \mapsto \mathrm{d}f$, $\xi_i^j \mapsto \mathrm{d}\xi_i^j$, $\mathrm{d}f \mapsto 0$ and $\mathrm{d}\xi_i^j \mapsto 0$, and is extended to the whole algebra as a derivation of bidegree $(0, 1)$.
- $\pounds_Q\colon \mathcal{C}^{\infty}(T[1]\mathcal{M})^{\bullet} \to \mathcal{C}^{\infty}(T[1]\mathcal{M})^{\bullet+1}$ is the *Lie derivative* with respect to the vector field $\mathcal{Q}$, i.e. the graded commutator $\pounds_Q = [i_Q, \mathbf{d}] = i_Q \circ \mathbf{d} - \mathbf{d} \circ i_Q$, and it is a derivation of bidegree $(1, 0)$. Here, i_Q is the bidegree $(1, -1)$-derivation on $T[1]\mathcal{M}$, which sends $\xi \in \mathcal{C}^{\infty}(\mathcal{M})$ to 0, $\mathbf{d}\xi$ to $\mathcal{Q}(\xi)$ for $\xi \in \mathcal{C}^{\infty}(\mathcal{M})$, and is extended to the whole algebra as a derivation of bidegree $(1, -1)$. The differential $\pounds_Q$ can be seen as a [1]-shifted version of the tangent lift of the vector field Q from $\mathcal{M}$ to $T[1]\mathcal{M}$.

By checking their values on local generators, it is easy to see that $\pounds_Q^2 = 0$, $\mathbf{d}^2 = 0$ and $[\pounds_Q, \mathbf{d}] = \pounds_Q \circ \mathbf{d} + \mathbf{d} \circ \pounds_Q = 0$. Hence,

$$W^{p,q}(\mathcal{M}) := \left\{\text{elements of } \mathcal{C}^{\infty}(T[1]\mathcal{M}) \text{ of bidegree } (p, q)\right\}$$

[6] Note that here there is a sign difference in the notation with [33, 34]. $T[1]\mathcal{M}$ here is the same as $T[-1]\mathcal{M}$ in these papers.

together with $\pounds_Q$ and $\mathbf{d}$ forms a double complex.

Definition 3.11 The Weil algebra of a Lie n-algebroid $(\mathcal{M}, \mathcal{Q})$ is the differential graded algebra given by the total complex of $W^{p,q}(\mathcal{M})$:

$$W(\mathcal{M}) := \left(\bigoplus_{i\in\mathbb{Z}} \bigoplus_{i=p+q} W^{p,q}(\mathcal{M}), \pounds_{\mathcal{Q}} + \mathbf{d} \right).$$

In the case of a Lie 1-algebroid $A \to M$, this is the Weil algebra from [33, 34]. For the 1-algebroid case, see also [2] for an approach without the language of supergeometry.

4 Differential graded modules

This section defines the notion of a differential graded module over a Lie n-algebroid $(\mathcal{M}, \mathcal{Q})$ and gives the two fundamental examples of modules which come canonically with any Lie n-algebroid, namely the adjoint and the coadjoint modules. Note that the case of differential graded modules over a Lie 1-algebroid $A \to M$ is studied in detail in [35].

4.1 The category of differential graded modules

Let $A \to M$ be a Lie 1-algebroid. A *Lie algebroid module* [45] over A is defined as a sheaf $\mathscr{E}$ of locally freely and finitely generated graded $\Omega(A)$-modules over M together with a map $\mathcal{D}\colon \mathscr{E} \to \mathscr{E}$ which squares to zero and satisfies the Leibniz rule

$$\mathcal{D}(\alpha\eta) = (\mathrm{d}_A\alpha)\eta + (-1)^{|\alpha|}\alpha\mathcal{D}(\eta),$$

for $\alpha \in \Omega(A)$ and $\eta \in \mathscr{E}$. For a Lie n-algebroid $(\mathcal{M}, \mathcal{Q})$ over M, this is generalised to the following definitions.

Definition 4.1 A *(left) differential graded module of* $(\mathcal{M}, \mathcal{Q})$ is a sheaf $\mathscr{E}$ of locally freely and finitely generated left graded $C^\infty(\mathcal{M})$-modules over M together with a map $\mathcal{D}\colon \mathscr{E} \to \mathscr{E}$ of degree 1, such that $\mathcal{D}^2 = 0$ and

$$\mathcal{D}(\xi\eta) = \mathcal{Q}(\xi)\eta + (-1)^{|\xi|}\xi\mathcal{D}(\eta)$$

for all $\xi \in C^\infty(\mathcal{M})$ and $\eta \in \mathscr{E}(\mathcal{M})$.

Note that a *right differential graded module of* $(\mathcal{M}, \mathcal{Q})$ is a sheaf $\mathscr{E}$ of right graded modules as above together with a map $\mathcal{D}\colon \mathscr{E} \to \mathscr{E}$ of degree 1, such that $\mathcal{D}^2 = 0$ and

$$\mathcal{D}(\eta\xi) = \mathcal{D}(\eta)\xi + (-1)^{|\eta|}\eta\mathcal{Q}(\xi)$$

for all $\xi \in C^\infty(\mathcal{M})$ and $\eta \in \mathscr{E}(\mathcal{M})$. Any left-module can be made into a right-module (and vice versa) by setting $\eta \cdot \xi := (-1)^{|\eta|\cdot|\xi|}\xi \cdot \eta$ for $\xi \in C^\infty(\mathcal{M})$ and $\eta \in \mathscr{E}$.

A *differential graded bimodule of* $(\mathcal{M}, \mathcal{Q})$ is then a sheaf $\mathscr{E}$ as above together with left and right differential graded module structures such that the gradings and the differentials coincide, and the two module structures commute: $(\xi_1\eta)\xi_2 = \xi_1(\eta\xi_2)$ for all $\xi_1, \xi_2 \in \mathcal{C}^\infty(\mathcal{M})$ and $\eta \in \mathscr{E}$. Occasionally, a module structure naturally arises in a given direction and so, although left and right modules are essentially equivalent, considering them distinctly helps to minimize the signs in the formulas.

For short we write *(left or right) DG* $(\mathcal{M}, \mathcal{Q})$*-module*, or simply *(left or right) DG* $\mathcal{M}$*-module*. The cohomology of the induced complexes is denoted by $H_L^\bullet(\mathcal{M}, \mathcal{Q}; \mathscr{E})$ and $H_R^\bullet(\mathcal{M}, \mathcal{Q}; \mathscr{E})$, respectively, or simply by $H_L^\bullet(\mathcal{M}, \mathscr{E})$ and $H_R^\bullet(\mathcal{M}, \mathscr{E})$. If there is no danger of confusion, the prefixes "left" and "right", as well as the subscripts "L" and "R", will be omitted.

Definition 4.2 Let $(\mathscr{E}_1, \mathcal{D}_1)$ and $(\mathscr{E}_2, \mathcal{D}_2)$ be two differential graded modules over the Lie n-algebroids $(\mathcal{M}, \mathcal{Q}_\mathcal{M})$ and $(\mathcal{N}, \mathcal{Q}_\mathcal{N})$, respectively, and let $k \in \mathbb{Z}$. A degree 0 morphism, or simply a morphism, from $\mathscr{E}_1$ to $\mathscr{E}_2$ consists of a morphism of Lie n-algebroids $\phi\colon \mathcal{N} \to \mathcal{M}$ and a degree preserving map $\mu\colon \mathscr{E}_1 \to \mathscr{E}_2$ which is linear: $\mu(\xi\eta) = \phi^\star(\xi)\mu(\eta)$, for all $\xi \in \mathcal{C}^\infty(\mathcal{M})$ and $\eta \in \mathscr{E}(\mathcal{M})$, and commutes with the differentials $\mathcal{D}_1$ and $\mathcal{D}_2$.

Remark 4.3 The sheaves $\mathscr{E}_1$ and $\mathscr{E}_2$ in the definition above are equivalent to sheaves of linear functions on $\mathcal{Q}$-vector bundles over $\mathcal{M}$ [33]. From this point of view, it is natural that the definition of a morphism of differential graded modules has a contravariant nature.

As in the case of Lie algebroids, new examples of DG $\mathcal{M}$-modules of Lie n-algebroids are obtained by considering the usual algebraic constructions. In the following, we describe these constructions only for left DG modules but the case of right DG modules can be deduced from this.

Definition 4.4 (*Dual module*) Given a DG $\mathcal{M}$-module $\mathscr{E}$ with differential $\mathcal{D}_\mathscr{E}$, one defines a right-DG $\mathcal{M}$-module structure on the dual sheaf $\mathscr{E}^* := \underline{\mathrm{Hom}}(\mathscr{E}, \mathcal{C}^\infty)$ with differential $\mathcal{D}_{\mathscr{E}^*}$ defined via the property

$$\mathcal{Q}(\langle\psi, \eta\rangle) = \langle\mathcal{D}_{\mathscr{E}^*}(\psi), \eta\rangle + (-1)^{|\psi|}\langle\psi, \mathcal{D}_\mathscr{E}(\eta)\rangle,$$

for all $\psi \in \mathscr{E}^*(\mathcal{M})$ and $\eta \in \mathscr{E}(\mathcal{M})$, where $\langle\cdot\,,\cdot\rangle$ is the pairing of $\mathscr{E}^*$ and $\mathscr{E}$ [33].

Definition 4.5 (*Tensor product*) For DG $\mathcal{M}$-modules $\mathscr{E}$ and $\mathscr{F}$ with operators $\mathcal{D}_\mathscr{E}$ and $\mathcal{D}_\mathscr{F}$, the corresponding operator $\mathcal{D}_{\mathscr{E}\otimes\mathscr{F}}$ on $\mathscr{E}\otimes\mathscr{F}$ is uniquely characterised by the formula

$$\mathcal{D}_{\mathscr{E}\otimes\mathscr{F}}(\eta\otimes\eta') = \mathcal{D}_\mathscr{E}(\eta)\otimes\eta' + (-1)^{|\eta|}\eta\otimes\mathcal{D}_\mathscr{F}(\eta'),$$

for all $\eta \in \mathscr{E}(\mathcal{M})$ and $\eta' \in \mathscr{F}(\mathcal{M})$.

Definition 4.6 (Hom module) For DG $\mathcal{M}$-modules $\mathscr{E}$, $\mathscr{F}$ with operators $\mathcal{D}_\mathscr{E}$ and $\mathcal{D}_\mathscr{F}$, the differential $\mathcal{D}_{\underline{\mathrm{Hom}}(\mathscr{E},\mathscr{F})}$ on $\underline{\mathrm{Hom}}(\mathscr{E}, \mathscr{F})$ is defined via

$$\mathcal{D}_{\mathscr{F}}(\psi(\eta)) = \mathcal{D}_{\underline{\mathrm{Hom}}(\mathscr{E},\mathscr{F})}(\psi)(\eta) + (-1)^{|\psi|}\psi(\mathcal{D}_{\mathscr{E}}(\eta)),$$

for all $\psi \in \underline{\mathrm{Hom}}(\mathscr{E}(\mathcal{M}), \mathscr{F}(\mathcal{M}))$ and $\eta \in \mathscr{E}(\mathcal{M})$.

Definition 4.7 ((*Anti*)*symmetric powers*) For a DG $\mathcal{M}$-module $\mathscr{E}$ with operator $\mathcal{D}_{\mathscr{E}}$, the corresponding operator $\mathcal{D}_{\underline{S}(\mathscr{E})}$ on $\underline{S}^k(\mathscr{E})$ is uniquely characterised by the formula

$$\mathcal{D}_{\underline{S}(\mathscr{E})}(\eta_1\eta_2\ldots\eta_k) = \sum_{i=1}^{k}(-1)^{|\eta_1|+\ldots+|\eta_{i-1}|}\eta_1\ldots\mathcal{D}_{\mathscr{E}}(\eta_i)\ldots\eta_k,$$

for all $\eta_1, \ldots, \eta_k \in \mathscr{E}(\mathcal{M})$. A similar formula gives also the characterisation for the operator $\mathcal{D}_{\underline{A}(\mathscr{E})}$ of the antisymmetric powers $\underline{A}^q(\mathscr{E})$.

Definition 4.8 (*Direct sum*) For DG $\mathcal{M}$-modules $\mathscr{E}$, $\mathscr{F}$ with operators $\mathcal{D}_{\mathscr{E}}$ and $\mathcal{D}_{\mathscr{F}}$, the differential operator $\mathcal{D}_{\mathscr{E}\oplus\mathscr{F}}$ on $\mathscr{E}\oplus\mathscr{F}$ is defined as

$$\mathcal{D}_{\mathscr{E}\oplus\mathscr{F}} = \mathcal{D}_{\mathscr{E}} \oplus \mathcal{D}_{\mathscr{F}}.$$

Definition 4.9 (*Shifts*) For $k \in \mathbb{Z}$, the DG $\mathcal{M}$-module $\mathbb{R}[k]$ is defined as $C^\infty(\mathcal{M}) \otimes \Gamma(M \times \mathbb{R}[k])$ with differential given by $\mathcal{Q}$; here, $M \times \mathbb{R}[k]$ is the $[k]$-shift of the trivial line bundle over M, i.e. $M \times \mathbb{R}$ in degree $-k$ and zero otherwise. Given now a module $\mathscr{E}$ with differential $\mathcal{D}_{\mathscr{E}}$, we define the shifted module $\mathscr{E}[k] := \mathscr{E} \otimes \mathbb{R}[k]$. Due to the definition of the tensor module, its differential $\mathcal{D}[k]$ acts via

$$\mathcal{D}_{\mathscr{E}}[k](\eta \otimes 1) = \mathcal{D}_{\mathscr{E}}(\eta) \otimes 1$$

for all $\eta \in \mathscr{E}(\mathcal{M})$. Abbreviating the element $\eta \otimes 1$ simply as η, the shifted differential $\mathcal{D}_{\mathscr{E}}[k]$ coincides[7] with $\mathcal{D}_{\mathscr{E}}$.

Definition 4.10 Let $(\mathcal{M}, \mathcal{Q}_{\mathcal{M}})$ and $(\mathcal{N}, \mathcal{Q}_{\mathcal{N}})$ be Lie n-algebroids, and suppose that $\mathscr{E}_1$ and $\mathscr{E}_2$ are DG-modules over $\mathcal{M}$ and $\mathcal{N}$, respectively. A degree k-morphism, for $k \in \mathbb{Z}$, from $\mathscr{E}_1$ to $\mathscr{E}_2$ is defined as a degree 0 morphism $\mu : \mathscr{E}_1 \to \mathscr{E}_2[k]$; that is, a map sending elements of degree i in $\mathscr{E}_1$ to elements of degree $i + k$ in $\mathscr{E}_2$, such that it is linear over a Lie n-algebroid morphism $\phi : \mathcal{N} \to \mathcal{M}$ and commutes with the differentials. A k-isomorphism is a k-morphism with an inverse.

Remark 4.11 (1) The inverse of a k-isomorphism is necessarily a$-k$-morphism.
(2) For all $k \in \mathbb{Z}$ and all DG $\mathcal{M}$-modules $\mathscr{E}$, there is an obvious k-isomorphism $\mathscr{E} \to \mathscr{E}[k]$ over the identity on $\mathcal{M}$.

Considering the special case of $\mathcal{M} = \mathcal{N}$ in the definition above yields k-morphisms between DG $\mathcal{M}$-modules over the same Lie n-algebroid. The resulting graded category of DG $\mathcal{M}$-modules is denoted by $\underline{\mathbb{M}\mathrm{od}}(\mathcal{M}, \mathcal{Q})$ or simply by $\underline{\mathbb{M}\mathrm{od}}(\mathcal{M})$. The isomorphism classes of these categories are denoted by $\underline{\mathrm{Mod}}(\mathcal{M}, \mathcal{Q})$, or simply by $\underline{\mathrm{Mod}}(\mathcal{M})$.

[7] Again one could choose to tensor with $\mathbb{R}[k]$ from the left. Then on elements of the form $1 \otimes \eta$, the resulting differential would act as $\mathcal{D}_{\mathscr{E}}[k] = (-1)^k\mathcal{D}_{\mathscr{E}}$.

