

10-1-2012

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Recommended Citation

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Body-and-cad geometric constraint systems

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ABSTRACT

Motivated by constraint-based CAD software, we develop the foundation for the rigidity theory of a very general model: the *body-and-cad structure*, composed of rigid bodies in 3D constrained by pairwise coincidence, angle and distance constraints. We identify 21 relevant geometric constraints and develop the corresponding infinitesimal rigidity theory for these structures. The classical body-and-bar rigidity model can be viewed as a body-and-cad structure that uses only one constraint from this new class.

As a consequence, we identify a new, necessary but not sufficient, counting condition for *minimal rigidity* of body-and-cad structures: *nested sparsity*. This is a slight generalization of the well-known sparsity condition of Maxwell.

Categories and Subject Descriptors: I.3.5 [Computing Methodologies]: Computer Graphics—*Computational Geometry and Object Modeling*; G.2.3 [Discrete Mathematics]: Applications; G.2.2 [Discrete Mathematics]: Graph Theory; J.6 [Computer-Aided Engineering]: Computer-aided design (CAD)

Keywords: rigidity theory, computer aided design, geometric constraint system

1. INTRODUCTION

This paper initiates the study and sets up the foundation for the rigidity theory of body-and-cad structures, a class of 3D geometric frameworks with specific coincidence, angle

*Partially supported by NSF CCF-0728783.

†Partially supported by a Research Grant from SolidWorks 2007.

‡Partially supported by NSF CCF-0728783.

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SAC'09 March 8-12, 2009, Honolulu, Hawaii, U.S.A.

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and distance constraints between rigid bodies. To the best of our knowledge, these constraints have not been studied before from this perspective.

Motivation. Popular computer aided design (CAD) software based on geometric constraint solvers allow users to design complex 3D systems by placing geometric constraints among sets of rigid body building blocks. The constraints are specified by identifying *primitive geometries* (points, lines, planes, or splines) on participating rigid bodies. Analyzing all of these simultaneously is a very difficult problem. *In this paper, we focus on a subset of these constraints that are amenable to a rigidity-theoretical investigation.*

Underlying classical rigidity theory results is a general proof pattern, spanning algebraic geometry (for *rigidity*), linear algebra (for *infinitesimal rigidity*) and graph theory (for *combinatorial rigidity*). The ultimate goal is a full combinatorial characterization of *generic infinitesimally minimally rigid structures*, but such results are extremely rare: 3d bar-and-joint rigidity remains a conspicuously open problem, while the 2d version is fully understood. An important step along the way is identifying a pattern in the rigidity matrix developed as part of the *infinitesimal rigidity theory* for the structures. While this is straightforward for the well-known bar-and-joint model, it is more complicated in the body-and-bar model. In this abstract, we formulate the even more involved rigidity matrix for the *body-and-cad* model.

Results. We define a *body-and-cad structure* to be composed of rigid bodies connected by **pairwise coincidence**, **angle** (parallel, perpendicular, or arbitrary fixed angle) and **distance** constraints. These may only be placed on the primitive geometries of points, lines or planes. We develop the pattern of the **rigidity matrix** and identify a necessary combinatorial counting property called *nested sparsity*, which is the counterpart of the well-known Maxwell condition for fixed length rigidity. We also show that this condition is *not* sufficient.

Structure. Only a sketch of our results is presented in this short abstract. Full details can be found in the second author's PhD dissertation [4]. Section 2 gives a very brief overview of the required mathematical background. Section 3 develops the foundation for the infinitesimal rigidity theory, providing the basic building blocks used for each new constraint. Due to lack of space, we only include the

full development for one example: line-line coincidence constraints. Finally, Section 4 identifies a new combinatorial property resulting from the structure of the rigidity matrix; this *nested sparsity* condition, while necessary, is unfortunately not sufficient as illustrated with a counterexample.

Related work Classical rigidity theory [2] focuses on distance constraints between points [3] or rigid bodies [13, 16]. *Direction* constraints (where 2 points are required to define a fixed direction, with respect to a global coordinate system) are well-understood and arise from parallel redrawing applications [17]. Motivated by CAD systems, Servatius and Whiteley present a characterization, which can be viewed as a generalized Laman counting property, for 2D systems with both length and direction constraints [11].

