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Vertex Pops and Popturns

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1 Introduction

This paper considers transformations of a planar polygon P according to two types of operations. A *vertex pop* (or a *pop*) reflects a vertex v_i, $i \in \{1, \ldots, n\}$, across the line through the two adjacent vertices v_{i-1} and v_{i+1} (where index arithmetic is modulo n). A *popturn* rotates v_i in the plane by 180° about the midpoint of the line segment $v_{i-1}v_{i+1}$. Pops and popturns are polygon reconfiguration moves similar to "Erdős pocket flips" and "flipturns" [5, 3] in that they preserve the lengths of the polygon edges.

Our goal in this paper is to study which polygons can be convexified by a series of pops or popturns, under various intersection restrictions and definitional variants.

We distinguish between three types of polygons. A *simple polygon* is non-self-intersecting, in that edges intersect only at common endpoints. A polygon is *weakly simple* if its boundary does not "properly cross" itself. Finally, a general *polygon* may be self-intersecting with proper crossings. Pops and popturns can easily introduce weak or proper crossings, so the latter two classes are often more natural to study.

We also focus on two subclasses of polygons. In an *orthogonal polygon*, adjacent edges meet at right angles. In an *equilateral* or *unit polygon*, all edge lengths are equal, say, to 1. In unit polygons, pops and popturns become identical operations.

We will see that a vertex pop can create a *hairpin vertex* (or a *pin*): a vertex v_i whose incident edges overlap collinearly. If also $v_{i-1}{=}v_{i+1}$ (which arises naturally in unit polygons), then the reflection line for a pop of v_i is not determined. Whether to allow a pop of such a pin, and if so, how to define it, leads to many interesting variations, detailed in Section 3 below.

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Polygons	Moves	Convexifiable?
arbitrary	popturns	yes, always
simple	popturns	yes iff no purse
weakly simple, unit, orthogonal	pops+180° rot. or pops+untwists	yes, always

Table 1: Summary of our results.

Our results. Table 1 lists our results. If crossings are permitted, it remains unresolved whether every polygon can be convexified via vertex pops, but we show that popturns suffice. Restricting to simple polygons, it is known that every star-shaped polygon can be convexified by popturns [1, Thm. 3.2]. We characterize precisely the class of polygons that can be convexified by simple popturns: those without a "purse." Our final result is specialized to unit orthogonal polygons, which can be reconfigured under various hairpin move restrictions.

2 Popturns

The polygon P with clockwise vertices $(v_1, v_2, \ldots, v_n)$ can be seen as a cyclic sequence of rooted vectors $(e_1, \ldots, e_n)$, where $e_i = (v_{i-1}, v_i)$. A sequence of vectors is *simple* if they form a simple polygon, and *clockwise* if each vector has the interior of P on its right side. In the following, we will use the terms *sequence of vectors* and *polygon* interchangeably. A popturn then corresponds to swapping two adjacent vectors in their cyclic ordering. We call the popturn *crossing-free* if the resulting polygon is simple. The two vectors to be popturned and their images form a parallelogram. We call the popturn *simple* if it is crossing-free and this parallelogram does not contain P. This is the case if and only if the resulting polygon is simple and clockwise. A crossing-free popturn can turn a polygon "inside-out" whereas the more restrictive simple popturn cannot.

If we permit crossings, popturns can convexify by simulating bubble sort on edge directions, where each adjacent swap corresponds to a popturn; see [2, p. 32].

Theorem 1 *Any polygon of n vertices can be convexified (permitting crossings) by a sequence of at most $\frac{1}{2}\binom{n}{2}$ popturns.*