4.2 Adjoint and coadjoint modules

Recall that every $[n]$-manifold $\mathcal{M}$ comes with the sheaf of graded derivations $\underline{\mathrm{Der}}(C^\infty(\mathcal{M}))$ of $C^\infty(\mathcal{M})$, which is called *the sheaf of vector fields over* $\mathcal{M}$. It is a natural sheaf of locally freely and finitely generated graded $C^\infty(\mathcal{M})$-modules over the smooth manifold M, with (left) $C^\infty(\mathcal{M})$-module structure defined by the property $(\xi_1\mathcal{X})(\xi_2) = \xi_1\mathcal{X}(\xi_2)$ for all $\xi_1, \xi_2 \in C^\infty(\mathcal{M})$ and $\mathcal{X} \in \underline{\mathrm{Der}}(C^\infty(\mathcal{M}))$. In addition to the left module structure, the space of vector fields are also endowed with a right $C^\infty(\mathcal{M})$-module structure. The right multiplication with functions in $C^\infty(\mathcal{M})$ is obtained by viewing the elements of $\underline{\mathrm{Der}}(C^\infty(\mathcal{M}))$ of degree i as functions of bidegree $(i, 1)$ of the graded manifold $T^*[1]\mathcal{M}$, similarly to the shifted tangent bundle defined before. The two module structures on $\underline{\mathrm{Der}}(C^\infty(\mathcal{M}))$ are related by[8] $\mathcal{X}\xi = (-1)^{|\xi|(|\mathcal{X}|+1)}\xi\mathcal{X}$, for all homogeneous $\xi \in C^\infty(\mathcal{M})$ and $\mathcal{X} \in \underline{\mathrm{Der}}(C^\infty(\mathcal{M}))$. In particular, we emphasize that the two structures are not simply the same module structure viewed from the right and left.

Suppose now that $\mathcal{M}$ is endowed with a homological vector field $\mathcal{Q}$, i.e. $(\mathcal{M}, \mathcal{Q})$ is a Lie n-algebroid. Then the Lie derivative on the space of vector fields $\pounds_{\mathcal{Q}} := [\mathcal{Q}, \cdot]$ is a degree 1 operator which squares to zero and has both the left and right Leibniz identities with respect to the left and right module structures explained above. That is, the sheaf of vector fields over $(\mathcal{M}, \mathcal{Q})$ has a canonical DG $\mathcal{M}$-bimodule structure. It is called the *adjoint module* of $\mathcal{M}$ and denoted by

$$(\mathfrak{X}(\mathcal{M}), \pounds_{\mathcal{Q}}).$$

The dual module $\bigoplus_p C^\infty(T[1]\mathcal{M})_{(p,1)}$ of 1-forms over $\mathcal{M}$ carries the grading obtained from the horizontal grading of the Weil algebra – that is, the elements of $C^\infty(T[1]\mathcal{M})_{(p,1)}$ have degree p. Its structure operator as a (left) DG module is given by the Lie derivative $\pounds_{\mathcal{Q}} = [i_{\mathcal{Q}}, \mathbf{d}]$. This DG $\mathcal{M}$-module is called the *coadjoint module* of $(\mathcal{M}, \mathcal{Q})$ and denoted by

$$(\Omega^1(\mathcal{M}), \pounds_{\mathcal{Q}}).$$

The corresponding right-DG module has the structure operator $-\pounds_{\mathcal{Q}}$.

4.3 Poisson Lie *n*-algebroids: coadjoint vs adjoint modules

This section shows that a compatible pair of a homological vector field and a Poisson bracket on an $[n]$-manifold gives rise to a degree $-n$ map from the coadjoint to the adjoint module which is an morphism of right DG $\mathcal{M}$-modules.

Let $k \in \mathbb{Z}$. A degree k Poisson bracket on an $[n]$-manifold $\mathcal{M}$ is a degree k $\mathbb{R}$-bilinear map $\{\cdot\,,\cdot\}\colon C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$, i.e. $|\{\xi_1, \xi_2\}| = |\xi_1| + |\xi_2| + k$, such that $\{\xi_1, \xi_2\} = -(-1)^{(|\xi_1|+k)(|\xi_2|+k)}\{\xi_2, \xi_1\}$ and it satisfies the graded Leibniz

[8] Here, $|\chi|$ is the degree of χ as an element of $\underline{\mathrm{Der}}(C^\infty(\mathcal{M}))$. Its degree as a function on $T^*[1]\mathcal{M}$ equals then $|\chi| + 1$.

and Jacobi identities

$$\{\xi_1, \xi_2\xi_3\} = \{\xi_1, \xi_2\}\,\xi_3 + (-1)^{(|\xi_1|+k)|\xi_2|}\xi_2\{\xi_1, \xi_3\}$$
$$\{\xi_1, \{\xi_2, \xi_3\}\} = \{\{\xi_1, \xi_2\}, \xi_3\} + (-1)^{(|\xi_1|+k)(|\xi_2|+k)}\,\{\xi_2, \{\xi_1, \xi_3\}\}\,,$$

for homogeneous elements $\xi_1, \xi_2, \xi_3 \in \mathcal{C}^\infty(\mathcal{M})$. We remark that the role of k in the above formulas can be explained by viewing the comma in the bracket as having degree k.

A morphism between two Poisson $[n]$-manifolds $(\mathcal{N}, \{\cdot\,,\cdot\}_\mathcal{N})$ and $(\mathcal{M}, \{\cdot\,,\cdot\}_\mathcal{M})$ is a morphism of $[n]$-manifolds $\mathcal{F}\colon \mathcal{N} \to \mathcal{M}$ which respects the Poisson brackets: $\mathcal{F}^\star\{\xi_1, \xi_2\}_\mathcal{M} = \{\mathcal{F}^*\xi_1, \mathcal{F}^*\xi_2\}_\mathcal{N}$ for all $\xi_1, \xi_2 \in \mathcal{C}^\infty(\mathcal{M})$.

As is the case for ordinary Poisson manifolds, a degree k Poisson bracket on $\mathcal{M}$ induces a degree k map

$$\mathrm{Ham}\colon \mathcal{C}^\infty(\mathcal{M}) \to \underline{\mathrm{Der}}\left(\mathcal{C}^\infty(\mathcal{M})\right)$$

which sends ξ to its *Hamiltonian vector field* $\mathcal{X}_\xi = \{\xi\,,\cdot\}$. An $[n]$-manifold is called *symplectic* if it is equipped with a degree k Poisson bracket whose Hamiltonian vector fields generate all of $\underline{\mathrm{Der}}(\mathcal{C}^\infty(\mathcal{M}))$.

If an $[n]$-manifold $\mathcal{M}$ carries both a homological vector field $\mathcal{Q}$ and a degree k Poisson bracket $\{\cdot\,,\cdot\}$, then the two structures are compatible if

$$\mathcal{Q}\{\xi_1, \xi_2\} = \{\mathcal{Q}(\xi_1), \xi_2\} + (-1)^{|\xi_1|+k}\,\{\xi_1, \mathcal{Q}(\xi_2)\}$$

for homogeneous $\xi_1 \in \mathcal{C}^\infty(\mathcal{M})$ and all $\xi_2 \in \mathcal{C}^\infty(\mathcal{M})$. Using the Hamiltonian map defined above, the compatibility of $\mathcal{Q}$ and $\{\cdot\,,\cdot\}$ can be rewritten as $\mathcal{X}_{\mathcal{Q}(\xi)} = [\mathcal{Q}, \mathcal{X}_\xi]$ for all $\xi \in \mathcal{C}^\infty(\mathcal{M})$.

Definition 4.12 A *Poisson Lie n-algebroid* $(\mathcal{M}, \mathcal{Q}, \{\cdot\,,\cdot\})$ is an $[n]$-manifold $\mathcal{M}$ endowed with a compatible pair of a homological vector field $\mathcal{Q}$ and a degree $-n$ Poisson bracket $\{\cdot\,,\cdot\}$. If in addition the Poisson bracket is symplectic, then it is called a *symplectic Lie n-algebroid*. A morphism of Poisson Lie n-algebroids is a morphism of the underlying $[n]$-manifolds which is also a morphism of Lie n-algebroids and a morphism of Poisson $[n]$-manifolds.

A Poisson (symplectic) Lie 0-algebroid is a usual Poisson (symplectic) manifold M. A Poisson Lie 1-algebroid corresponds to a Lie bialgebroid (A, A^*) and a symplectic Lie 1-algebroid is again a usual Poisson manifold—Sect. 7 explains this in detail. A result due to Ševera [42] and Roytenberg [40] shows that symplectic Lie 2-algebroids are in one-to-one correspondence with Courant algebroids.

In [31], it was shown that a Lie algebroid A with a linear Poisson structure satisfies the Lie bialgebroid compatibility condition if and only if the map $T^*A \to TA$ induced by the Poisson bivector is a Lie algebroid morphism from $T^*A = T^*A^* \to A^*$ to $TA \to TM$. This is now generalized to give a characterisation of Poisson Lie n-algebroids.

Let $\mathcal{M}$ be an $[n]$-manifold equipped with a homological vector field $\mathcal{Q}$ and a degree $-n$ Poisson bracket $\{\cdot\,,\cdot\}$. The Poisson bracket on $\mathcal{M}$ induces a map $\sharp\colon \Omega^1(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})[-n]$ defined on the generators via the property

$$(\sharp(\mathrm{d}\xi_1))\,\xi_2 = \{\xi_1, \xi_2\}\,, \tag{5}$$

for all $\xi_1, \xi_2 \in C^\infty(\mathcal{M})$, and extended odd linearly by the rules

$$\sharp\,(\xi_1\mathrm{d}\xi_2) = (-1)^{|\xi_1|}\xi_1\sharp\,(\mathrm{d}\xi_2) \qquad \text{and} \qquad \sharp\,(\mathrm{d}\xi_1\xi_2) = (-1)^{|\xi_2|}\sharp\,(\mathrm{d}\xi_1)\,\xi_2.$$

Theorem 4.13 *Let $\mathcal{M}$ be an $[n]$-manifold equipped with a homological vector field $\mathcal{Q}$ and a degree $-n$ Poisson bracket $\{\cdot\,,\cdot\}$. Then $(\mathcal{M}, \mathcal{Q}, \{\cdot\,,\cdot\})$ is a Poisson Lie n-algebroid if and only if $\sharp\colon \Omega^1(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ is a morphism of right DG $\mathcal{M}$-modules, i.e. $\sharp \circ £_{\mathcal{Q}} = -£_{\mathcal{Q}} \circ \sharp$.*

Proof From (5),

$$(£_{\mathcal{Q}}(\sharp(\mathrm{d}\xi_1)) + \sharp(£_{\mathcal{Q}}(\mathrm{d}\xi_1)))\,\xi_2 = \mathcal{Q}\{\xi_1, \xi_2\} - (-1)^{|\xi_1|-n}\{\xi_1, \mathcal{Q}(\xi_2)\} - \{\mathcal{Q}(\xi_1), \xi_2\}.$$

In other words, the compatibility of $\mathcal{Q}$ with $\{\cdot\,,\cdot\}$ is equivalent to $£_{\mathcal{Q}} \circ \sharp = -\sharp \circ £_{\mathcal{Q}}$.

A detailed analysis of this map in the cases of Poisson Lie algebroids of degree $n \leq 2$ is given in Section 7.2. The two following corollaries can be realised as obstructions for a Lie n-algebroid with a Poisson bracket to be symplectic. In particular, for $n = 2$ one obtains the corresponding results for Courant algebroids.

Corollary 4.14 *Let $\mathcal{M}$ be an $[n]$-manifold equipped with a homological vector field $\mathcal{Q}$ and a degree $-n$ Poisson bracket $\{\cdot\,,\cdot\}$. Then $(\mathcal{M}, \mathcal{Q}, \{\cdot\,,\cdot\})$ is symplectic if and only if $\sharp$ is an isomorphism of right DG $\mathcal{M}$-modules.*

Corollary 4.15 *For any Poisson Lie n-algebroid $(\mathcal{M}, \mathcal{Q}, \{\cdot\,,\cdot\})$ there is a natural degree $-n$ map in cohomologies $\sharp\colon H^\bullet_R(\mathcal{M}, \Omega^1) \to H^{\bullet-n}_R(\mathcal{M}, \mathfrak{X})$ which is an isomorphism if the bracket is symplectic.*

5 Representations up to homotopy

This section generalises the notion of representation up to homotopy of Lie algebroids from [2, 18] to representations of higher Lie algebroids. Some basic examples are given, and 3-term representations of a split Lie 2-algebroid are described in detail. The adjoint and coadjoint representations of a split Lie 2-algebroid are special examples, which this section describes with explicit formulas for their structure objects and their coordinate transformation. Lastly, it shows how to define these two representations together with their objects for general Lie n-algebroids for all n.

5.1 The category of representations up to homotopy

Recall that a representation up to homotopy of a Lie algebroid A is given by an A-module of the form $\Omega(A, \underline{E}) = \Omega(A) \otimes \Gamma(\underline{E})$ for a graded vector bundle $\underline{E}$ over M. In the same manner, a *(left) representation up to homotopy of a Lie n-algebroid* $(\mathcal{M}, \mathcal{Q})$ is defined as a (left) DG $\mathcal{M}$-module of the form $C^\infty(\mathcal{M}) \otimes \Gamma(\underline{E})$ for a graded vector bundle $\underline{E} \to M$.

Following the notation from [2], we denote the category of representations up to homotopy by $\underline{\mathbb{R}\mathrm{ep}}^\infty(\mathcal{M}, \mathcal{Q})$, or simply by $\underline{\mathbb{R}\mathrm{ep}}^\infty(\mathcal{M})$. The isomorphism classes of representations up to homotopy of this category are denoted by $\underline{\mathrm{Rep}}^\infty(\mathcal{M}, \mathcal{Q})$, or by $\underline{\mathrm{Rep}}^\infty(\mathcal{M})$. A representation of the form $\underline{E} = E_0 \oplus \cdots \oplus E_{k-1}$ is called a *k-term representation*, or simply a *k-representation*.

Remark 5.1 Any DG $\mathcal{M}$-module is non-canonically isomorphic to a representation up to homotopy of $(\mathcal{M}, \mathcal{Q})$. The proof, similar to that of the $n = 1$ case [35], is as follows: an $\mathcal{M}$-module is, by definition, the sheaf of sections $\Gamma(\mathcal{B})$ of a vector bundle $\mathcal{B}$ over $\mathcal{M}$ in the category of graded manifolds. The pull-back $0^*_{\mathcal{M}}\mathcal{B}$, where $0_{\mathcal{M}} \colon M \to \mathcal{M}$ is the zero embedding, is an ordinary graded vector bundle $\underline{E}$ over M and hence splits as $\underline{E} = \bigoplus_i E_i[i]$. According to [35, Theorem 2.1], the double pull-back $\pi^*_{\mathcal{M}} 0^*_{\mathcal{M}}\mathcal{B}$ is non-canonically isomorphic to $\mathcal{B}$ as vector bundles over $\mathcal{M}$, where $\pi_{\mathcal{M}} \colon \mathcal{M} \to M$ is the projection map. Then, as a sheaf over M, $\Gamma(\mathcal{B})$ is identified with $\Gamma(\pi^*_{\mathcal{M}} 0^*_{\mathcal{M}}\mathcal{B}) = \Gamma(\pi^*_{\mathcal{M}}\underline{E})$, which in turn is canonically isomorphic to $C^\infty(\mathcal{M}) \otimes \Gamma(\underline{E})$.