Incidence constraints have been studied in the literature, mostly in connection with Geometric Theorem Proving [7, 8] for projective incidence theorems. Angle constraints also have received some attention. Zhou and Sitharam [19] characterize a large class of 2D angle constraint systems along with a set of combinatorial construction rules that maintain generic independence. Saliola and Whiteley [9] prove that, even in the plane, the complexity of determining the independence of a set of circle intersection angles is the same as that of generic bar-and-joint rigidity in 3D.

Combinatorial *sparsity* conditions [5, 6, 12] are intimately tied with rigidity theory, appearing often as necessary conditions (as for 3D bar-and-joint rigidity) and sometimes even as complete characterizations (as for 2D bar-and-joint, body-and-bar in arbitrary dimension) [3, 13, 18].

2. PRELIMINARIES

Geometric constraints. Besides the well-studied distance constraint between points (as in body-and-bar structures), we identify 20 new pairwise coincidence, distance and angle constraints between points, lines and planes. We label constraints by the geometries involved, e.g., a line-plane perpendicular constraint between bodies A and B indicates that a line on A is perpendicular to a plane on B . Here is the full set of body-and-cad constraints that we study:

- **Plane-plane constraints.** Parallel, perpendicular, fixed angle, coincidence, distance.
- **Plane-line constraints.** Parallel, perpendicular, fixed angle, coincidence, distance.
- **Plane-point constraints.** Coincidence, distance.
- **Line-line constraints.** Parallel, perpendicular, fixed angle, coincidence, distance.
- **Line-point constraints.** Coincidence, distance.
- **Point-point constraints.** Coincidence, distance.

Terminology and notation. Our results rely on the same mathematical background as in the work on body-and-bar rigidity by Tay [13] and White and Whiteley [16]. We use Grassmann-Cayley algebra, Plücker coordinates and instantaneous screw theory (see, e.g., [14, 15] and [10]). In this paper, we restrict ourselves to dimension 3; 2 -tensors in the Grassmann-Cayley algebra (see, e.g., [14, 15]) are identified with vectors in \mathbb{R}^6 . The Grassmann-Cayley *join* operator is represented with \vee . The dot product of two vectors u and v is denoted $\langle u, v \rangle$.

Instantaneous rigid body motions. An *instantaneous screw motion* (see page 24 of [10]) is defined by an instantaneous translation and rotation about a *screw axis*. It is used

to represent an instantaneous rigid body motion. An instantaneous screw is represented by a 6-vector $\mathbf{t} = \{-\boldsymbol{\omega}, \mathbf{v}\}$, where $\boldsymbol{\omega}, \mathbf{v} \in \mathbb{R}^3$; the minus sign in front of $\boldsymbol{\omega}$ is a convenient, technical convention. The first component $\boldsymbol{\omega}$ encodes the angular velocity and, as a vector, gives the direction of the screw axis. The translational velocity can be computed from $\boldsymbol{\omega}$ and \mathbf{v} , but we skip the details as they are not relevant for the rest of the paper. There is an exact correspondence between 2-tensors and instantaneous screws.

Notation. We will work with a body-and-cad structure composed of n bodies. Each body i will have associated to it an instantaneous screw \mathbf{s}_i . The *star operator* $*$ swaps the first and last 3 coordinates of a 6-vector; i.e., for a screw $\mathbf{t} = \{-\boldsymbol{\omega}, \mathbf{v}\}$, $\mathbf{t}^* = \{\mathbf{v}, -\boldsymbol{\omega}\}$. Let $\mathbf{s} = \{\mathbf{s}_1, \dots, \mathbf{s}_n\} \in \mathbb{R}^{6n}$ and $\mathbf{s}^* = \{\mathbf{s}_1^*, \dots, \mathbf{s}_n^*\}$.

The vector \mathbf{s} is an *infinitesimal motion* of a body-and-cad structure if it infinitesimally respects the constraints. This can be expressed with the help of a matrix (called the *rigidity matrix*), fully described in Section 3. An infinitesimal motion is a vector in the kernel of the rigidity matrix. When $\mathbf{s}_i = \mathbf{s}_j$ for all i and j , this is a *trivial infinitesimal motion*.

Body-and-cad rigidity. A body-and-cad structure is *rigid* if the only motions respecting the constraints are the *trivial* 3D motions (rotation and translation); otherwise, it is *flexible*. It is *infinitesimally rigid* if the only infinitesimal motions are trivial. Infinitesimal rigidity is the linearized version of rigidity and is the only type we study in this paper.