In the remainder of this section, we concentrate on simple polygons and simple popturns. The *turning angle* $\tau_i = \tau_{v_i}$ at vertex v_i is the clockwise angle between the vectors e_i and e_{i+1} ($-\pi < \tau_i < \pi$), and the *total turning angle* $\tau_{i,j} = \tau_{e_i,e_j}$ between edges e_i and e_j is $\sum_{l=i}^{j-1} \tau_l$. Because the polygon is closed, simple, and clockwise, $\tau_{i,j} + \tau_{j,i} = 2\pi$. Notice that the total turning angle τ_{e_i,e_j} between two edges, e_i and e_j does not change after a simple popturn unless e_i and e_j are adjacent, i.e., $j = i+1$ or $j = i-1$, and the popturn is performed at their common vertex. Consider, for example, Fig. 1(a), in which a popturn at v_3 reorders the sequence of vectors $\{\ldots, e_1, e_2, e_3, e_4, \ldots\}$ to $\{\ldots, e_1, e_3, e_2, e_4, \ldots\}$. Only two edge-turning angles change: $\tau_{e_2,e_3} = \frac{1}{4}\pi$ becomes $\tau'_{e_2,e_3} = (2+\frac{1}{4})\pi$, wrapping around the entire polygon; meanwhile, $\tau_{e_3,e_2} = 1\frac{3}{4}\pi$ becomes $\tau'_{e_3,e_2} = -\frac{1}{4}\pi$.

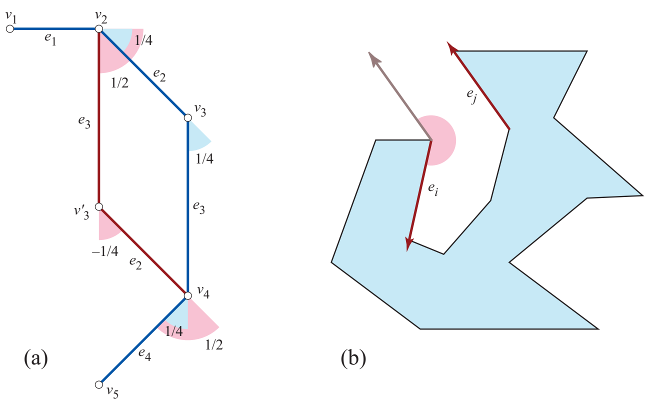

Figure 1: (a) Popturn at v_3 of $\{e_2, e_3\}$. (b) Purse $e_i, \ldots, e_j$.

A *purse* is a (cyclic) subsequence $e_i, \ldots, e_j$ such that $\tau_{i,j} \leq -\pi$; see Fig. 1(b). We show the following:

Lemma 2 *If $e_i, \ldots, e_j$ is a purse, then e_i and e_j can never be made adjacent by any sequence of simple popturns; and so $\tau_{i,j}$ is constant.*

Proof: As stated previously, τ_{e_i,e_j} will be affected by a popturn only if e_i and e_j are adjacent, i.e., $j = i+1$ or $j = i-1$. In the first case, $\tau_{i,j} = \tau_{i,i+1}$ must be strictly between $-\pi$ and π. In the second case, $\tau_{i,j} = \tau_{j+1,j} = 2\pi - \tau_{j,j+1}$, which is strictly between π and 3π. However, purse $e_i, \ldots, e_j$ has $\tau_{i,j} \leq -\pi$, meeting neither case. Before e_i and e_j become adjacent, $\tau_{i,j}$ must change, but before $\tau_{i,j}$ can change, e_i and e_j must become adjacent. Thus e_i and e_j will never become adjacent. □

A vertex v_i is *reflex* if $\tau_i < 0$. A popturn at a reflex vertex is called a *reflex popturn.*

Lemma 3 *Given a simple clockwise polygon, if the popturn at a reflex vertex v_i is not simple, then the polygon has a purse.*

Proof: Let v'_i be the position of v_i after the popturn. If the popturn at v_i is not simple, then the parallelogram $v_{i-1}v_iv_{i+1}v'_i$ intersects P. Suppose that P has no purse. It follows that edge e_{i+2} is outside of the parallelogram. Then by the Jordan curve theorem, there is a proper intersection between the boundary of P and $v_{i+1}v'_i$ or v'_iv_{i-1}. Assume by symmetry that there is such an intersection on the edge $v_{i+1}v'_i$ and let q be the first proper intersection encountered while walking from v_{i+1} to v'_i; see Fig. 2.

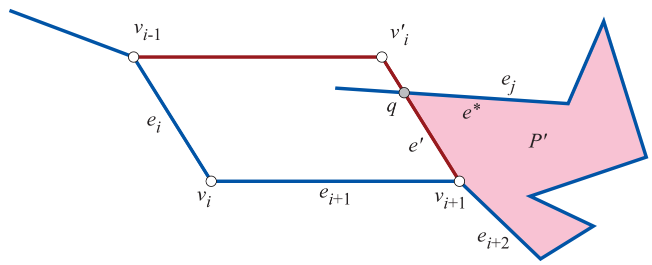

Figure 2: Proof of Lem. 3.