Example 5.2 ($\mathcal{Q}$-closed functions) Let $(\mathcal{M}, \mathcal{Q})$ be a Lie n-algebroid and suppose $\xi \in C^\infty(\mathcal{M})^k$ such that $\mathcal{Q}(\xi) = 0$. Then one can construct a representation up to homotopy $C^\infty(\mathcal{M}) \otimes \Gamma(\underline{E}_\xi)$ of $\mathcal{M}$ on the graded vector bundle $\underline{E}_\xi = (\mathbb{R}[0] \oplus \mathbb{R}[1-k]) \times M \to M$ (i.e. $\mathbb{R}$ in degrees 0 and $k - 1$, and zero otherwise). Its differential $\mathcal{D}_\xi$ is given in components by the map

$$\mathcal{D}_\xi = \sum_i \mathcal{D}^i_\xi,$$

where

$$\mathcal{D}^i_\xi \colon C^\infty(\mathcal{M})^i \oplus C^\infty(\mathcal{M})^{i-k+1} \to C^\infty(\mathcal{M})^{i+1} \oplus C^\infty(\mathcal{M})^{i-k+2}$$

is defined by the formula[9]

$$\mathcal{D}^i_\xi (\zeta_1, \zeta_2) = \left(\mathcal{Q}(\zeta_1) + (-1)^{i-k+1}\zeta_2\xi, \mathcal{Q}(\zeta_2)\right).$$

If there is an element $\xi' \in C^\infty(\mathcal{M})^k$ which is $\mathcal{Q}$-cohomologous to ξ, i.e. $\xi - \xi' = \mathcal{Q}(\xi'')$ for some $\xi'' \in C^\infty(\mathcal{M})^{k-1}$, then the representations $\underline{E}_\xi$ and $\underline{E}_{\xi'}$ are isomorphic via

[9] Note that, up to some signs, this construction can be understood as a mapping cone construction for the chain map $f_\xi \colon C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})[k]$, $\eta \mapsto \eta \cdot \xi$.

the isomorphism $\mu\colon \underline{E}_{\xi} \to \underline{E}_{\xi'}$ defined in components by

$$\mu^i : \mathcal{C}^\infty(\mathcal{M})^i \oplus \mathcal{C}^\infty(\mathcal{M})^{i-k+1} \to \mathcal{C}^\infty(\mathcal{M})^i \oplus \mathcal{C}^\infty(\mathcal{M})^{i-k+1}$$

given by the formula

$$\mu^i\,(\zeta_1, \zeta_2) = \left(\zeta_1 + \zeta_2 \xi'', \zeta_2\right).$$

Hence, one obtains a well-defined map $H^\bullet(\mathcal{M}) \to \underline{\text{Rep}}^\infty(\mathcal{M})$. In particular, if $\mathcal{M}$ is a Lie algebroid, the above construction recovers Example 3.5 in [2].

5.2 The case of (split) Lie 2-algebroids

Fix now a split Lie 2-algebroid $\mathcal{M}$, and recall that from the analysis of Sect. 3.1, $\mathcal{M}$ is given by the sum $Q[1] \oplus B^*[2]$ which forms the complex

$$B^* \xrightarrow{\ell} Q \xrightarrow{\rho_Q} TM.$$

Unravelling the data of the definition of representations up to homotopy for the special case where E is concentrated only in degree 0 yields the following characterisation.

Proposition 5.3 *A representation of the Lie* 2*-algebroid* $Q[1] \oplus B^*[2]$ *consists of a (non-graded) vector bundle* E *over* M*, together with a* Q*-connection* ∇ *on* E *such that*[10] *:*

(i) ∇ *is flat, i.e.* $R_\nabla = 0$ *on* $\Gamma(E)$*,*
(ii) $\partial_B \circ \mathrm{d}_\nabla = 0$ *on* $\Gamma(E)$.

Proof Let $(E, \mathcal{D})$ be a representation of the Lie 2-algebroid. Due to the Leibniz rule, $\mathcal{D}$ is completely characterised by what it does on $\Gamma(E)$. By definition, it sends $\Gamma(E)$ into $\Omega^1(Q, E)$. Using the Leibniz rule once more together with the definition of the homological vector field $\mathcal{Q}$ on $\Omega^1(Q)$, for all $f \in C^\infty(M)$ and all $e \in \Gamma(E)$ yields

$$\mathcal{D}(fe) = \left(\rho_Q^* \mathrm{d} f\right) \otimes e + f\mathcal{D}(e),$$

which implies that $\mathcal{D} = \mathrm{d}_\nabla$ for a Q-connection ∇ on $\Gamma(E)$. Moreover, by definition of $\mathcal{D}$ one must have $\mathcal{D}^2(e) = 0$ for all $e \in \Gamma(E)$. On the other hand, a straightforward computation yields

$$\mathcal{D}^2(e) = \mathcal{D}\,(\mathrm{d}_\nabla e) = \mathrm{d}_\nabla^2 e + \partial_B\,(\mathrm{d}_\nabla e) \in \Omega^2(Q, E) \oplus \Gamma(B \otimes E).$$

□

[10] Note that all the objects that appear in the following equations act via the generalised wedge products that were discussed before. For example, $\partial(\mathrm{d}_\nabla e)$ and $\omega_2(\omega_2(e))$ mean $\partial \wedge \mathrm{d}_\nabla e$ and $\omega_2 \wedge \omega_2(e)$, respectively. This is explained in detail in the Appendix of [2].

Example 5.4 (Trivial line bundle) The trivial line bundle $\mathbb{R}[0]$ over M with Q-connection defined by

$$\mathrm{d}_\nabla f = \mathrm{d}_Q f = \rho_Q^* \mathrm{d} f$$

is a representation of the Lie 2-algebroid $Q[1] \oplus B^*[2]$. The operator $\mathcal{D}$ is given by the homological vector field $\mathcal{Q}$ and thus the cohomology induced by the representation is the Lie 2-algebroid cohomology: $H^\bullet(\mathcal{M}, \mathbb{R}) = H^\bullet(\mathcal{M})$. The shifted version of this example was used before to define general shifts of DG $\mathcal{M}$-modules.

Example 5.5 More generally, for all $k > 0$, the trivial vector bundle $\mathbb{R}^k$ of rank k over M with Q-connection defined component-wise as in the example above becomes a representation with cohomology $H^\bullet(\mathcal{M}, \mathbb{R}^k) = H^\bullet(\mathcal{M}) \oplus \cdots \oplus H^\bullet(\mathcal{M})$ (k-times).

Remark 5.6 Given a split Lie n-algebroid $A_1[1] \oplus \cdots \oplus A_n[n]$ over a smooth manifold M, with $n \geq 2$, the vector bundle $A_1 \to M$ carries a skew-symmetric dull algebroid structure induced by the 2-bracket and the anchor $\rho\colon A_1 \to TM$ given by $\mathcal{Q}(f) = \rho^* \mathrm{d} f$, for $f \in C^\infty(M)$. Hence, Proposition 5.3, Examples 5.4 and 5.5 can be carried over verbatim to the general case.

We will be particularly interested in the case of 3-term representations of (split) Lie 2-algebroids. As we will see later, such representations correspond to VB-Lie 2-algebroids. In particular, the adjoint and coadjoint representations are 3-term representations.

The reader should note the similarity of the following proposition with the description of 2-term representations of Lie algebroids from [2].

Proposition 5.7 *A 3-term representation up to homotopy* ($\underline{E} = E_0 \oplus E_1 \oplus E_2, \mathcal{D}$) *of* $Q[1] \oplus B^*[2]$ *is equivalent to the following data:*

(i) *A degree 1 map* $\partial\colon \underline{E} \to \underline{E}$ *such that* $\partial^2 = 0$,
(ii) *a* Q*-connection* ∇ *on the complex* $\partial\colon E_\bullet \to E_{\bullet+1}$,
(iii) *an element* $\omega_2 \in \Omega^2(Q, \underline{End}^{-1}(\underline{E}))$,
(iv) *an element* $\omega_3 \in \Omega^3(Q, \underline{End}^{-2}(\underline{E}))$, *and an element* $\phi_j \in \Gamma(B) \otimes \Omega^j(Q, \underline{End}^{-j-1}(\underline{E}))$ *for* $j = 0, 1$

such that[11]

(1) $\partial \circ \omega_2 + \mathrm{d}_\nabla^2 + \omega_2 \circ \partial = 0$,
(2) $\partial \circ \phi_0 + \partial_B \circ \mathrm{d}_\nabla + \phi_0 \circ \partial = 0$,
(3) $\partial \circ \omega_3 + \mathrm{d}_\nabla \circ \omega_2 + \omega_2 \circ \mathrm{d}_\nabla + \omega_3 \circ \partial = \langle \omega, \phi_0 \rangle$,
(4) $\mathrm{d}_{\overline{\nabla}} \phi_0 + \partial \circ \phi_1 + \partial_B \circ \omega_2 + \phi_1 \circ \partial = 0$,
(5) $\mathrm{d}_\nabla \circ \omega_3 + \omega_2 \circ \omega_2 + \omega_3 \circ \mathrm{d}_\nabla = \langle \omega, \phi_1 \rangle$,
(6) $\mathrm{d}_{\overline{\nabla}} \phi_1 + \omega_2 \circ \phi_0 + \partial_B \circ \omega_3 + \phi_0 \circ \omega_2 = 0$,
(7) $\phi_0 \circ \phi_0 + \partial_B \circ \phi_1 = 0$,

[11] In the following equations, the map $\partial_B\colon \Omega^1(Q) \to \Gamma(B)$ extends to $\partial_B\colon \Omega^k(Q) \to \Omega^{k-1}(Q, B)$ by the rule $\partial_B(\tau_1 \wedge \cdots \wedge \tau_k) = \sum_{i=1}^k (-1)^{i+1} \tau_1 \wedge \ldots \wedge \hat{\tau_i} \wedge \cdots \wedge \tau_k \wedge \partial_B \tau_i$, for $\tau_i \in \Omega^1(Q)$.

where $\overline{\nabla}$ is the Q-connection on $B \otimes \underline{End}^{-j-1}(\underline{E})$ induced by ∇ on B and $\nabla^{\underline{End}}$ on $\underline{End}(\underline{E})$.

Remark 5.8 (1) If both of the bundles E_1 and E_2 are zero, the equations agree with those of a 1-term representation.

(2) The equations in the statement can be summarised as follows:

$$[\partial, \phi_0] + \partial_B \circ \mathrm{d}_\nabla = 0, \qquad \phi_0 \circ \phi_0 + \partial_B \circ \phi_1 = 0,$$

and for all i:

$$\begin{aligned}&[\partial, \omega_i] + [\mathrm{d}_\nabla, \omega_{i-1}] + \omega_2 \circ \omega_{i-2} + \omega_3 \circ \omega_{i-3} + \cdots + \omega_{i-2} \circ \omega_2 = \langle \omega, \phi_{i-3} \rangle,\\ &\partial_B \circ \omega_{i+2} + [\partial, \phi_{i+1}] + \mathrm{d}_{\overline{\nabla}} \phi_i + \sum_{j \geq 2} [\omega_j, \phi_{i-j+1}] = 0.\end{aligned}$$

(3) Of course, there are similar descriptions of higher term representations up to homotopy of general split Lie n-algebroids. The proof below can easily be adapted to higher degrees. Since only the 3-term representations of split Lie 2-algebroids are explicitly needed later on, only this setting is worked out in detail here.

Proof It is enough to check that $\mathcal{D}$ acts on $\Gamma(\underline{E})$. Since $\mathcal{D}$ is of degree 1, it maps each $\Gamma(E_i)$ into the direct sum

$$\Gamma(E_{i+1}) \oplus \left(C^\infty(\mathcal{M})^1 \otimes \Gamma(E_i)\right) \oplus \left(C^\infty(\mathcal{M})^2 \otimes \Gamma(E_{i-1})\right) \oplus \left(C^\infty(\mathcal{M})^3 \otimes \Gamma(E_{i-2})\right).$$

Considering the components of $\mathcal{D}$, this translates to the following three equations:

$$\mathcal{D}(e) = \partial(e) + d(e) \in \Gamma(E_1) \oplus \Omega^1(Q, E_0)$$

for $e \in \Gamma(E_0)$,

$$\begin{aligned}\mathcal{D}(e) &= \partial(e) + d(e) + \omega_2(e) + \phi_0(e) \in \Gamma(E_2) \oplus \Omega^1(Q, E_1) \oplus \Omega^2(Q, E_0)\\ &\quad \oplus (\Gamma(B) \otimes \Gamma(E_0))\end{aligned}$$

for $e \in \Gamma(E_1)$, and

$$\begin{aligned}\mathcal{D}(e) &= d(e) + \omega_2(e) + \phi_0(e) + \omega_3(e) + \phi_1(e)\\ &\in \Omega^1(Q, E_2) \oplus \Omega^2(Q, E_1) \oplus (\Gamma(B) \otimes \Gamma(E_1))\\ &\oplus \Omega^3(Q, E_0) \oplus \left(\Gamma(B) \otimes \Omega^1(Q, E_0)\right)\end{aligned}$$

for $e \in \Gamma(E_2)$. Due to the correspondence in (5) and the Leibniz rule for $\mathcal{D}$, $\partial \in \underline{\mathrm{End}}^1(\underline{E})$, $d = \mathrm{d}_\nabla$ where ∇ are Q-connections on the vector bundles E_i for $i = 0, 1, 2$, $\omega_i \in \Omega^i(Q, \underline{\mathrm{End}}^{1-i}(\underline{E}))$ for $i = 2, 3$, and $\phi_i \in \Gamma(B) \otimes \Omega^i(Q, \underline{\mathrm{End}}^{-i-1}(\underline{E}))$ for $i = 0, 1$.

A straightforward computation and a degree count in the expansion of the equation $\mathcal{D}^2 = 0$ shows that $(\underline{E}, \partial)$ is a complex, ∇ commutes with ∂, and the equations in the statement hold.

5.3 Adjoint representation of a Lie 2-algebroid

This section shows that any split Lie 2-algebroid $Q[1] \oplus B^*[2]$ admits a 3-term representation up to homotopy which is called *the adjoint representation*. It is a generalisation of the adjoint representation of a (split) Lie 1-algebroid studied in [2].

Proposition 5.9 *Any split Lie 2-algebroid $Q[1]\oplus B^*[2]$ admits a 3-term representation up to homotopy as follows: Choose arbitrary TM-connections on Q and B^* and denote both by ∇. Then the structure objects are*[12]

(i) *the adjoint complex $B^*[2] \to Q[1] \to TM[0]$ with maps $-\ell$ and ρ_Q,*
(ii) *the two Q-connections ∇^{bas} on Q and TM, and the Q-connection ∇^* on B^* given by the split Lie 2-algebroid,*
(iii) *the element $\omega_2 \in \Omega^2(Q, Hom(Q, B^*) \oplus Hom(TM, Q))$ defined by*

$$\omega_2(q_1, q_2)q_3 = -\omega(q_1, q_2, q_3) \in \Gamma(B^*) \text{ and } \omega_2(q_1, q_2)X = -R_{\nabla}^{bas}(q_1, q_2)X \in \Gamma(Q)$$

for $q_1, q_2, q_3 \in \Gamma(Q)$ and $X \in \mathfrak{X}(M)$,
(iv) *the element $\omega_3 \in \Omega^3(Q, Hom(TM, B^*))$ defined by*

$$\omega_3(q_1, q_2, q_3)X = -(\nabla_X\omega)(q_1, q_2, q_3) \in \Gamma(B^*)$$

for $q_1, q_2, q_3 \in \Gamma(Q)$ and $X \in \mathfrak{X}(M)$,
(v) *the element $\phi_0 \in \Gamma(B) \otimes (Hom(Q, B^*) \oplus Hom(TM, Q))$ defined by*

$$\phi_0(\beta)X = \ell(\nabla_X\beta) - \nabla_X(\ell(\beta)) \in \Gamma(Q) \text{ and } \phi_0(\beta)q = \nabla_{\rho(q)}\beta - \nabla_q^*\beta \in \Gamma(B^*)$$

for $\beta \in \Gamma(B^*)$, $q \in \Gamma(Q)$, $X \in \mathfrak{X}(M)$,
(vi) *the element $\phi_1 \in \Gamma(B) \otimes \Omega^1(Q, Hom(TM, B^*))$ defined by*

$$\phi_1(\beta, q)X = \nabla_X\nabla_q^*\beta - \nabla_q^*\nabla_X\beta - \nabla_{\nabla_X q}^*\beta + \nabla_{\nabla_q^{\text{bas}}X}\beta \in \Gamma(B^*)$$

for $\beta \in \Gamma(B^*)$, $q \in \Gamma(Q)$, $X \in \mathfrak{X}(M)$.

The proof can be done in two ways. First, one could check explicitly that all the conditions of a 3-representation of $Q[1]\oplus B^*[2]$ are satisfied. This is an easy but long computation and it can be found in [37]. Instead, the following section shows that given a splitting and TM-connections on the vector bundles Q and B^*, there exists an isomorphism of sheaves of $C^\infty(\mathcal{M})$-modules between the adjoint module $\mathfrak{X}(\mathcal{M})$ and

[12] Some signs are chosen so that the map given in 5.4 is an isomorphism for the differential of the adjoint module defined earlier.

$C^\infty(\mathcal{M}) \otimes \Gamma(TM[0] \oplus Q[1] \oplus B^*[2])$, such that the objects defined above correspond to the differential $\pounds_Q$. Another advantage of this approach is that it gives a precise recipe for the definition and the explicit formulas for the components of the adjoint representation of a Lie n-algebroid for general n.