Body-and-cad minimal rigidity.

The concept of *minimal rigidity* is usually defined as follows: a rigid structure is *minimally rigid* if the removal of any constraint results in a flexible structure. However, in our case, geometric constraints may correspond to more than one “primitive” constraint. Formally, a *primitive* constraint yields only one row in the rigidity matrix (defined in Section 3), while the body-and-cad constraints may yield several rows. In our setting, we define *minimal rigidity* as above, but referring to the removal of primitive constraints only.

The **example** in Figure 1 illustrates the subtleties of this concept. Let A and B be two dice rigidly stacked with the following constraints: (i) (**Plane-plane parallel**) A 's Face 1 is parallel to B 's Face 1, (ii) (**Plane-plane perpendicular**) A 's Face 2 is perpendicular to B 's Face 3, (iii) (**Plane-line distance**) The distance between A 's Face 1 and B 's Line 12 (intersection of Faces 1 and 2) is 1, and (iv) (**Point-point coincidence**) A 's Corner 236 (the point defined by Faces 2, 3 and 6) is coincident to B 's Corner 123. This structure is *rigid*. We say the structure is *overconstrained* since it remains rigid even after removal of constraint (iii). The resulting structure is now *minimally rigid*. As we will see in Section 3, constraints (i), (ii) and (iv) correspond to 6

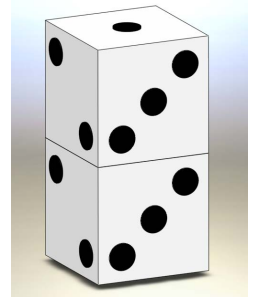


Figure 1: Two dice rigidly stacked; die A is above B . Faces are labeled by the number of dots, and face 6 lies at the bottom (opposite 1). The length of an edge is 1.

primitive constraints. Thus, the removal of any primitive constraint results in a flexible structure.

Now consider stacking the dice with the following two constraints: (i) (**Line-line coincidence**) A 's Line 26 is coincident to B 's Line 12 and (ii) (**Line-line coincidence**) A 's Line 36 is coincident to B 's Line 13. This structure is still rigid. While it becomes flexible after the removal of either constraint (i) or (ii), it is *not* minimally rigid. As we will see in Section 3, a line-line coincidence constraint corresponds to 4 primitive constraints. Thus, this structure has 8 primitive constraints and is overconstrained. To give some intuition, note that a structure composed of 2 rigid bodies has 12 degrees of freedom. Of these, 6 are trivial, so we fix body A to factor them out. Now consider constraint (i); the structure is left with 2 degrees of freedom, as B may slide along the line and rotate about it. This indicates that a line-line coincidence constraint is somehow “killing” 4 degrees of freedom. This intuition is formalized by the 4 rows of the rigidity matrix developed in Section 3 for the line-line coincidence constraint.

Rigidity theory roadmap. To develop the rigidity theory for a new model, three steps must be accomplished. (1) *Algebraic theory.* Formulate the rigidity concept in algebraic terms, resulting in an algebraic variety. (2) *Infinitesimal theory.* Analyze the local behavior at some point on the algebraic variety. This reduces to the study of a *rigidity matrix*. (3) *Combinatorial rigidity.* Whenever possible, find a combinatorial characterization of minimal rigidity in terms of properties of an underlying graph structure. This is usually derived from properties of the rigidity matrix at a *generic* point on the algebraic variety. In this paper, we skip Step 1 and jump immediately to the infinitesimal rigidity theory for body-and-cad structures as it is natural to reason directly in the world of instantaneous screw motions.

3. INFINITESIMAL THEORY

The previous example shows that body-and-cad constraints are more complicated to analyze than classical distance constraints. We introduce two new concepts to simplify the analysis: *primitive angular* and *blind* constraints. We then define, as building blocks, 4 *basic* angular and blind constraints and develop their infinitesimal theory. All of the 21 body-and-cad constraints can be studied using these **building blocks**, leading to the body-and-cad rigidity matrix.

3.1 Primitive constraints

A *primitive* constraint is one that may affect at most one degree of freedom. We also classify constraints into two types: *angular* and *blind*; as the theory is developed, it will become more clear why these classifications are appropriate, as they correspond to constraints demonstrating different algebraic behaviors.