Let P' be a counterclockwise polygon formed by taking the portion of P between v_{i+1} and q and a vector e' from q to v_{i+1}. Let e^* be the vector preceding e' in P'. The polygon P' is closed, simple, and counterclockwise. Thus the total turning angle $\tau_{e',e^*} + \tau_{e^*,e'} = -2\pi$. But the vector e^* is part of a vector of P, say e_j, and e_i is parallel to e'; thus, $\tau_{i,j} = \tau_{e',e^*}$. Finally, e^* and e' are adjacent, so $\tau_{e^*,e'}$ must be strictly between $-\pi$ and π, and $\tau_{i,j} = \tau_{e',e^*} = -\tau_{e^*,e'} - 2\pi < -\pi$. Thus $e_i, \ldots, e_j$ is a purse. □

Theorem 4 *A simple polygon P can be convexified by a finite sequence of simple popturns if and only if P contains no purse.*

Proof: If P contains a purse $e_i, \ldots, e_j$, then by definition, $\tau_{i,j} < -\pi$ and by Lem. 2, e_i and e_j will never become adjacent, which implies that the value of $\tau_{i,j}$ will remain the same after any sequence of simple popturns. In a clockwise convex polygon, the total turning angle between every pair of edges is non-negative. This implies P can never become convex after any sequence of simple popturns.

Note that applying any sequence of popturns to P will result in a polygon which is a permutation of the original vectors. If P contains no purse, then by Lem. 3, the popturn at any reflex vertex is simple. Such a popturn will increase the area of the polygon, so the same permutation of vectors will never be repeated. Since the number of different permutations of vectors is finite, any sequence of reflex popturns will have to be finite as well. At the end of such a maximal sequence, no reflex vertex remains and the polygon is convex. □

Now we can more precisely bound the number of popturns needed to convexify a polygon:

Lemma 5 *Let P be a polygon that has no purse. Any maximal sequence of reflex popturns will convexify an n-gon P after exactly $|\{(i,j)|\tau_{i,j} < 0\}| \leq \binom{n}{2}$ popturns.*

The situation is significantly more complex in the case of crossing-free popturns. In the full version we prove:

Theorem 6 *Deciding if a polygon can be convexified by a sequence of crossing-free popturns is NP-Hard.*

3 Unit Orthogonal Polygons

When restricted to simple pops, even the 12-vertex polygon in Fig. 3(a) cannot be convexified. Here we loosen that restriction and allow hairpin vertices. A hairpin vertex v_i in a unit polygon has $v_{i-1}{=}v_{i+1}$, which leaves a pop of v_i undefined. We feel it is natural to define the pop of a pin v_i when $v_{i-1} = v_{i+1}$ as the reflection across the line L perpendicular to the pin edges and through their common endpoint. This permits convexifying the previous example; see Fig. 3(b-d). Through an extension of the argument in Thm. 4, we can show that pops together with pin pops still do not suffice to convexify all unit polygons while remaining weakly simple. But rather than detail this argument, we turn instead to positive results.

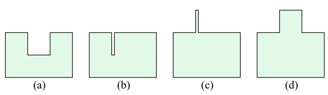

Figure 3: (a) A unit polygon that cannot be convexified by pure pops. (b,c,d) Convexifying by pin popping.

3.1 Pin-move Extensions

There are three natural pin-move extensions: rotating a pin 90°, rotating 180°, or "untwisting" a pin. The first is related to the work of Dumitrescu and Pach [4], in that their "coin moves" can be simulated in certain contexts with the help of 90° pin rotations. However, we do not pursue this connection, and only observe that 90° rotations are subsumed by 180° rotations. We next show that the second two pin movements, and therefore the first, permit convexifying any unit orthogonal polygon while remaining weakly simple.