Remark 5.10 The adjoint representation of a Courant algebroid $E \to M$ can be deduced from the formulas above and from Example 3.10. Choose a linear connection $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ that preserves the metric underlying the Courant algebroid structure on E. As in Example 3.10, define the basic connection $\nabla^{\text{bas}}\colon \Gamma(E) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, $\nabla^{\text{bas}}_e X = [\rho(e), X] + \rho(\nabla_X e)$. Recall that ∇ defines as well the dull bracket $[\cdot\,,\cdot]$ on sections of E:

$$[e_1, e_2] = [\![e_1, e_2]\!] - \rho^* \langle \nabla_\cdot e_1, e_2 \rangle .$$

The dull bracket and the TM-connection on E defines the basic E-connection $\nabla^{\text{bas}}\colon \Gamma(E) \times \Gamma(E) \to \Gamma(E)$, $\nabla^{\text{bas}}_{e_1} e_2 = [e_1, e_2] + \nabla_{\rho(e_2)} e_1 = [\![e_1, e_2]\!] - \rho^*\langle \nabla_\cdot e_1, e_2\rangle + \nabla_{\rho(e_2)} e_1$. Choose in addition a TM-connection ∇ on TM.

The complex for the adjoint representation is $T^*M[2] \to E[1] \to TM[0]$ with maps $-\rho^*$ and ρ. The E-connections on T^*M, E and TM are $\nabla^{\text{bas}*}$, ∇^{bas} and ∇^{bas} defined as above, respectively. The form $\omega_2 \in \Omega^2(E, \operatorname{Hom}(E, T^*M))$ is given by

$$\begin{aligned}\left\langle \omega_2\left(e, e'\right), X\right\rangle = {} & \nabla_X \left[\!\left[e, e'\right]\!\right] - \left[\!\left[\nabla_X e, e'\right]\!\right] - \left[\!\left[e, \nabla_X e'\right]\!\right] - \nabla_{\nabla^{\text{bas}}_{e'} X} e + \nabla_{\nabla^{\text{bas}}_{e} X} e' \\ & + P^{-1} \left\langle \nabla_{\nabla^{\text{bas}}_{\cdot} X} e, e' \right\rangle\end{aligned}$$

for $e, e' \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$, while the second summand $\omega_2 \in \Omega^2(E, \operatorname{Hom}(TM, E))$ is given by

$$\omega_2\left(e, e'\right) X = \nabla_X \left[e, e'\right] - \left[e, \nabla_X e'\right] - \left[\nabla_X e, e'\right] - \nabla_{\nabla^{\text{bas}}_{e'} X} e + \nabla_{\nabla^{\text{bas}}_{e} X} e'.$$

This data can be compared with the components of the representation up to homotopy of a Lie algebroid $A \to M$, after the choice of a TM-connection on A, see [18]. The remaining terms are given by (iv), (v) and (vi) in Proposition 5.9, and as they do not seem more instructive than the general form in the proposition, they are not computed in more detail here.

5.4 Adjoint module vs adjoint representation

Recall that for a split $[n]$-manifold $\mathcal{M} = \bigoplus E_i[i]$, the space of vector fields over $\mathcal{M}$ is generated as a $C^\infty(\mathcal{M})$-module by two special kinds of vector fields. Namely, the degree $-i$ vector fields $\hat{e}$ for $e \in \Gamma(E_i)$, and the family of vector fields $\nabla^1_X \oplus \cdots \oplus \nabla^n_X$ for $X \in \mathfrak{X}(M)$ and a choice of TM-connection ∇^i on each vector bundle E_i, for $i = 1, \ldots, n$.

Consider now a Lie 2-algebroid $(\mathcal{M}, \mathcal{Q})$ together with a splitting $\mathcal{M} \simeq Q[1] \oplus B^*[2]$ and a choice of TM-connections ∇^{B^*} and ∇^Q on B^* and Q, respectively.

These choices give as follows the adjoint representation ad_∇, whose complex is given by $TM[0]\oplus Q[1]\oplus B^*[2]$. Define a map $\mu_\nabla\colon \mathcal{C}^\infty(\mathcal{M})\otimes\Gamma(TM[0]\oplus Q[1]\oplus B^*[2])\to \mathfrak{X}(\mathcal{M})$ on the generators by

$$\Gamma(B^*)\ni\beta\mapsto\hat{\beta},\qquad \Gamma(Q)\ni q\mapsto\hat{q},\qquad \mathfrak{X}(M)\ni X\mapsto\nabla_X^{B^*}\oplus\nabla_X^{Q}$$

and extend $\mathcal{C}^\infty(\mathcal{M})$-linearly to the whole space to obtain a degree-preserving isomorphism of sheaves of $\mathcal{C}^\infty(\mathcal{M})$-modules.

A straightforward computation shows that

$$\begin{aligned}
&\pounds_{\mathcal{Q}}(\hat{\beta})=\mu_\nabla\left(-\ell(\beta)+\mathrm{d}_{\nabla^*}\beta\right),\\
&\pounds_{\mathcal{Q}}(\hat{q})=\mu_\nabla\left(\rho_Q(q)+\mathrm{d}_{\nabla^{\mathrm{bas}}}q+\omega_2(\cdot\,,\cdot)q+\phi_0(\cdot)q\right),\\
&\pounds_{\mathcal{Q}}(\nabla_X^{B^*}\oplus\nabla_X^{Q})=\mu_\nabla\left(\mathrm{d}_{\nabla^{\mathrm{bas}}}X+\phi_0(\cdot)X+\omega_2(\cdot\,,\cdot)X+\omega_3(\cdot\,,\cdot\,,\cdot)X+\phi_1(\cdot\,,\cdot)X\right)
\end{aligned}$$

and therefore, the objects in the statement of Proposition 5.9 define the differential $\mathcal{D}_{\mathrm{ad}_\nabla}:=\mu_\nabla^{-1}\circ\pounds_{\mathcal{Q}}\circ\mu_\nabla$ of a 3-representation of $Q[1]\oplus B^*[2]$, called the *adjoint representation* and denoted as $(\mathrm{ad}_\nabla,\mathcal{D}_{\mathrm{ad}_\nabla})$. The adjoint representation is hence, up to isomorphism, independent of the choice of splitting and connections (see the following section for the precise transformations), and so gives a well-defined class $\mathrm{ad}\in\underline{\mathrm{Rep}}^\infty(\mathcal{M})$.

Due to the result above, one can also define the *coadjoint representation* of a Lie 2-algebroid $(\mathcal{M},\mathcal{Q})$ as the isomorphism class $\mathrm{ad}^*\in\underline{\mathrm{Rep}}^\infty(\mathcal{M})$. To find an explicit representative of ad^*, suppose that $Q[1]\oplus B^*[2]$ is a splitting of $\mathcal{M}$, and consider its adjoint representation ad_∇ as above for some choice of TM-connections ∇ on B^* and Q. Recall that given a representation up to homotopy $(\underline{E},\mathcal{D})$ of $(\mathcal{M},\mathcal{Q})$, its dual $\underline{E}^*$ becomes a representation up to homotopy with operator $\mathcal{D}^*$ characterised by the formula

$$\mathcal{Q}\left(\xi\wedge\xi'\right)=\mathcal{D}^*(\xi)\wedge\xi'+(-1)^{|\xi|}\xi\wedge\mathcal{D}\left(\xi'\right),$$

for all $\xi\in\mathcal{C}^\infty(\mathcal{M})\otimes\Gamma(\underline{E}^*)$ and $\xi'\in\mathcal{C}^\infty(\mathcal{M})\otimes\Gamma(\underline{E})$. Here, $\wedge=\wedge_{\langle\cdot\,,\cdot\rangle}$, with $\langle\cdot\,,\cdot\rangle$ the pairing of $\underline{E}$ with $\underline{E}^*$. Unravelling the definition of the dual for the representation ad_∇, one finds that the structure differential of $\mathrm{ad}_\nabla^*=\mathcal{C}^\infty(\mathcal{M})\otimes\Gamma(B[-2]\oplus Q^*[-1]\oplus T^*M[0])$ is given by the following objects:

(1) the *coadjoint complex* $T^*M\to Q^*\to B$ obtained by $-\rho_Q^*$ and $-\ell^*$,
(2) the Q-connections ∇ on B and $\nabla^{\mathrm{bas},*}$ on Q^* and T^*M,
(3) the elements

$$\begin{aligned}
&\omega_2^*(q_1,q_2)\tau=\tau\circ\omega_2(q_1,q_2), &&\omega_2^*(q_1,q_2)b=-b\circ\omega_2(q_1,q_2),\\
&\phi_0^*(\beta)\tau=\tau\circ\phi_0(\beta), &&\phi_0^*(\beta)b=-b\circ\phi_0(\beta),\\
&\omega_3^*(q_1,q_2,q_3)b=-b\circ\omega_3(q_1,q_2,q_3), &&\phi_1^*(\beta,q)b=-b\circ\phi_1(\beta,q),
\end{aligned}$$

for all $q,q_1,q_2,q_3\in\Gamma(Q)$, $\tau\in\Gamma(Q^*)$, $b\in\Gamma(B)$ and $\beta\in\Gamma(B^*)$.

Remark 5.11 The coadjoint representation can also be obtained from the coadjoint module $\Omega^1(\mathcal{M})$ by the right $C^\infty(\mathcal{M})$-module isomorphism $\mu_\nabla^\star\colon \Omega^1(\mathcal{M}) \to \Gamma(B[-2]\oplus Q^*[-1]\oplus T^*M[0])\otimes C^\infty(\mathcal{M})$ which is dual to $\mu_\nabla\colon C^\infty(\mathcal{M})\otimes\Gamma(TM[0]\oplus Q[1]\oplus B^*[2]) \to \mathfrak{X}(\mathcal{M})$ above. Explicitly, it is defined as the pull-back map $\mu_\nabla^\star(\omega) = \omega\circ\mu_\nabla$ for all $\omega\in\Omega^1(\mathcal{M})$, whose inverse is given on the generators by $\Gamma(B[-2]\oplus Q^*[-1]\oplus T^*M[0])\otimes C^\infty(\mathcal{M}) \to \Omega^1(\mathcal{M})$,

$$\Gamma(B)\ni b\mapsto \mathbf{d}b - \mathrm{d}_{\nabla^*}b, \quad \Gamma(Q^*)\ni\tau\mapsto \mathbf{d}\tau - \mathrm{d}_{\nabla^*}\tau, \quad \text{and} \quad \Omega^1(M)\ni\theta\mapsto\theta.$$

5.5 Coordinate transformation of the adjoint representation

The adjoint representation up to homotopy of a Lie 2-algebroid was constructed after a choice of splitting and TM-connections. This section explains how the adjoint representation transforms under different choices.

First, a morphism of 3-representations of a split Lie 2-algebroid can be described as follows.

Proposition 5.12 *Let $(\underline{E}, \mathcal{D}_{\underline{E}})$ and $(\underline{F}, \mathcal{D}_{\underline{F}})$ be 3-term representations up to homotopy of the split Lie 2-algebroid $Q[1]\oplus B^*[2]$. A morphism $\mu\colon \underline{E}\to\underline{F}$ is equivalent to the following data:*

(i) *For each $i = 0, 1, 2$, an element $\mu_i\in\Omega^i(Q, \underline{Hom}^{-i}(\underline{E}, \underline{F}))$.*
(ii) *An element $\mu^b\in\Gamma(B\otimes\underline{Hom}^{-2}(\underline{E}, \underline{F}))$.*

The above objects are subject to the relations

(1) $[\partial, \mu_i] + [\mathrm{d}_\nabla, \mu_{i-1}] + \sum_{j+k=i,\, i\geq 2} [\omega_j, \mu_k] = \langle\omega, \mu_{i-3}^b\rangle,$
(2) $[\partial, \mu^b] + [\phi_0, \mu_0] + \partial_B\circ\mu_1 = 0,$
(3) $\mathrm{d}_{\overline{\nabla}}\mu^b + [\phi_0, \mu_1] + [\phi_1, \mu_0] + \partial_B\circ\mu_2 = 0.$

Proof As before it suffices to check how μ acts on $\Gamma(\underline{E})$, by the same arguments. Then it must be of the type

$$\mu = \mu_0 + \mu_1 + \mu_2 + \mu^b,$$

where $\mu_i\in\Omega^i(Q, \underline{\mathrm{Hom}}^{-i}(\underline{E}, \underline{F}))$ and $\mu^b\in\Gamma(B)\otimes\Gamma(\underline{\mathrm{Hom}}^{-2}(\underline{E}, \underline{F}))$. It is easy to see that the three equations in the statement come from the expansion of $\mu\circ\mathcal{D}_{\underline{E}} = \mathcal{D}_{\underline{F}}\circ\mu$ when μ is written in terms of the components defined before.

The transformation of ad $\in\underline{\mathrm{Rep}}^\infty(\mathcal{M})$ for a fixed splitting $Q[1]\oplus B^*[2]$ of $\mathcal{M}$ and different choices of TM-connections is given by their difference. More precisely, let ∇ and ∇' be the two TM-connections. Then the map $\mu = \mu_{\nabla'}^{-1}\circ\mu_\nabla\colon \mathrm{ad}_\nabla\to\mathrm{ad}_{\nabla'}$ is defined by $\mu = \mu_0 + \mu_1 + \mu^b$, where

$$\mu_0 = \mathrm{id}$$
$$\mu_1(q)X = \nabla'_X q - \nabla_X q$$

$$\mu^b(\beta)X = \nabla'_X\beta - \nabla_X\beta,$$

for $X \in \mathfrak{X}(M)$, $q \in \Gamma(Q)$ and $\beta \in \Gamma(B^*)$. The equations in Proposition 5.12 are automatically satisfied since by construction

$$\mathcal{D}_{\mathrm{ad}_{\nabla'}} \circ \mu = \mathcal{D}_{\mathrm{ad}_{\nabla'}} \circ \mu_{\nabla'}^{-1} \circ \mu_\nabla = \mu_{\nabla'}^{-1} \circ \pounds_Q \circ \mu_\nabla = \mu_{\nabla'}^{-1} \circ \mu_\nabla \circ \mathcal{D}_{\mathrm{ad}_\nabla} = \mu \circ \mathcal{D}_{\mathrm{ad}_\nabla}.$$

This yields the following result.

Proposition 5.13 *Given two pairs of TM-connections on the bundles B^* and Q, the isomorphism $\mu\colon ad_\nabla \to ad_{\nabla'}$ between the corresponding adjoint representations is given by $\mu = id \oplus \left(\nabla' - \nabla\right)$.*

The next step is to show how the adjoint representation transforms after a change of splitting of the Lie 2-algebroid. Fix a Lie 2-algebroid $(\mathcal{M}, Q)$ over the smooth manifold M and choose a splitting $Q[1] \oplus B^*[2]$, with structure objects $(\ell, \rho, [\cdot\,,\cdot]_1, \nabla^1, \omega^1)$ as before. Recall that a change of splitting does not change the vector bundles B^* and Q, and it is equivalent to a section $\sigma \in \Omega^2(Q, B^*)$. The induced isomorphism of [2]-manifolds over the identity on M is given by: $\mathcal{F}_\sigma^\star(\tau) = \tau$ for all $\tau \in \Gamma(Q^*)$ and $\mathcal{F}_\sigma^\star(b) = b + \sigma^\star b \in \Gamma(B) \oplus \Omega^2(Q)$ for all $b \in \Gamma(B)$. If $(\ell, \rho, [\cdot\,,\cdot]_2, \nabla^2, \omega^2)$ is the structure objects of the second splitting, then the compatibility of σ with the homological vector fields reads the following:

- The skew-symmetric dull brackets are related by $[q_1, q_2]_2 = [q_1, q_2]_1 - \ell(\sigma(q_1, q_2))$.
- The connections are related by $\nabla_q^2 b = \nabla_q^1 b + \partial_B\langle\sigma(q,\cdot), b\rangle$, or equivalently on the dual by $\nabla_q^{2*}\beta = \nabla_q^{1*}\beta - \sigma(q, \ell(\beta))$.
- The curvature terms are related by $\omega^2 = \omega^1 + \mathrm{d}_{2,\nabla^1}\sigma$, where the operator

  $$\mathrm{d}_{2,\nabla^1}\sigma\colon \Omega^\bullet(Q, B^*) \to \Omega^{\bullet+1}(Q, B^*)$$

  is defined by the usual Koszul formula using the dull bracket $[\cdot\,,\cdot]_2$ and the connection ∇^{1*}.

The above equations give the following identities between the structure data for the adjoint representations[13] ad_∇^1 and ad_∇^2.