A rigid body in 3D has 6 degrees of freedom, 3 of which are rotational and 3 of which are translational. A *primitive angular* constraint may affect only a rotational degree of freedom, whereas a *primitive blind* constraint may affect either a rotational or a translational degree of freedom. We will associate a set of primitive angular and a set of primitive blind constraints with each body-and-cad constraint.

3.2 Rigidity matrix

The rigidity matrix for a body-and-cad structure has 6 columns for each body i , corresponding to the components of the instantaneous screw \mathbf{s}_i , as was done for the original body-and-bar rigidity matrix¹. There is a row for each primitive constraint associated to the original body-and-cad structure. A primitive angular constraint results in a row containing zero entries in the first 3 columns for each body, and a primitive blind constraint may have non-zero entries in any of the 6 columns for each body. In the schematic below, gray cells indicate potentially non-zero entries, and red cells highlight the zero entries for angular constraints.

		\mathbf{s}_1^*			\mathbf{s}_i^*			\mathbf{s}_n^*		
		\mathbf{v}_1	$\boldsymbol{\omega}_1$	\dots	\mathbf{v}_i	$\boldsymbol{\omega}_i$	\dots	\mathbf{v}_n	$\boldsymbol{\omega}_n$	\dots
Angular constraints		0		\dots	0		\dots	0		\dots
		\vdots	\vdots	\dots	\vdots	\vdots	\dots	\vdots	\vdots	\vdots
		0		\dots	0		\dots	0		\dots
		\vdots	\vdots	\dots	\vdots	\vdots	\dots	\vdots	\vdots	\vdots
Blind constraints		0		\dots	0		\dots	0		\dots
		\vdots	\vdots	\dots	\vdots	\vdots	\dots	\vdots	\vdots	\vdots

Since the trivial motions corresponding to the 3-dimensional rigid motions are necessarily in the kernel of R , the maximum rank of R is $6n - 6$. By definition, a structure is *infinitesimally rigid* if its rigidity matrix has rank $6n - 6$.

3.3 Building blocks

We now define 4 very specific **basic** angular and blind constraints (2 of each) and develop the infinitesimal theory for them. This section is the most technical part of our paper; everything else is derived from these basic **building blocks**.

Angular building blocks.

All body-and-cad angular constraints can be reduced to the following two basic ones between pairs of lines: (i) *line-line non-parallel fixed angle* and (ii) *line-line parallel*.

(i) *Basic line-line non-parallel fixed angle.* A line-line non-parallel angle constraint between bodies i and j is defined by identifying a pair of non-parallel lines, each rigidly affixed to one body. Let \mathbf{d}_i and \mathbf{d}_j be the directions of the lines affixed to bodies i and j , respectively. Then the constraint is infinitesimally maintained if the axis of the relative screw $\mathbf{s}_i - \mathbf{s}_j$ is in a direction lying in the plane determined by \mathbf{d}_i and \mathbf{d}_j , i.e.,

$$\langle (\boldsymbol{\omega}_i - \boldsymbol{\omega}_j), \mathbf{d}_i \times \mathbf{d}_j \rangle = 0$$

Since $-\boldsymbol{\omega}_i$ is composed of the last three coordinates of \mathbf{s}_i^* , this is equivalent to

$$\langle (\mathbf{s}_i^* - \mathbf{s}_j^*), ((0, 0, 0), -(\mathbf{d}_i \times \mathbf{d}_j)) \rangle = 0 \quad (1)$$

This corresponds to one row in the rigidity matrix:

		\mathbf{s}_i^*			\mathbf{s}_j^*			\dots		
		\mathbf{v}_i	$-\boldsymbol{\omega}_i$	\dots	\mathbf{v}_j	$-\boldsymbol{\omega}_j$	\dots	\dots	\dots	\dots
\dots	\dots	0	$-(\mathbf{d}_i \times \mathbf{d}_j)$	\dots	0	$\mathbf{d}_j \times \mathbf{d}_i$	\dots	\dots	\dots	\dots

¹The starred version \mathbf{s}_i^* will be used to conveniently order the columns of the rigidity matrix.