Let P be a unit orthogonal polygon. We define a U-shaped boundary piece $(v_i, v_{i+1}, ..., v_{j-1}, v_j)$ to be a *cup* if v_{i+1} and v_{j-1} are both reflex or both convex and $v_{i+1}, \ldots v_{j-1}$ are collinear. The line segment $v_i v_j$ is the cup *lid*. A cup is *open* if no piece of ∂P lies along its lid. A *horizontal cup* (or H-cup) is an upright or upside down U-shape; a *vertical cup* (or V-cup) is a U-shape on its side.

Our reconfiguration algorithm converts P to a canonical form by moving pins around ∂P. If $[v_{i-1}, v_i, v_{i+1}]$ is a pin, call v_i its *tip* and $v_{i-1} = v_{i+1}$ its *base*. We distinguish two types of pins. A *flat* pin has the tip vertex v_i coincident with either v_{i-2} or v_{i+2}; see Figs. 4(a,b). A *barb* pin has a tip vertex v_i distinct from both v_{i-2} and v_{i+2}; see Fig. 4(c,d).

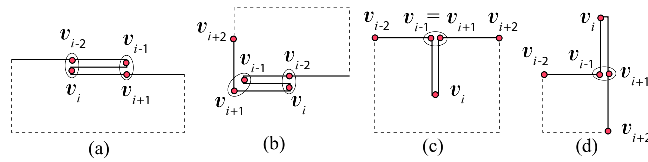

Figure 4: (a,b) Flat pins (c,d) Barb pins.

Now we relax the condition that pops preserve simplicity of the polygon, and allow for simple pin "twists" in a small neighborhood around their base point. A *twisted* pin (e.g. Fig. 5(b)) is the result of pop(v_i) applied in the following two conditions: (i) $[v_{i-1}, v_i, v_{i+1}]$ is a simple (untwisted) flat pin, and (ii) $v_i v_{i+1}$ and $v_{i+1} v_{i+2}$ are orthogonal (cf. Figs. 4(b), 5(a)). Once a pin becomes twisted, we immediately untwist it (cf. Fig. 5(c)). Note that our pop operations apply on the simple polygon obtained by separating the pin base into two points within an epsilon-disk of the base, as illustrated in the pin drawings. Although pin untwisting may seem like "cheating," in fact the operation is quite natural, for the coincidence of v_{i-1} with v_{i+1} means that Fig. 5(b,c) are geometrically identical. Although

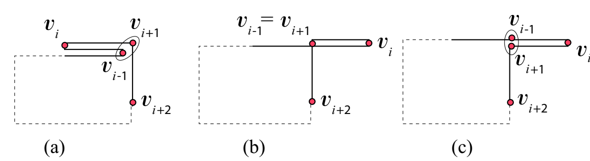

Figure 5: (a) Initial pin $[v_{i-1}, v_i, v_{i+1}]$ (b) Pin twisted after pop(v_i) (c) Untwisted pin.

it may appear from Fig. 5(a,c) that the result of popping/untwisting a pin is the same as rotating the pin 180° about its reflex base point, the pop and the 180°-turn operations are not always identical. Nevertheless, we show that they are equivalent in the sense that the composite operations (see Sec 3.3) used by our algorithm can be defined in terms of either pop/untwist pin operations or pop/180°-turn pin rotations.

3.2 Canonical Form

Let P be a polygon with $2x$ horizontal edges and $2y$ vertical edges. The *canonical form* of P is a rectangle of length x and height y. It is used as an intermediate stage in reconfiguring P into another polygon with a same number of horizontal and vertical edges.

3.3 Composite Operations

We define three composite operations used by the reconfiguration algorithm. Each can be implemented using pop/untwist operations or pop/180°-turn operations. SLIDE(Π): Moves the pin Π one lattice edge cw around

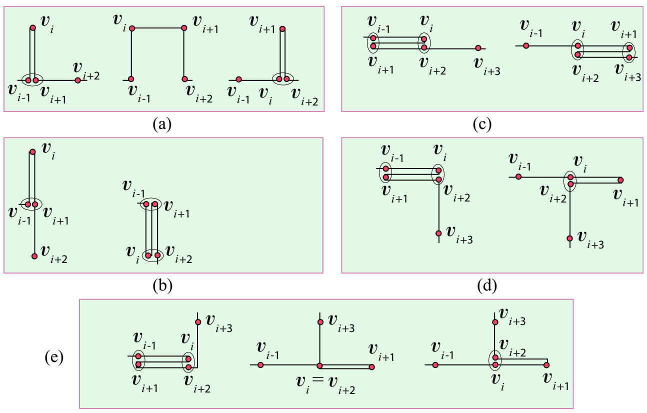

Figure 6: (a, b) Sliding a barb pin (c, d, e) Sliding a flat pin.