Lemma 5.14 *Let $q, q_1, q_2 \in \Gamma(Q)$, $\beta \in \Gamma(B^*)$ and $X \in \mathfrak{X}(M)$. Then*

(i) $\ell_2 = \ell_1$ *and* $\rho_2 = \rho_1$.
(ii) $\nabla_{q_1}^{2,bas} q_2 = \nabla_{q_1}^{1,bas} q_2 - \ell(\sigma(q_1, q_2))$

$$\nabla_q^{2,bas} X = \nabla_q^{1,bas} X$$
$$\nabla_q^{2,*}\beta = \nabla_q^{1,*}\beta - \sigma(q, \ell(\beta)).$$

[13] Note that the two pairs of TM-connections are identical.

(iii) $\omega_2^2(q_1, q_2)q_3 = \omega_2^1(q_1, q_2)q_3 + \mathrm{d}_{2,\nabla^1}\sigma(q_1, q_2, q_3)$ $\omega_2^2(q_1, q_2)X = \omega_2^1(q_1, q_2)X + \nabla_X(\ell(\sigma(q_1, q_2))) - \ell(\sigma(q_1, \nabla_X q_2)) + \ell(\sigma(q_2, \nabla_X q_1))$.

(iv) $\omega_3^2(q_1, q_2, q_3)X = \omega_3^1(q_1, q_2, q_3)X + (\nabla_X(\mathrm{d}_{2,\nabla^1}\sigma))(q_1, q_2, q_3)$.

(v) $\phi_0^2(\beta)q = \phi_0^1(\beta)q + \sigma(q, \ell(\beta))$ $\phi_0^2(\beta)X = \phi_0^1(\beta)X$.

(vi) $\phi_1^2(\beta, q)X = \phi_1^1(\beta, q)X - \sigma(\nabla_X q_1, \ell(\beta)) - \sigma(q, \ell(\nabla_X\beta)) + \nabla_X(\sigma(q, \ell(\beta)))$.

Consider now two Lie 2-algebroids $\mathcal{M}_1$ and $\mathcal{M}_2$ over M, and an isomorphism

$$\mathcal{F}\colon (\mathcal{M}_1, \mathcal{Q}_1) \to (\mathcal{M}_2, \mathcal{Q}_2)$$

given by the maps $\mathcal{F}_Q\colon Q_1 \to Q_2, \mathcal{F}_B\colon B_1^* \to B_2^*$, and $\mathcal{F}_0\colon \wedge^2 Q_1 \to B_2^*$. Recall that a 0-morphism between two representations up to homotopy $(\underline{E}_1, \mathcal{D}_1)$ and $(\underline{E}_2, \mathcal{D}_2)$ of $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, is given by a degree 0 map

$$\mu\colon C^\infty(\mathcal{M}_2) \otimes \Gamma\left(\underline{E}_2\right) \to C^\infty(\mathcal{M}_1) \otimes \Gamma\left(\underline{E}_1\right),$$

which is $C^\infty(\mathcal{M}_2)$-linear: $\mu(\xi \otimes e) = \mathcal{F}^\star\xi \otimes \mu(e)$ for all $\xi \in C^\infty(\mathcal{M}_2)$ and $e \in \Gamma(\underline{E}_2)$, and makes the following diagram commute

$$\begin{tikzcd}
C^\infty(\mathcal{M}_2) \otimes \Gamma(\underline{E}_2) \arrow[r, "\mu"] \arrow[d, "\mathcal{D}_2"'] & C^\infty(\mathcal{M}_1) \otimes \Gamma(\underline{E}_1) \arrow[d, "\mathcal{D}_1"] \\
C^\infty(\mathcal{M}_2) \otimes \Gamma(\underline{E}_2) \arrow[r, "\mu"'] & C^\infty(\mathcal{M}_1) \otimes \Gamma(\underline{E}_1).
\end{tikzcd}$$

The usual analysis as before implies that μ must be given by a morphism of complexes $\mu_0\colon (\underline{E}_2, \partial_2) \to (\underline{E}_1, \partial_1)$ and elements

$$\begin{aligned}
\mu_1 &\in \Omega^1\left(Q_1, \underline{\mathrm{Hom}}^{-1}\left(\underline{E}_2, \underline{E}_1\right)\right), \\
\mu_2 &\in \Omega^2\left(Q_1, \underline{\mathrm{Hom}}^{-2}\left(\underline{E}_2, \underline{E}_1\right)\right), \\
\mu^b &\in \Gamma(B) \otimes \Gamma\left(\underline{\mathrm{Hom}}^{-2}\left(\underline{E}_2, \underline{E}_1\right)\right),
\end{aligned}$$

which satisfy equations similar to the set of equations in Proposition 5.12.

A change of splitting of the Lie 2-algebroid transforms as follows the adjoint representation. Since changes of choices of connections are now fully understood, choose the same connection for both splittings $\mathcal{M}_1 \simeq Q[1] \oplus B^*[2] \simeq \mathcal{M}_2$. Suppose that $\sigma \in \Omega^2(Q, B^*)$ is the change of splitting and denote by $\mathcal{F}_\sigma$ the induced isomorphism of the split Lie 2-algebroids whose components are given by $\mathcal{F}^\star_{\sigma,Q} = \mathrm{id}_{Q^*}, \mathcal{F}^\star_{\sigma,B} = \mathrm{id}_B, \mathcal{F}^\star_{\sigma,0} = \sigma^\star$. The composition map $\mu^\sigma : \mathrm{ad}^1_\nabla \to \mathfrak{X}(\mathcal{M}) \to \mathrm{ad}^2_\nabla$ is given in components by

$$\mu_0^\sigma = \mathrm{id}$$

$$\mu_1^\sigma(q_1)q_2 = \sigma(q_1, q_2)$$
$$\mu_2^\sigma(q_1, q_2)X = (\nabla_X\sigma)(q_1, q_2).$$

A similar argument as before implies that μ^σ is a morphism between the two adjoint representations and therefore the following result follows.

Proposition 5.15 *Given two splittings of a Lie 2-algebroid with induced change of splitting $\sigma \in \Omega^2(Q, B^*)$ and a pair of TM-connections on the vector bundles B^* and Q, the isomorphism between the corresponding adjoint representations is given by $\mu = id \oplus \ \sigma \oplus \nabla.\sigma$.*

5.6 Adjoint representation of a Lie *n*-algebroid

The construction of the adjoint representation up to homotopy of a Lie n-algebroid $(\mathcal{M}, \mathcal{Q})$ for general n is similar to the $n = 2$ case. Specifically, choose a splitting $\mathcal{M} \simeq \bigoplus_{i=1}^n E_i[i]$ and TM-connections ∇^{E_i} on the bundles E_i. Then there is an induced isomorphism of $C^\infty(\mathcal{M})$-modules

$$\mu_\nabla : C^\infty(\mathcal{M}) \otimes \Gamma\,(TM[0] \oplus E_1[1] \oplus \cdots \oplus E_n[n]) \to \mathfrak{X}(\mathcal{M}),$$

which at the level of generators is given by

$$\Gamma(E_i) \ni e \mapsto \hat{e} \quad \text{and} \quad \mathfrak{X}(M) \ni X \mapsto \nabla_X^{E_n} \oplus \cdots \oplus \nabla_X^{E_1}.$$

Then μ is used to transfer $\pounds_\mathcal{Q}$ from $\mathfrak{X}(\mathcal{M})$ to obtain the differential $\mathcal{D}_{\mathrm{ad}_\nabla} := \mu^{-1} \circ \pounds_\mathcal{Q} \circ \mu$ on $C^\infty(\mathcal{M}) \otimes \Gamma(TM[0] \oplus E_1[1] \oplus \cdots \oplus E_n[n])$.

6 Split VB-Lie *n*-algebroids

This section gives a picture of representations up to homotopy in more "classical" geometric terms. That is, in terms of linear Lie n-algebroid structures on double vector bundles. It introduces the notion of split VB-Lie n-algebroids and explains how they correspond to $(n+1)$-representations of Lie n-algebroids. In particular, the tangent of a Lie n-algebroid is a VB-Lie n-algebroid which is linked to the adjoint representation. The main result in this section is a generalisation of the correspondence between decomposed VB-algebroids and 2-representations in [18].

6.1 Double vector bundles

Recall that a double vector bundle (D, V, F, M) is a commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{\pi_F} & F \\ {\scriptstyle\pi_V}\downarrow & & \downarrow{\scriptstyle q_F} \\ V & \xrightarrow[q_V]{} & M \end{array}$$

such that all the arrows are vector bundle projections and the structure maps of the bundle $D \to V$ are vector bundle morphisms over the corresponding structure maps of $F \to M$ (see [30]). This is equivalent to the same condition holding for the structure maps of $D \to F$ over $V \to M$. The bundles V and F are called the side bundles of D. The intersection of the kernels $C := \pi_V^{-1}(0^V) \cap \pi_F^{-1}(0^F)$ is the *core* of D and is naturally a vector bundle over M, with projection denoted by $q_C : C \to M$. The inclusion $C \hookrightarrow D$ is denoted by $C_m \ni c_m \mapsto \overline{c} \in \pi_V^{-1}(0_m^V) \cap \pi_F^{-1}(0_m^F)$.

A morphism (G_D, G_V, G_F, g) of two double vector bundles (D, V, F, M) and (D', V', F', M') is a commutative cube

$$\begin{tikzcd}
& D \arrow[rr] \arrow[dl, "G_D"] \arrow[dd] & & F \arrow[dl, "G_F"] \arrow[dd] \\
D' \arrow[rr, crossing over] \arrow[dd] & & F' & \\
& V \arrow[rr] \arrow[dl, "G_V"] & & M \arrow[dl, "g"] \\
V' \arrow[rr] & & M' \arrow[from=uu, crossing over] &
\end{tikzcd}$$

such that all the faces are vector bundle maps.

Given a double vector bundle (D, V, F, M), the space of sections of D over V, denoted by $\Gamma_V(D)$, is generated as a $C^\infty(V)$-module by two special types of sections, called *core* and *linear* sections and denoted by $\Gamma_V^c(D)$ and $\Gamma_V^l(D)$, respectively (see [30]). The core section $c^\dagger \in \Gamma_V^c(D)$ corresponding to $c \in \Gamma(C)$ is defined as

$$c^\dagger(v_m) = 0_{v_m}^D +_F \overline{c(m)}, \quad \text{for } m \in M \text{ and } v_m \in V_m.$$

A section $\delta \in \Gamma_V(D)$ is linear over $f \in \Gamma(F)$, if $\delta : V \to D$ is a vector bundle morphism $V \to D$ over $f : M \to F$.

Finally, a section $\psi \in \Gamma(V^* \otimes C)$ defines a linear section $\psi^\wedge : V \to D$ over the zero section $0^F : M \to F$ by

$$\psi^\wedge(v_m) = 0_{v_m}^D +_F \overline{\psi(v_m)}$$

for all $m \in M$ and $v_m \in V_m$. This type of linear section is called a *core-linear* section. In terms of the generators $\theta \otimes c \in \Gamma(V^* \otimes C)$, the correspondence above reads $(\theta \otimes c)^\wedge = \ell_\theta \cdot c^\dagger$, where ℓ_θ is the linear function on V associated to $\theta \in \Gamma(V^*)$.

Example 6.1 (Decomposed double vector bundle) Let V, F, C be vector bundles over the same manifold M. Set $D := V \times_M F \times_M C$ with vector bundle structures $D = q_V^!(F \oplus C) \to V$ and $D = q_F^!(V \oplus C) \to F$. Then (D, V, F, M) is a double vector bundle, called the decomposed double vector bundle with sides V and F and with core C. Its core sections have the form $c^\dagger : f_m \mapsto (0_m^V, f_m, c(m))$, for $m \in M$, $f_m \in F_m$ and $c \in \Gamma(C)$, and the space of linear sections $\Gamma_V^l(D)$ is naturally identified with

$\Gamma(F) \oplus \Gamma(V^* \otimes C)$ via $(f, \psi)\colon v_m \mapsto (f(m), v_m, \psi(v_m))$ where $\psi \in \Gamma(V^* \otimes C)$ and $f \in \Gamma(F)$. This yields the canonical *linear horizontal lift* $h\colon \Gamma(F) \hookrightarrow \Gamma^l_V(D)$.

Example 6.2 (Tangent bundle of a vector bundle) Given a vector bundle $q\colon E \to M$, its tangent bundle TE is naturally a vector bundle over the manifold E. In addition, applying the tangent functor to all the structure maps of $E \to M$ yields a vector bundle structure on $Tq\colon TE \to TM$ which is called the *tangent prolongation* of E. Hence, (TE, TM, E, M) has a natural double vector bundle structure with sides TM and E. Its core is naturally identified with $E \to M$ and the inclusion $E \hookrightarrow TE$ is given by $E_m \ni e_m \mapsto \frac{d}{dt}\big|_{t=0} te_m \in T^q_{0^E_m}E$. For $e \in \Gamma(E)$, the section $Te \in \Gamma^l_{TM}(TE)$ is linear over e. The core vector field $e^\dagger \in \Gamma_{TM}(TE)$ is defined by $e^\dagger(v_m) = T_m 0^E(v_M) +_E \frac{d}{dt}\mid_{t=0} te(m)$ for $m \in M$ and $v_m \in T_M M$ and the *vertical lift* $e^\uparrow \in \Gamma_E(TE) = \mathfrak{X}(E)$ is the (core) vector field defined by the flow $\mathbb{R} \times E \to E$, $(t, e'_m) \mapsto e'_m + te(m)$. Elements of $\Gamma^l_E(TE) =: \mathfrak{X}^l(E)$ are called *linear vector fields* and are equivalent to derivations $\delta\colon \Gamma(E) \to \Gamma(E)$ over some element in $\mathfrak{X}(M)$ [30]. The linear vector field which corresponds to the derivation δ is written X_δ.

6.2 Linear splittings, horizontal lifts and duals

A *linear splitting* of a double vector bundle (D, V, F, M) with core C is a double vector bundle embedding Σ of the decomposed double vector bundle $V \times_M F$ into D over the identities on V and F. It is well-known that every double vector bundle admits a linear splitting, see [14, 16, 38] or [21] for the general case. Moreover, a linear splitting is equivalent to a *decomposition* of D, i.e. to an isomorphism of double vector bundles $S : V \times_M F \times_M C \to D$ over the identity on V, F and C. Given Σ, the decomposition is obtained by setting $S(v_m, f_m, c_m) = \Sigma(v_m, f_m) +_F (0_{f_m} +_V \overline{c_m})$, and conversely, given S, the splitting is defined by $\Sigma(v_m, f_m) = S(v_m, f_,, 0^C_m)$.

A linear splitting of D, and consequently a decomposition, is also equivalent to a *horizontal lift*, i.e. a right splitting of the short exact sequence

$$0 \to \Gamma(V^* \otimes C) \to \Gamma^l_V(D) \to \Gamma(F) \to 0$$

of $C^\infty(M)$-modules. The correspondence is given by $\sigma_F(f)(v_m) = \Sigma(f(m), b_m)$ for $f \in \Gamma(F)$, $m \in M$ and $b_m \in B(m)$. Note that all the previous constructions can be done similarly if one interchanges the roles of V and F.

Example 6.3 For the tangent bundle TE of a vector bundle $E \to M$, a linear splitting is equivalent to a choice of a TM-connection on E. Specifically, given a horizontal lift $\sigma : \mathfrak{X}(M) \to \mathfrak{X}^l(E)$, the corresponding connection ∇ is defined by $\sigma(Y) = X_{\nabla_Y}$ for all $Y \in \mathfrak{X}(M)$.

Double vector bundles can be dualized in two ways, namely, as the dual of D either over V or over F [30]. Precisely, from a double vector bundle (D, V, F, M) with core C, one obtains the double vector bundles

$$\begin{tikzcd} D_V^* \arrow[r, "\pi_{C^*}"] \arrow[d, "\pi_V"'] & C^* \arrow[d, "q_{C^*}"] \\ V \arrow[r, "q_V"'] & M \end{tikzcd} \qquad \begin{tikzcd} D_F^* \arrow[r, "\pi_F"] \arrow[d, "\pi_{C^*}"'] & F \arrow[d, "q_F"] \\ C^* \arrow[r, "q_{C^*}"'] & M \end{tikzcd}$$

with cores F^* and V^*, respectively.