(ii) *Basic line-line parallel constraint.* A line-line parallel constraint between bodies i and j is defined by identifying a pair of parallel lines, each rigidly affixed to one body. Let $\mathbf{d} = (a, b, c)$ be the direction of the parallel lines. Then the constraint is infinitesimally maintained if the axis of the relative screw $\mathbf{s}_i - \mathbf{s}_j$ is in the same direction as \mathbf{d} , i.e., $(\boldsymbol{\omega}_i - \boldsymbol{\omega}_j) = \alpha \mathbf{d}$, for some scalar α . This can be expressed by the following two linear equations, where $\boldsymbol{\omega} = \boldsymbol{\omega}_i - \boldsymbol{\omega}_j$:

$$\begin{aligned}\omega^x b - \omega^y a &= 0 \\ \omega^y c - \omega^z b &= 0\end{aligned}$$

Since $-\boldsymbol{\omega}_i$ is composed of the last three coordinates of \mathbf{s}_i^* , these are equivalent to

$$\langle (\mathbf{s}_i^* - \mathbf{s}_j^*), (0, 0, 0, -b, a, 0) \rangle \quad (2)$$

$$\langle (\mathbf{s}_i^* - \mathbf{s}_j^*), (0, 0, 0, 0, -c, b) \rangle \quad (3)$$

and correspond to **two rows** in the rigidity matrix:

	\mathbf{s}_i^*			\mathbf{s}_j^*		
...	\mathbf{v}_i	$-\boldsymbol{\omega}_i$...	\mathbf{v}_j	$-\boldsymbol{\omega}_j$...
...0...	0	$(-b, a, 0)$...0...	0	$(b, -a, 0)$...0...
...0...	0	$(0, -c, b)$...0...	0	$(0, c, -b)$...0...

Blind building blocks.

Let \mathbf{p} be a point and \mathbf{p}' its instantaneous velocity resulting from the instantaneous screw associated with a 2-tensor $\mathbf{t} \in \mathbb{R}^6$. Let $\mathbf{c} \in \mathbb{R}^3$ be an arbitrary direction vector. We either constrain the velocity \mathbf{p}' to be orthogonal or parallel to \mathbf{c} . This yields the remaining basic constraints (iii) *basic blind orthogonality* and (iv) *basic blind parallel*. Expressing both of them becomes straightforward using the following fact:

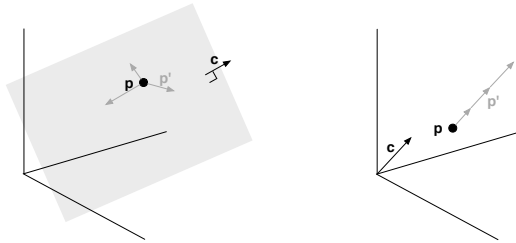
FACT 1. Let $\mathbf{t} \in \mathbb{R}^6$ be an instantaneous screw, $\mathbf{p} \in \mathbb{R}^3$ a point and \mathbf{p}' the velocity of \mathbf{p} under the screw motion \mathbf{t} . Then

$$\mathbf{t} \vee (\mathbf{p} : 1) = (\mathbf{p}', -\langle \mathbf{p}, \mathbf{p}' \rangle) \quad (4)$$

and, for any $\mathbf{q} \in \mathbb{R}^3$ and $q^w \in \mathbb{R}$,

$$\mathbf{t} \vee (\mathbf{p} : 1) \vee (\mathbf{q} : q^w) = \langle \mathbf{p}', \mathbf{q} \rangle - q^w \langle \mathbf{p}, \mathbf{p}' \rangle \quad (5)$$

The derivation of this fact can be found in the appendix.



(a) Allowable \mathbf{p}' velocity vectors for \mathbf{p} when constrained to be orthogonal to a direction \mathbf{c} must lie in the plane through \mathbf{p} with normal \mathbf{c} . (b) Allowable \mathbf{p}' velocity vectors for \mathbf{p} when constrained to be in the same direction as \mathbf{c} .

Figure 2: Basic blind geometric constraints.