the boundary. See Fig. 6.
WALK(Π, c): Applies a sequence of SLIDE operations to walk the pin Π cw along ∂P until its base coincides with corner point c.
POPSWEEP(Π, c): Here Π is an outward pointing barb pin whose base vertex b is connected to vertex c by a straight boundary segment. This operation pops all vertices on the boundary segment, starting with b.

3.4 Converting P To Canonical Form

Let $T = \ell r$ be the leftmost among the topmost maximal horizontal sections of ∂P, with ℓ (r) the left (right) endpoint of T. The algorithm uses the composite operations to convert P into a canonical rectangle R that has its lower-right corner at r (see Fig. 7(d)). Initially, R is degenerate and coincides with line segment $[\ell, r]$.

The algorithm repeatedly creates a pin Π and walks it around ∂P to the top left corner t_ℓ of R (initially $t_\ell = \ell$), where it uses a POPSWEEP operation to expand Π into a new (top) row or (left) column of R. A pin is created by popping all base vertices of an open cup, which always exists (Lem. 7). E.g., in Fig. 7(d), popping base vertex b_2 of cup (a_2, b_2, c_2, d_2) creates a pin (see Fig. 7(e)). The cup must be open, for otherwise popping the base vertices results in ∂P touching along non-pin edges. In the first iteration, the algorithm uses a pin Π corresponding to an upright open H-cup; this ensures that, once it reaches ℓ, Π expands into a row extending from ℓ to r, turning R into a one-row rectangle ($\ell = b_\ell, t_\ell, t_r, r$). Figs. 7(a-g) show this for two pins.

Lemma 7 *If P is not a rectangle, it has at least one open cup in the halfplane H bounded above by T.*

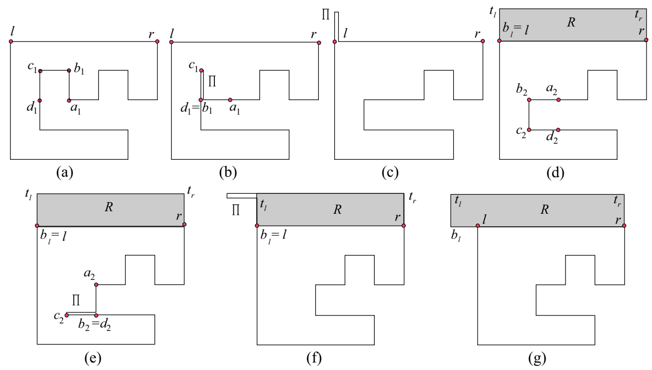

Figure 7: (a) P (b) H-cup (a_1, b_1, c_1, d_1) turned into pin Π by pop(b_1) (c) Pin Π after WALK (d) Rectangle R after POPSWEEP (e-g) Same steps for next pin.

Proof: If P is not a rectangle, then the reconfiguration is not complete and some part of ∂P lies in the interior of H. Therefore, P has at least one upright horizontal cup in H, namely the H-cup with a lowest horizontal edge as base. Of all upright horizontal cups in H, let $C = (v_i, v_{i+1}, \ldots v_{j-1}, v_j)$ be one with a highest base. Assume for the sake of contradiction that C is not open. Then its lid contains some (maximal) horizontal section $v_k, ..., v_{k+s}$ of ∂P. Assume w.l.o.g. that $k \geq j$. If v_k and v_j coincide, then $(v_{j-2}, v_{j-1}, v_j = v_k, v_{k+1})$ is an open V-cup in H and the proof is finished, and similarly if v_{k+s} and v_i coincide. So assume $k \neq j$ and $k + s \neq i$. Then $v_k, ..., v_{k+s}$ must be the base of an upright H-cup, call it D. Simple arguments show that D lies in H and is higher than C, a contradiction. □

Theorem 8 *The described algorithm transforms P into a rectangle in $\Theta(n^2)$ pop operations.*

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