Given a linear splitting $\Sigma\colon V \times_M F \to D$, the dual splitting $\Sigma^*\colon V \times_M C^* \to D_V^*$ is defined by

$$\left\langle \Sigma^*(v_m, \gamma_m), \Sigma(v_m, f_m)\right\rangle = 0 \text{ and } \left\langle \Sigma^*(v_m, \gamma_m), c^\dagger(v_m)\right\rangle = \langle \gamma_m, c(m)\rangle\,,$$

for all $m \in M$ and $v_m \in V_m$, $f_m \in F_m$, $\gamma_m \in C_m^*$, $c \in \Gamma(C)$.

6.3 VB-Lie n-algebroids and $(n+1)$-representations

Suppose now that $(\underline{D}, V, \underline{A}, M)$ is a double vector bundle together with graded vector bundle decompositions $\underline{D} = D_1[1] \oplus \cdots \oplus D_n[n]$ and $\underline{A} = A_1[1] \oplus \cdots \oplus A_n[n]$, over V and M, respectively, which are compatible with the projection $\underline{D} \to \underline{A}$. This means that each of the individual squares (D_i, V, A_i, M) also forms a double vector bundle. Schematically, this yields the following sequence of diagrams

$$\begin{tikzcd} & D_1[1] \arrow[rr] \arrow[ddddl] & & A_1[1] \arrow[ddddl] \\ & \oplus & & \oplus \\ & D_2[2] \arrow[rr] \arrow[ddl] & & A_2[2] \arrow[ddl] \\ & \oplus & & \oplus \\ V \arrow[rr] & \vdots & M & \vdots \\ & \oplus & & \oplus \\ & D_n[n] \arrow[rr] \arrow[uul] & & A_n[n] \arrow[uul] \end{tikzcd}$$

where all the “planes" are double vector bundles. This yields that the core of $(\underline{D}, V, \underline{A}, M)$ is the graded vector bundle $\underline{C} = C_1[1] \oplus \cdots \oplus C_n[n]$, where C_i is the core of (D_i, V, A_i, M), for $i = 1, \ldots, n$.

Definition 6.4 The quadruple $(\underline{D}, V, \underline{A}, M)$ is a *(split) VB-Lie n-algebroid* if

(1) the graded vector bundle $\underline{D} \to V$ is endowed with a homological vector field $\mathcal{Q}_{\underline{D}}$,
(2) the Lie n-algebroid structure of $\underline{D} \to V$ is *linear*, in the sense that

  (a) the anchor $\rho_D\colon D_1 \to TV$ is a double vector bundle morphism,
  (b) the map ∂_{D_i} fits into a morphism of double vector bundles $(\partial_{D_i}, \mathrm{id}_V, \partial_{A_i}, \mathrm{id}_M)$ between (D_i, V, A_i, M) and (D_{i+1}, V, A_{i+1}, M) for all i,
  (c) the multi-brackets of $\underline{D}$ satisfy the following relations:

(i) the i-bracket of i linear sections is a linear section;
(ii) the i-bracket of $i-1$ linear sections with a core section is a core section;
(iii) the i-bracket of $i-k$ linear sections with k core sections, $i \geq k \geq 2$, is zero;
(iv) the i-bracket of i core sections is zero.

Remark 6.5 (1) A VB-Lie n-algebroid structure on the double vector bundle $(\underline{D}, V, \underline{A}, M)$ defines uniquely a Lie n-algebroid structure on $\underline{A} \to M$ as follows: the anchor $\rho_D\colon D_1 \to TV$ is linear over the anchor $\rho\colon A_1 \to TM$, and if all $d_k \in \Gamma^l_V(\underline{D})$ cover $a_k \in \Gamma(\underline{A})$ for $k = 1, 2, \ldots, i$, then $[\![d_1, \ldots, d_i]\!]_{\underline{D}} \in \Gamma^l_V(\underline{D})$ covers $[\![a_1, \ldots, a_i]\!]_{\underline{A}} \in \Gamma(\underline{A})$. Therefore, the graded vector bundles $\underline{D} \to V$ and $\underline{A} \to M$ are endowed with homological vector fields $\mathcal{Q}_{\underline{D}}$ and $\mathcal{Q}_{\underline{A}}$ for which the bundle projection $\underline{D} \to \underline{A}$ is a morphism of Lie n-algebroids over the projection $V \to M$.

(2) A VB-Lie 1-algebroid as in the definition above is just a VB-algebroid.

(3) More concisely, a VB-Lie n-algebroid can be defined as an $[n]$-vector bundle $\mathcal{E}$ over an $[n]$-manifold $\mathcal{M}$, together with a homological vector field $\mathcal{Q}_{\mathcal{E}}$ on $\mathcal{E}$, that is linear over a homological vector field $\mathcal{Q}$ on $\mathcal{M}$. Then the set of sections of $\mathcal{E}^*$, equipped with $\mathcal{Q}_{\mathcal{E}}$, is a DG $(\mathcal{M}, \mathcal{Q})$-module, see also Remark 5.1. The goal of this section is however to provide explicit formulas in the split case, for the convenience of the reader.

Example 6.6 (Tangent prolongation of a (split) Lie n-algebroid) The basic example of a split VB-Lie n-algebroid is obtained by applying the tangent functor to a split Lie n-algebroid $\underline{A} = A_1[1] \oplus \cdots \oplus A_n[n] \to M$. The double vector bundle is given by the diagram

$$\begin{array}{ccc} \underline{TA} & \longrightarrow & \underline{A} \\ \downarrow & & \downarrow \\ TM & \longrightarrow & M \end{array}$$

where the Lie n-algebroid structure of $\underline{TA} = T\underline{A} = TA_1[1] \oplus \cdots \oplus TA_n[n]$ over the manifold TM is defined by the relations

(1) $\rho_{TA} = J_M \circ T_{\rho_A}\colon TA_1 \to TTM$, where $J_M\colon TTM \to TTM$ is the canonical involution, see e.g. [30],
(2) $[\![Ta_{k_1}, \ldots, Ta_{k_i}]\!] = T[\![a_{k_1}, \ldots, a_{k_i}]\!]$,
(3) $[\![Ta_{k_1}, \ldots, Ta_{k_{i-1}}, a^\dagger_{k_i}]\!] = [\![a_{k_1}, \ldots, a_{k_{i-1}}, a_{k_i}]\!]^\dagger$,
(4) $[\![Ta_{k_1}, \ldots, Ta_{k_j}, a^\dagger_{k_{j+1}}, \ldots, a^\dagger_{k_i}]\!] = 0$ for all $1 \leq j \leq i-2$,
(5) $[\![a^\dagger_{k_1}, \ldots, a^\dagger_{k_i}]\!] = 0$,

for all sections $a_{k_j} \in \Gamma(A_{k_j})$ with pairwise distinct k_j and all i.

Applying the above construction to a split Lie 2-algebroid $Q[1] \oplus B^*[2] \to M$ with structure $(\rho_Q, \ell, \nabla^*, \omega)$ yields as follows the objects $(\rho_{TQ}, T\ell, T\nabla^*, T\omega)$ of the split Lie 2-algebroid structure of $TQ[1] \oplus TB^*[2] \to TM$: The complex $TB^* \to$

$TQ \to TTM$ consists of the anchor of TQ given by $\rho_{TQ} = J_M \circ T\rho_Q$, and the vector bundle map $T\ell\colon TB^* \to TQ$. The bracket of TQ is defined by the relations

$$[Tq_1, Tq_2]_{TQ} = T[q_1, q_2]_Q, \qquad [Tq_1, q_2^\dagger]_{TQ} = [q_1, q_2]_Q^\dagger, \qquad [q_1^\dagger, q_2^\dagger]_{TQ} = 0,$$

for $q_1, q_2 \in \Gamma(Q)$. The TQ-connection $T\nabla^*\colon \Gamma_{TM}(TQ) \times \Gamma_{TM}(TB^*) \to \Gamma_{TM}(TB^*)$ is defined by

$$(T\nabla^*)_{Tq}(T\beta) = T(\nabla_q^*\beta), \quad (T\nabla^*)_{Tq}(\beta^\dagger) = (\nabla_q^*\beta)^\dagger = (T\nabla^*)_{q^\dagger}\beta,$$
$$(T\nabla^*)_{q^\dagger}(\beta^\dagger) = 0,$$

for $q \in \Gamma(Q)$ and $\beta \in \Gamma(B^*)$. Finally, the 3-form $T\omega \in \Omega^3(TQ, TB^*)$ is defined by

$$(T\omega)(Tq_1, Tq_2, Tq_3) = T(\omega(q_1, q_2, q_3)), \quad (T\omega)(Tq_1, Tq_2, q_3^\dagger) = \omega(q_1, q_2, q_3)^\dagger,$$
$$(T\omega)(q_1, q_2^\dagger, q_3^\dagger) = 0 = (T\omega)(q_1^\dagger, q_2^\dagger, q_3^\dagger),$$

for $q_1, q_2, q_3 \in \Gamma(Q)$.

As it is shown in [18], an interesting fact about the tangent prolongation of a Lie algebroid is that it encodes its adjoint representation. The same holds for a split Lie n-algebroid $A_1[1] \oplus \cdots \oplus A_n[n]$, since by definition the adjoint module is exactly the space of sections of the $\mathcal{Q}$-vector bundle $T(A_1[1] \oplus \cdots \oplus A_n[n]) \to A_1[1] \oplus \cdots \oplus A_n[n]$. The next example shows this correspondence explicitly in the case of split Lie 2-algebroids $Q[1] \oplus B^*[2]$.

Example 6.7 Choose two TM-connections on Q and B^*, both denoted by ∇. These choices induce the horizontal lifts $\Gamma(Q) \to \Gamma_{TM}^l(TQ)$ and $\Gamma(B^*) \to \Gamma_{TM}^l(TB^*)$, both denoted by h. More precisely, given a section $q \in \Gamma(Q)$, its lift is defined as $h(q) = Tq - (\nabla_. q)^\wedge$. A similar formula holds for $h(\beta)$ as well. Then an easy computation yields the following:

(1) $\rho_{TQ}(q^\dagger) = \rho(q)^\uparrow$ and $(T\ell)(\beta^\dagger) = \ell(\beta)^\uparrow$
(2) $\rho_{TQ}(h(q)) = X_{\nabla_q^{\text{bas}}}$
(3) $(T\ell)(h(\beta)) = h(\ell(\beta)) + (\nabla_.(\ell(\beta)) - \ell(\nabla_.\beta))^\wedge$
(4) $[h(q_1), h(q_2)]_{TQ} = h[q_1, q_2]_Q - R_\nabla^{\text{bas}}(q_1, q_2)^\wedge$
(5) $[h(q_1), q_2^\dagger]_{TQ} = (\nabla_{q_1}^{\text{bas}} q_2)^\dagger$
(6) $(T\nabla^*)_{h(q)}(\beta^\dagger) = (\nabla_q^*\beta)^\dagger$
(7) $(T\nabla^*)_{q^\dagger}(h(\beta)) = (\nabla_q^*\beta - \nabla_{\rho(q)}\beta)^\dagger$
(8) $(T\nabla^*)_{h(q)}(h(\beta)) = h(\nabla_q^*\beta) + \Big(\nabla_{\nabla_. q}^*\beta - \nabla_{\rho(\nabla_. q)}\beta + \nabla_q^*\nabla_.\beta - \nabla_.\nabla_q^*\beta - \nabla_{[\rho(q),\cdot]}\beta\Big)^\wedge$
(9) $(T\omega)(h(q_1), h(q_2), h(q_3)) = h(\omega(q_1, q_2, q_3)) + ((\nabla_.\omega)(q_1, q_2, q_3))^\wedge$
(10) $(T\omega)(h(q_1), h(q_2), q_3^\dagger) = (\omega(q_1, q_2, q_3))^\dagger$.

In fact, this result is a special case of a correspondence between VB-Lie n-algebroid structures on a decomposed graded double vector bundle $(\underline{D}, V, \underline{A}, M)$ and $(n+1)$-representations of $\mathcal{M} = \underline{A}$ on the complex $\underline{E} = V[0] \oplus C_1[1] \oplus \cdots \oplus C_n[n]$. In the general case (for n-arbitrary), it is easier to give the correspondence in terms of the homological vector field on $\underline{D}$ and the dual representation on $\underline{E}^* = C_n^*[-n] \oplus \cdots \oplus C_1^*[-1] \oplus V^*[0]$.

Suppose that $(\underline{D}, V, \underline{A}, M)$ is a VB-Lie n-algebroid with homological vector fields $\mathcal{Q}_{\underline{D}}$ and $\mathcal{Q}_{\underline{A}}$, and choose a decomposition for each double vector bundle (D_i, V, A_i, M)[14], and consequently for $(\underline{D}, V, \underline{A}, M)$. Consider the dual $\underline{D}_V^*$ and recall that the spaces $\Gamma_V(D_i^*)$ are generated as $C^\infty(V)$-modules by core and linear sections. For the latter, use the identification $\Gamma_V^l(D_i^*) = \Gamma(A_i^* \otimes V^*) \oplus \Gamma(C_i^*)$ induced by the decomposition. Accordingly, the element $\alpha \in \Gamma(A_i^*)$ is identified with the core section $\pi_{\underline{A}}^\star(\alpha) \in \Gamma_V^c(\underline{D}^*)$.

For all $\psi \in \Gamma(V^*)$, the 1-form $\mathrm{d}\ell_\psi$ is a linear section of $T^*V \to V$ over ψ and the anchor $\rho_{D_1}\colon D_1 \to TV$ is a morphism of double vector bundles. This implies that the degree 1 function $\mathcal{Q}_{\underline{D}}(\ell_\psi) = \rho_{D_1}^* \mathrm{d}\ell_\psi$ is a linear section of $\Gamma_V(\underline{D}^*)$ and thus

$$\mathcal{Q}_{\underline{D}}(\ell_\psi) \in \Gamma_V^l\left(D_1^*\right) = \Gamma\left(A_1^* \otimes V^*\right) \oplus \Gamma\left(C_1^*\right).$$

Moreover, due to the decomposition, $D_i = q_V^!(A_i \oplus C_i)$ as vector bundles over V for all $i = 1, \ldots, n$. Given $\gamma \in \Gamma(C_i^*)$, the function $\mathcal{Q}_{\underline{D}}(\gamma)$ lies in $\Gamma(\underline{S}^{i+1}(\underline{D}_V^*))$, where $\underline{D}_V^* = q_V^!(A_1^* \oplus C_1^*) \oplus \cdots \oplus q_V^!(A_n^* \oplus C_n^*)$. A direct computation shows that the components of $\mathcal{Q}_{\underline{D}}(\gamma)$ which lie in spaces with two or more sections of the form $\Gamma(q_V^! C_i^*)$ and $\Gamma(q_V^! C_j^*)$ vanish due to the bracket conditions of a VB-Lie n-algebroid. Therefore, define the representation $\mathcal{D}^*$ of $\underline{A}$ on the dual complex $\underline{E}^*$ by the equations

$$\mathcal{Q}_{\underline{D}}(\ell_\psi) = \mathcal{D}^*(\psi) \text{ and } \mathcal{Q}_{\underline{D}}(\gamma) = \mathcal{D}^*(\gamma),$$

for all $\psi \in \Gamma(V^*)$ and all $\gamma \in \Gamma(C_i^*)$.

Conversely, given a representation $\mathcal{D}^*$ of $\underline{A}$ on $\underline{E}^*$, the above equations together with

$$\mathcal{Q}_{\underline{D}}\left(q_V^* f\right) = \pi_{\underline{A}}^\star\left(\mathcal{Q}_{\underline{A}}(f)\right) \text{ and } \mathcal{Q}_{\underline{D}}\left(\pi_{\underline{A}}^\star(\alpha)\right) = \pi_{\underline{A}}^\star\left(\mathcal{Q}_{\underline{A}}(\alpha)\right)$$

for all $f \in C^\infty(M)$ and $\alpha \in \Gamma(\underline{A}^*)$, define a VB-Lie n-algebroid structure on the double vector bundle $(\underline{D}, V, \underline{A}, M)$. As discussed in Remark 6.5, this yields the following theorem.

Theorem 6.8 *Let $(\underline{D}, V, \underline{A}, M)$ be a decomposed graded double vector bundle as above with core $\underline{C}$. There is a 1-1 correspondence between VB-Lie n-algebroid structures on $(\underline{D}, V, \underline{A}, M)$ and $(n+1)$-representations up to homotopy of $\underline{A}$ on the complex $V[0] \oplus C_1[1] \oplus \cdots \oplus C_n[n]$.*

[14] In the case of the tangent Lie n-algebroid, this corresponds to choosing the TM-connections on the vector bundles of the adjoint complex.

7 Constructions in terms of splittings

This section presents in terms of splittings two of the applications of the adjoint and coadjoint representations that were defined before. First, there is an explicit description of the Weil algebra of a split Lie n-algebroid together with its structure differentials, in terms of vector bundles and connections, similarly to [2]. Second, the map between the coadjoint and the adjoint representations in the case of a Poisson Lie n-algebroid for degrees $n \leq 2$ is examined in detail.

7.1 The Weil algebra of a split Lie *n*-algebroid

Suppose first that $\mathcal{M} = Q[1] \oplus B^*[2]$ is a split Lie 2-algebroid and consider two TM-connections on the vector bundles Q and B^*, both denoted by ∇. Recall from Sect. 5.4 the (non-canonical) isomorphism of DG $\mathcal{M}$-modules

$$\mathfrak{X}(\mathcal{M}) \cong C^\infty(\mathcal{M}) \otimes \Gamma\left(TM[0] \oplus Q[1] \oplus B^*[2]\right).$$

This implies that

$$\Omega^1(\mathcal{M}) \cong C^\infty(\mathcal{M}) \otimes \Gamma\left(B[-2] \oplus Q^*[-1] \oplus T^*M[0]\right),$$

and thus the generators of the Weil algebra can be identified with

$$\underbrace{C^\infty(\mathcal{M})^t}_{(t,0)}, \underbrace{\Gamma(\wedge^u T^*M)}_{(0,u)}, \underbrace{\Gamma(S^v Q^*)}_{(v,v)}, \underbrace{\Gamma(\wedge^w B)}_{(2w,w)}.$$

Using also that $C^\infty(\mathcal{M})^t = \bigoplus_{t=r+2s} \Gamma(\wedge^r Q^*) \otimes \Gamma(S^s B)$, the space of (p,q)-forms is decomposed as

$$\begin{aligned} W^{p,q}(\mathcal{M},\nabla) &= \bigoplus_{\substack{p=t+v+2w\\ q=u+w+v}} C^\infty(\mathcal{M})^t \otimes \Gamma\left(\wedge^u T^*M \otimes S^v Q^* \otimes \wedge^w B\right) \\ &= \bigoplus_{\substack{p=r+2s+v+2w\\ q=u+w+v}} \Gamma\left(\wedge^u T^*M \otimes \wedge^r Q^* \otimes S^v Q^* \otimes \wedge^w B \otimes S^s B\right). \end{aligned}$$

Therefore, after a choice of splitting and TM-connections ∇ on Q and B^*, the total space of the Weil algebra of $\mathcal{M}$ can be written as

$$W(\mathcal{M},\nabla) = \bigoplus_{r,s,u,v,w} \Gamma\left(\wedge^u T^*M \otimes \wedge^r Q^* \otimes S^v Q^* \otimes \wedge^w B \otimes S^s B\right).$$

The next step is to express the differentials $\pounds_Q$ and $\mathbf{d}$ on $W(\mathcal{M},\nabla)$ in terms of the two TM-connections ∇. For the horizontal differential, recall that by definition the q-th row of the double complex $W(\mathcal{M},\nabla)$ equals the space of q-forms $\Omega^q(\mathcal{M})$ on

$\mathcal{M}$ with differential given by the Lie derivative $\pounds_Q$. Due to the identification of DG $\mathcal{M}$-modules

$$\Omega^q(\mathcal{M}) = \Omega^1(\mathcal{M}) \wedge \ldots \wedge \Omega^1(\mathcal{M}) = C^\infty(\mathcal{M}) \otimes \Gamma\left(\mathrm{ad}^*_\nabla \wedge \cdots \wedge \mathrm{ad}^*_\nabla\right)$$

(q-times) and the Leibniz identity for $\pounds_Q$, it follows that the q-th row of $W(\mathcal{M}, \nabla)$ becomes the q-symmetric power of the coadjoint representation $\underline{S}^q(\mathrm{ad}^*_\nabla)$ and $\pounds_Q = \mathcal{D}_{\underline{S}^q(\mathrm{ad}^*_\nabla)}$.

The vertical differential $\mathbf{d}$ is built from two 2-representations of the tangent Lie algebroid TM, namely the dualization of the TM-representations on the graded vector bundles $\underline{E}_Q = Q[0] \oplus Q[-1]$ and $\underline{E}_{B^*} = B^*[0] \oplus B^*[-1]$ whose differentials are given by the chosen TM-connections $(\mathrm{id}_Q, \nabla, R_\nabla)$ and $(\mathrm{id}_{B^*}, \nabla, R_\nabla)$, respectively. Indeed, suppose first that $\tau \in \Gamma(Q^*)$ and $b \in \Gamma(B)$ are functions on $\mathcal{M}$, i.e. 0-forms. Then from Remark 5.11, it follows that $\mathbf{d}$ acts via

$$\mathbf{d}\tau = \tau + \mathrm{d}_{\nabla^*}\tau \qquad \text{and} \qquad \mathbf{d}b = b + \mathrm{d}_{\nabla^*}b.$$

If now $\tau \in \Gamma(Q^*)$, $b \in \Gamma(B)$ are 1-forms on $\mathcal{M}$, then

$$\begin{aligned}\mathbf{d}\tau &= \mathbf{d}\left(\tau + \mathrm{d}_{\nabla^*}\tau - \mathrm{d}_{\nabla^*}\tau\right) = \mathbf{d}^2\tau - \mathbf{d}\left(\mathrm{d}_{\nabla^*}\tau\right) = \mathrm{d}_{\nabla^*}\tau - \mathrm{d}^2_{\nabla^*}\tau,\\ \mathbf{d}b &= \mathbf{d}\left(b + \mathrm{d}_{\nabla^*}b - \mathrm{d}_{\nabla^*}b\right) = \mathbf{d}^2 b - \mathbf{d}\left(\mathrm{d}_{\nabla^*}b\right) = \mathrm{d}_{\nabla^*}b - \mathrm{d}^2_{\nabla^*}b.\end{aligned}$$

Remark 7.1 Note that if $B^* = 0$, i.e. $\mathcal{M}$ is an ordinary Lie algebroid $A \to M$, the above construction recovers (up to isomorphism) the connection version of the Weil algebra $W(A, \nabla)$ from [1, 2, 34].

In the general case of a split Lie n-algebroid $\mathcal{M} = A_1[1] \oplus \cdots \oplus A_n[n]$ with a choice of TM-connections on all the bundles A_i, one may apply the same procedure as above to obtain the (non-canonical) DG $\mathcal{M}$-module isomorphisms

$$\begin{aligned}&\mathfrak{X}(\mathcal{M}) \cong C^\infty(\mathcal{M}) \otimes \Gamma\left(TM[0] \oplus A_1[1] \oplus \cdots \oplus A_n[n]\right)\\ &\Omega^1(\mathcal{M}) \cong C^\infty(\mathcal{M}) \otimes \Gamma\left(A_n^*[-n] \oplus \cdots \oplus A_1^*[-1] \oplus T^*M[0]\right),\end{aligned}$$

and hence the identification of the generators of the Weil algebra with

$$\underbrace{C^\infty(\mathcal{M})^t}_{(t,0)}, \underbrace{\Gamma(\wedge^u T^*M)}_{(0,u)}, \underbrace{\Gamma(S^{v_1} A_1^*)}_{(v_1,v_1)}, \underbrace{\Gamma(\wedge^{v_2} A_2^*)}_{(2v_2,v_2)}, \ldots, \underbrace{\Gamma(\wedge^{v_n} A_n^*)}_{(nv_n,v_n)}.$$

This then yields

$$\begin{aligned}W^{p,q}(\mathcal{M}, \nabla) &= \bigoplus_{\substack{p=t+v_1+2v_2+\ldots\\ q=u+v_1+v_2+\ldots}} C^\infty(\mathcal{M})^t \otimes \Gamma\left(\wedge^u T^*M \otimes S^{v_1} A_1^* \otimes \wedge^{v_2} A_2^* \otimes \ldots\right)\\ &= \bigoplus_{\substack{p=r_1+v_1+2r_2+2v_2+\ldots\\ q=u+v_1+v_2+\ldots}} \Gamma\left(\wedge^u T^*M \otimes \wedge^{r_1} A_1^* \otimes S^{v_1} A_1^* \otimes S^{r_2} A_2^* \otimes \wedge^{v_2} A_2^* \otimes \ldots\right).\end{aligned}$$

Similar considerations as before imply that the q-th row of $W(\mathcal{M}, \nabla)$ is given by $\underline{S}^q(\mathrm{ad}^*_\nabla)$ with $\pounds_Q = \mathcal{D}_{\underline{S}^q(\mathrm{ad}^*_\nabla)}$, and that $\mathbf{d}$ is built again by the dualization of the 2-representations of TM on the graded vector bundles $\underline{E}_{A_i} = A_i[0] \oplus A_i[-1]$, for $i = 1, \ldots, n$, whose differentials are given by $(\mathrm{id}_{A_i}, \nabla, R_\nabla)$.

7.2 Poisson Lie algebroids of low degree

This section describes in detail the degree $-n$ morphism $\sharp\colon \mathrm{ad}^*_\nabla \to \mathrm{ad}_\nabla$ of right n-representations in the case of Poisson Lie n-algebroids for $n = 0, 1, 2$. Recall that the map $\sharp$ sends an exact 1-form $\mathrm{d}\xi$ of the graded manifold $\mathcal{M}$ to the vector field $\{\xi, \cdot\}$.

First, consider a Poisson Lie 0-algebroid, i.e. a usual Poisson manifold $(M, \{\cdot\,, \cdot\})$. Then the Lie 0-algebroid is just M, with a trivial homological vector field – it can be thought of as a trivial Lie algebroid $A = 0 \times M \to M$ with trivial differential $\mathrm{d}_A = 0$, and consequently trivial homological vector field. The coadjoint and adjoint representations are just the vector bundles $T^*M[0]$ and $TM[0]$, respectively, with zero module differentials, and the map $\sharp$ simply becomes the usual vector bundle map induced by the Poisson bivector field that corresponds to the Poisson bracket

$$\sharp\colon T^*M[0] \to TM[0].$$

Consider a Lie algebroid $A \to M$ with anchor $\rho\colon A \to TM$ and a linear Poisson structure $\{\cdot\,, \cdot\}$, i.e. a Lie algebroid structure on the dual $A^* \to M$. It is easy to see that this means that the [1]-manifold $A[1]$ has a Poisson structure of degree -1. This Poisson structure is the Schouten bracket defined on $\Omega^\bullet(A)$ by the Lie algebroid bracket on A^*. Then it is immediate that $(A[1], \mathrm{d}_A, \{\cdot\,, \cdot\})$ is a Poisson Lie 1-algebroid if and only if (A, A^*) is a Lie bialgebroid. The latter is equivalent to $(A, \{\cdot\,, \cdot\})$ being a Poisson Lie algebroid [32].

Let $\rho'\colon A^* \to TM$, $\alpha \mapsto \{\alpha, \cdot\}|_{C^\infty(M)}$ and $[\cdot\,, \cdot]_* := \{\cdot\,, \cdot\}|_{\Omega^1(A)\times\Omega^1(A)}$ be the anchor and bracket of A^*, respectively. After a choice of a TM-connection ∇ on the vector bundle A, the map $\sharp\colon \mathrm{ad}^*_\nabla \to \mathrm{ad}_\nabla$ acts via

$$\begin{aligned}
\sharp(\mathrm{d}f) &= \sharp_0(\mathrm{d}f) = \{f, \cdot\} = -\rho'^*(\mathrm{d}f) \in \Gamma(A) \\
\sharp(\beta) &= \sharp_0(\beta) + \sharp_1(\cdot)\beta \in \mathfrak{X}(M) \oplus (\Gamma(A^*) \otimes \Gamma(A))
\end{aligned}$$

for all $f \in C^\infty(M)$, $\beta \in \Omega^1(A)$, where we identify β with $\mathbf{d}\beta - \mathrm{d}_{\nabla^*}\beta$, i.e. $\sharp(\beta) = \sharp(\mathbf{d}\beta) - \sharp(\mathrm{d}_{\nabla^*}\beta) = \{\beta, \cdot\} - \sharp(\mathrm{d}_{\nabla^*}\beta)$. Computing how these act on $\alpha \in \Omega^1(M)$ and $g \in C^\infty(M)$, viewed as functions of the graded manifold $A[1]$, gives the components of $\sharp(\beta)$: From the right-hand-side of the equation we obtain

$$(\sharp_0(\beta) + \sharp_1(\cdot)\beta)\, g = \sharp_0(\beta)g \in C^\infty(M)$$

while from the left-hand-side we obtain

$$\sharp(\beta)g = \sharp(\mathbf{d}\beta - \mathrm{d}_{\nabla^*}\beta)g = \{\beta, g\} - \sharp(\mathrm{d}_{\nabla^*}\beta)g = \rho'(\beta)g.$$

From this, it follows that $\sharp_0(\beta) = \rho'(\beta)$. Using now this, the right-hand-side gives

$$(\sharp_0(\beta) + \sharp_1(\cdot)\beta)\,\alpha = \nabla^*_{\rho'(\beta)}\alpha + (\sharp_1(\cdot)\beta)\,\alpha \in \Gamma(A^*) \oplus \Gamma(A^*)$$

while the left-hand-side gives

$$\sharp(\beta)\alpha = \sharp(\mathbf{d}\beta - \mathrm{d}_{\nabla^*}\beta)\alpha = \{\beta, \alpha\} - \sharp(\mathrm{d}_{\nabla^*}\beta)\alpha = [\beta, \alpha]_* + \nabla^*_{\rho'(\alpha)}\beta = (\nabla^*)^{\mathrm{bas}}_\beta\alpha.$$

This implies that $(\sharp_1(\cdot)\beta)\alpha = (\nabla^*)^{\mathrm{bas}}_\beta\alpha - \nabla^*_{\rho'(\alpha)}\beta$ and thus $\sharp$ consists of the (-1)-chain map $\sharp_0$ given by the anti-commutative diagram

$$\begin{tikzcd} T^*M[0] \arrow[r, "-\rho^*"] \arrow[d, "-\rho'^*"] & A^*[-1] \arrow[d, "\rho'"] \\ A[1] \arrow[r, "\rho"] & TM[0] \end{tikzcd}$$

together with $\sharp_1(a)\beta = \langle(\nabla^*)^{\mathrm{bas}}_\beta(\cdot) - \nabla^*_{\rho'(\beta)}(\cdot), a\rangle \in \Gamma(A^{**}) \simeq \Gamma(A)$, for all $\beta \in \Gamma(A^*)$ and $a \in \Gamma(A)$.

By Theorem 4.13, $\sharp$ is an anti-morphism of 2-representations if and only if $(A[1], \mathrm{d}_A, \{\cdot\,,\cdot\})$ is a Poisson Lie 1-algebroid. Hence, $\sharp$ is an anti-morphism of 2-representations if and only if (A, A^*) is a Lie bialgebroid. Similarly, Ref. [17] shows that ad^*_∇ and ad_∇ form a *matched pair* if and only if (A, A^*) is a Lie bialgebroid.

Note that $(A, \{\cdot\,,\cdot\})$ is a Poisson Lie algebroid if the induced vector bundle morphism $\sharp\colon T^*A \to TA$ over A is a VB-algebroid morphism over $\rho'\colon A^* \to TM$ [32]. Then the fact that $\sharp\colon \mathrm{ad}^*_\nabla \to \mathrm{ad}_\nabla$ is an anti-morphism of 2-representations follows immediately [15], since ad^*_∇ and ad_∇ are equivalent to decompositions of the VB-algebroids $(T^*A \to A^*, A \to M)$ and $(TA \to TM, A \to M)$, respectively.

Now consider the case of 2-algebroids. First recall that a symplectic Lie 2-algebroid over a point, that is, a Courant algebroid over a point, is a usual Lie algebra $(\mathfrak{g}, [\cdot\,,\cdot])$ together with a non-degenerate pairing $\langle\cdot\,,\cdot\rangle\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, such that

$$\langle[x, y], z\rangle + \langle y, [x, z]\rangle = 0 \text{ for all } x, y, z \in \mathfrak{g}.$$

Using the adjoint and coadjoint representations $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, $x \mapsto [x, \cdot]$, and $\mathrm{ad}^*\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*)$, $x \mapsto -\mathrm{ad}(x)^*$, and denoting the canonical linear isomorphism induced by the pairing by $P\colon \mathfrak{g} \to \mathfrak{g}^*$, the equation above reads

$$P(\mathrm{ad}(x)y) = \mathrm{ad}^*(x)P(y) \text{ for all } x, y \in \mathfrak{g}.$$

In other words, this condition is precisely what is needed to turn the vector space isomorphism P into an isomorphism of Lie algebra representations between ad and ad^*. In fact, the map $\sharp\colon \mathrm{ad}^* \to \mathrm{ad}$ for Poisson Lie 2-algebroids is a direct generalisation of this construction.

Let $B \to M$ be a usual Lie algebroid with a 2-term representation $(\nabla^Q, \nabla^{Q^*}, R)$ on a complex $\partial_Q\colon Q^* \to Q$. The representation is called *self dual* [24] if it equals its dual, i.e. $\partial_Q = \partial_Q^*$, the connections ∇^Q and ∇^{Q^*} are dual to each other, and

$R^* = -R \in \Omega^2(B, \mathrm{Hom}(Q, Q^*))$, i.e. $R \in \Omega^2(B, \wedge^2 Q^*)$. Ref. [24] further shows that Poisson brackets $\{\cdot\,,\cdot\}$ on a split Lie 2-algebroid $Q[1] \oplus B^*[2]$ correspond to self dual 2-representations of B on $Q^*[1] \oplus Q[0]$ as follows: the bundle map $\partial_Q\colon Q^* \to Q$ is $\tau \mapsto \{\tau, \cdot\}|_{\Omega^1(Q)}$, the anchor $\rho_B\colon B \to TM$ is $b \mapsto \{b, \cdot\}|_{C^\infty(M)}$, the B-connection on Q^* is given by $\nabla_b^{Q^*}\tau = \{b, \tau\}$, and the 2-form R and the Lie bracket of B are defined by $\{b_1, b_2\} = [b_1, b_2] - R(b_1, b_2) \in \Gamma(B) \oplus \Omega^2(Q)$.

Fix now a Poisson Lie 2-algebroid $(\mathcal{M}, \mathcal{Q}, \{\cdot\,,\cdot\})$ together with a choice of a splitting $Q[1] \oplus B^*[2]$ for $\mathcal{M}$, a pair of TM-connections on B^* and Q, and consider the representations ad_∇ and ad_∇^*. Similarly as before, we have that

$$
\begin{aligned}
&\sharp(\mathrm{d}f) = \sharp_0(\mathrm{d}f) = \{f, \cdot\} = -\rho_B^*(\mathrm{d}f) \in \Gamma(B^*)\\
&\sharp(\tau) = \sharp_0(\tau) + \sharp_1(\cdot)\tau \in \Gamma(Q) \oplus (\Omega^1(Q) \otimes \Gamma(B^*))\\
&\sharp(b) = \sharp_0(b) + \sharp_1(\cdot)b + \sharp_2(\cdot\,,\cdot)b + \sharp^b(\cdot)b \in \mathfrak{X}(M) \oplus \Omega^1(Q, Q) \oplus \Omega^2(Q, B^*)\\
&\qquad\quad \oplus(\Gamma(B) \otimes \Gamma(B^*))
\end{aligned}
$$

for $f \in C^\infty(M)$, $\tau \in \Gamma(Q^*)$, $b \in \Gamma(B)$, where we identify τ with $\mathbf{d}\tau - \mathrm{d}_{\nabla^*}\tau$ and b with $\mathbf{d}b - \mathrm{d}_{\nabla^*}b$. Then the map $\sharp\colon \mathrm{ad}_\nabla^* \to \mathrm{ad}_\nabla$ consists of the (-2)-chain map given by the anti-commutative diagram

$$
\begin{tikzcd}
T^*M[0] \arrow[r, "-\rho_Q^*"] \arrow[d, "-\rho_B^*"] & Q^*[-1] \arrow[r, "-\partial_B"] \arrow[d, "\partial_Q"] & B[-2] \arrow[d, "\rho_B"] \\
B^*[2] \arrow[r, "-\partial_B^*"] & Q[1] \arrow[r, "\rho_Q"] & TM[0]
\end{tikzcd}
$$

and the elements

$$
\begin{aligned}
\sharp_1(q)\tau &= \langle \tau, \nabla_\cdot^Q q - \nabla_{\rho_B(\cdot)} q\rangle \in \Gamma(B^*)\\
\sharp_1(q)b &= \nabla_b^Q q - \nabla_{\rho_B(b)} q \in \Gamma(Q)
\end{aligned}
$$

for $q \in \Gamma(Q)$, $\tau \in \Gamma(Q^*)$, $b \in \Gamma(B)$,

$$\sharp_2(q_1, q_2)b = -\langle R(b, \cdot)q_1, q_2\rangle \in \Gamma(B^*)$$

for $q_1, q_1 \in \Gamma(Q)$, $b \in \Gamma(B)$, where R is the component that comes from the self-dual 2-representation of B from the Poisson structure,

$$\sharp^b(\beta)b = \langle \beta, \nabla_b^{\mathrm{bas}}(\cdot) - \nabla_{\rho_B(b)}^*(\cdot)\rangle \in \Gamma(B^*)$$

for $\beta \in \Gamma(B^*)$, $b \in \Gamma(B)$.

Suppose now that the split Lie 2-algebroid is symplectic, i.e. that it is of the form $E[1] \oplus T^*M[2]$ for a Courant algebroid $E \to M$. The only thing that is left from the construction in the Example 3.10 is a choice of a TM-connection on TM, and hence on the dual T^*M. The (anti-)isomorphism $\sharp\colon \mathrm{ad}_\nabla^* \to \mathrm{ad}_\nabla$ consists of the (-2)-chain map of the anti-commutative diagram

$$\begin{array}{ccccc} T^*M[0] & \xrightarrow{-\rho^*} & E^*[-1] & \xrightarrow{-\rho} & TM[-2] \\ \downarrow{\scriptstyle -\mathrm{id}} & & \downarrow{\scriptstyle P^{-1}} & & \downarrow{\scriptstyle \mathrm{id}} \\ T^*M[2] & \xrightarrow{-\rho^*} & E[1] & \xrightarrow{\rho} & TM[0] \end{array}$$

where $P\colon E \xrightarrow{\sim} E^*$ is the pairing, and the elements $\langle \natural_2(e_1, e_2)X, Y\rangle = -\langle R_\nabla(X, Y)e_1, e_2\rangle$ and $\langle \natural^b(\alpha)X, Y\rangle = -\langle \alpha, T_\nabla(X, Y)\rangle$. Its inverse consists of the 2-chain map given by the anti-commutative diagram

$$\begin{array}{ccccc} T^*M[2] & \xrightarrow{-\rho^*} & E[1] & \xrightarrow{\rho} & TM[0] \\ \downarrow{\scriptstyle -\mathrm{id}} & & \downarrow{\scriptstyle P} & & \downarrow{\scriptstyle \mathrm{id}} \\ T^*M[0] & \xrightarrow{-\rho^*} & E^*[-1] & \xrightarrow{-\rho} & TM[-2] \end{array}$$

and the elements $\langle \natural_2^{-1}(e_1, e_2)X, Y\rangle = -\langle R_\nabla(X, Y)e_1, e_2\rangle$ and $\langle (\natural^{-1})^b(\alpha)X, Y\rangle = -\langle \alpha, T_\nabla(X, Y)\rangle$. In other words, $\natural^2 = \mathrm{id}$. If the connection on TM is torsion-free, then the terms $\natural^b$ and $(\natural^{-1})^b$ vanish, as well. In particular, if the base manifold M is just a point, then the bundles TM and T^*M, and the elements $\natural_2$ and $\natural_2^{-1}$ are zero. Therefore, the map $\mathrm{ad}_\nabla^* \to \mathrm{ad}_\nabla$ reduces to the linear isomorphism of the pairing and agrees with the one explained above.

Acknowledgements The authors thank Miquel Cueca, Luca Vitagliano and Chenchang Zhu for interesting discussions and remarks, as well as the referees for useful suggestions and remarks.

Funding Open Access funding enabled and organized by Projekt DEAL.

References

1. Arias Abad, C., Crainic, M.: The Weil algebra and the Van Est isomorphism. Ann. Inst. Fourier (Grenoble) **61**(3), 927–970 (2011)
2. Arias Abad, C., Crainic, M.: Representations up to homotopy of Lie algebroids. J. Reine Angew. Math. **663**, 91–126 (2012)
3. Arias Abad, C., Crainic, M., Dherin, B.: Tensor products of representations up to homotopy. J. Homotopy Relat. Struct. **6**(2), 239–288 (2011)
4. Arias Abad, C., Schätz, F.: Deformations of Lie brackets and representations up to homotopy. Indag. Math. (N.S.), **22**(1–2), 27–54 (2011)
5. Arias Abad, C., Schätz, F.: The $\mathbf{A}_\infty$ de Rham theorem and integration of representations up to homotopy. Int. Math. Res. Not. IMRN **16**, 3790–3855 (2013)
6. Baez, J.C., Crans, A.S.: Higher-dimensional algebra. VI. Lie 2-algebras. Theory Appl. Categ. **12**, 492–538 (2004)
7. Bonavolontà, G., Poncin, N.: On the category of Lie n-algebroids. J. Geom. Phys. **73**, 70–90 (2013)

8. Brahic, O., Ortiz, C.: Integration of 2-term representations up to homotopy via 2-functors. Trans. Am. Math. Soc. **372**(1), 503–543 (2019)
9. Bursztyn, H., Cabrera, A., Ortiz, C.: Linear and multiplicative 2-forms. Lett. Math. Phys. **90**(1–3), 59–83 (2009)
10. Cabrera, A., Brahic, O., Ortiz, C.: Obstructions to the integrability of $\mathcal{VB}$-algebroids. J. Symplectic Geom. **16**(2), 439–483 (2018)
11. Caseiro, R., Laurent-Gengoux, C.: Modular class of Lie-∞-algebroids and adjoint representations. J. Geom. Mech. **14**(2), 273–305 (2022)
12. Courant, T.J., Weinstein, A.: Beyond Poisson structures. In: Action hamiltoniennes de groupes. Troisième théorème de Lie (Lyon, 1986), volume 27 of Travaux en Cours, pp. 39–49. Hermann, Paris (1988)
13. Crainic, M., Schätz, F., Struchiner, I.: A survey on stability and rigidity results for lie algebras. Indag. Math. **25**(5), 957–976 (2014). (**Poisson 2012: Poisson Geometry in Mathematics and Physics**)
14. del Carpio-Marek, F.: Geometric structures on degree 2 manifolds. PhD thesis, IMPA, available at www.impa.br/wp-content/uploads/2017/05/Fernando_Del_Carpio.pdf, Rio de Janeiro (2015). Accessed 15 Jun 2018
15. Drummond, T., Jotz, M., Ortiz, C.: VB-algebroid morphisms and representations up to homotopy. Differ. Geom. Appl. **40**, 332–357 (2015)
16. Grabowski, J., Rotkiewicz, M.: Higher vector bundles and multi-graded symplectic manifolds. J. Geom. Phys. **59**(9), 1285–1305 (2009)
17. Gracia-Saz, A., Jotz Lean, M., Mackenzie, K.C.H., Mehta, R.A.: Double Lie algebroids and representations up to homotopy. J. Homotopy Relat. Struct. **13**(2), 287–319 (2018)
18. Gracia-Saz, A., Mehta, R.A.: Lie algebroid structures on double vector bundles and representation theory of Lie algebroids. Adv. Math. **223**(4), 1236–1275 (2010)
19. Gualtieri, M.: Generalized complex geometry. PhD thesis (2003)
20. Gualtieri, M.: Generalized complex geometry. arXiv:math/0703298 (2007)
21. Heuer, M., Jotz Lean, M.: Multiple vector bundles: cores, splittings and decompositions. Theory Appl. Categ. **35**, 665–699 (2020)
22. Hitchin, N.: Generalized Calabi-Yau manifolds. Q. J. Math. **54**(3), 281–308 (2003)
23. Jotz Lean, M.: Dorfman connections and Courant algebroids. J. Math. Pures Appl. (9) **116**, 1–39 (2018)
24. Jotz Lean, M.: The geometrization of $\mathbb{N}$-manifolds of degree 2. J. Geom. Phys. **133**, 113–140 (2018)
25. Jotz Lean, M.: Lie 2-algebroids and matched pairs of 2-representations—a geometric approach. Pac. J. Math. **301**(1), 143–188 (2019)
26. Jotz Lean, M.: On LA-Courant algebroids and Poisson Lie 2-algebroids. Math. Phys. Anal. Geom. **23**(3), Paper No. 31, 40 (2020)
27. Jotz Lean, M., Ortiz, C.: Foliated groupoids and infinitesimal ideal systems. Indag. Math. (N.S.) **25**(5), 1019–1053 (2014)
28. Kajiura, Hiroshige, Stasheff, Jim: Homotopy algebras inspired by classical open-closed string field theory. Comm. Math. Phys. **263**(3), 553–581 (2006)
29. Liu, Z.-J., Weinstein, A., Xu, P.: Manin triples for Lie bialgebroids. J. Differ. Geom. **45**(3), 547–574 (1997)
30. Mackenzie, K.C.H.: General Theory of Lie Groupoids and Lie Algebroids. London Mathematical Society Lecture Note Series, vol. 213. Cambridge University Press, Cambridge (2005)
31. Mackenzie, K.C.H., Xu, P.: Lie bialgebroids and Poisson groupoids. Duke Math. J. **73**(2), 415–452 (1994)
32. Mackenzie, K.C.H., Xu, P.: Integration of Lie bialgebroids. Topology **39**(3), 445–467 (2000)
33. Mehta, R.: Supergroupoids, double structures, and equivariant cohomology. arXiv:math/0605356 (2006)
34. Mehta, R.A.: Q-algebroids and their cohomology. J. Symplectic Geom. **7**(3), 263–293 (2009)
35. Mehta, R.A.: Lie algebroid modules and representations up to homotopy. Indag. Math. New Ser. **25**(5), 1122–1134 (2014)
36. Mehta, R.A.: Modular classes of Lie groupoid representations up to homotopy. SIGMA Symmetry Integr. Geom. Methods Appl. **11**, Paper 058, 10 (2015)
37. Papantonis, T.: $\mathbb{Z}$-graded supergeometry: Differential graded modules, higher algebroid representations, and linear structures. arXiv:math/2106.14752 (2021)
38. Pradines, J.: Fibrés vectoriels doubles et calcul des jets non holonomes. Esquisses Mathématiques [Mathematical Sketches], vol. 29. Université d'Amiens U.E.R. de Mathématiques, Amiens (1977)

39. Quillen, D.: Superconnections and the Chern character. Topology **24**(1), 89–95 (1985)
40. Roytenberg, D.: On the structure of graded symplectic supermanifolds and Courant algebroids. In: Theodore Voronov Quantization, Poisson brackets and beyond (Manchester, 2001), volume 315 of Contemp. Math., pages 169–185. Amer. Math. Soc., Providence, RI (2002)
41. Schätz, F.: BFV-complex and higher homotopy structures. Comm. Math. Phys. **286**(2), 399–443 (2009)
42. Ševera, P.: Some title containing the words "homotopy" and "symplectic", e.g. this one. In Travaux mathématiques. Fasc. XVI, Trav. Math., XVI, pages 121–137. Univ. Luxemb., Luxembourg (2005)
43. Sheng, Y., Zhu, C.: Higher extensions of Lie algebroids. Commun. Contemp. Math. **19**(3), 1650034, 41 (2017)
44. Trentinaglia, G., Zhu, C.: Some remarks on representations up to homotopy. Int. J. Geom. Methods Mod. Phys. **13**(3), 1650024, 15 (2016)
45. Vaĭntrob, AYu.: Lie algebroids and homological vector fields. Uspekhi Mat. Nauk **52**(2(314)), 161–162 (1997)
46. van Est, W.T.: Local and global groups. I. Nederl. Akad. Wetensch. Proc. Ser. A 65 Indag. Math. **24**, 391–408 (1962)
47. van Est, W.T.: Local and global groups. II. Nederl. Akad. Wetensch. Proc. Ser. A 65 Indag. Math. **24**, 409–425 (1962)
48. Vitagliano, L.: Representations of homotopy Lie-Rinehart algebras. Math. Proc. Camb. Philos. Soc. **158**(1), 155–191 (2015)
49. Voronov, T.: Higher derived brackets and homotopy algebras. J. Pure Appl. Algebra **202**(1–3), 133–153 (2005)