(iii) Basic blind orthogonality constraint

To express that \mathbf{p}' is orthogonal to \mathbf{c} , we simply substitute $\mathbf{q} = \mathbf{c}$ and $q^w = 0$ into Equation 5. Then $\langle \mathbf{p}', \mathbf{c} \rangle = 0$ if and only if

$$\mathbf{t} \vee (\mathbf{p} : 1) \vee (\mathbf{c} : 0) = 0$$

	plane		line		point	
	angular	blind	angular	blind	angular	blind
plane						
coincidence	2	1	1	1	0	1
distance	2	1	1	1	0	1
parallel	2	0	1	0		
perpendicular	1	0	2	0		
fixed angle	1	0	1	0		
line						
coincidence			2	2	0	2
distance			0	1	0	1
parallel			2	0		
perpendicular			1	0		
fixed angle			1	0		
point						
coincidence					0	3
distance					0	1

Table 1: Association of body-and-cad (*coincidence*, *angular*, *distance*) constraints with the number of blind and angular primitive constraints.

if and only if

$$\langle \mathbf{t}^*, (\mathbf{p} : 1) \vee (\mathbf{c} : 0) \rangle = 0 \quad (6)$$

(iv) Basic blind parallel constraint

To express that \mathbf{p}' lies in the same direction as \mathbf{c} , we apply Equation 5 twice by substituting $\mathbf{q} = (c^y, -c^x, 0)$ and $q^w = 0$ first, then $\mathbf{q} = (0, c^z, -c^y)$ and $q^w = 0$. We obtain that $\mathbf{p}' = \alpha \mathbf{c}$ for some $\alpha \in \mathbb{R}$ if and only if

$$\mathbf{t} \vee (\mathbf{p} : 1) \vee (c^y, -c^x, 0, 0) = 0$$

$$\mathbf{t} \vee (\mathbf{p} : 1) \vee (0, c^z, -c^y, 0) = 0$$

if and only if

$$\langle \mathbf{t}^*, (\mathbf{p} : 1) \vee (c^y, -c^x, 0, 0) \rangle = 0 \quad (7)$$

$$\langle \mathbf{t}^*, (\mathbf{p} : 1) \vee (0, c^z, -c^y, 0) \rangle = 0 \quad (8)$$

3.4 Body-and-cad constraints

In the full version of this paper, we will present the complete analysis of all 21 body-and-cad constraints. The infinitesimal theory for each is derived from these 4 building blocks. We summarize the associations for each constraint to the number of primitive **angular** and **blind** constrains in Table 1. As an example of how to read the table, the second two columns of row 1 indicate that a plane-line coincidence constraint reduces to 1 angular and 1 blind primitive constraint.

In the rest of this section, we show just one example: the derivation of the **line-line coincidence** constraint.

Let $\mathbf{c}, \mathbf{p} \in \mathbb{R}^3$ define a line, with \mathbf{c} as its direction and \mathbf{p} a point on the line, affixed to bodies i and j . We begin with a **line-line parallel** angular constraint, resulting in 2 primitive *angular* constraints from Equations 2 and 3:

$$\langle (\mathbf{s}_i^* - \mathbf{s}_j^*), (0, 0, 0, -c^y, c^x, 0) \rangle \quad (9)$$

$$\langle (\mathbf{s}_i^* - \mathbf{s}_j^*), (0, 0, 0, 0, -c^z, c^y) \rangle \quad (10)$$

Then, to maintain coincidence, associate 2 primitive *blind* constraints from Equations 7 and 8 to force the relative velocity of \mathbf{p} to lie along \mathbf{c} :

$$\langle (\mathbf{s}_i - \mathbf{s}_j)^*, (\mathbf{p} : 1) \vee (c^y, -c^x, 0, 0) \rangle = 0 \quad (11)$$

$$\langle (\mathbf{s}_i - \mathbf{s}_j)^*, (\mathbf{p} : 1) \vee (0, c^z, -c^y, 0) \rangle = 0 \quad (12)$$

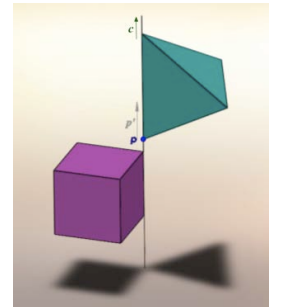


Figure 3: Line-line coincidence.

These 4 equations maintain the **line-line coincidence** constraint infinitesimally and correspond to **4 rows** in the following rigidity matrix schematic:

\mathbf{v}_i		\mathbf{s}_i^*	\mathbf{v}_j		\mathbf{s}_j^*
\mathbf{v}_i		$-\omega_i$	\mathbf{v}_j		$-\omega_j$
...	0	$(-c^y, c^x, 0)$...	0	$(c^y, -c^x, 0)$
...	0	$(0, -c^z, c^y)$...	0	$(0, c^z, -c^y)$
...	$(\mathbf{p} : 1) \vee$	$(c^y, -c^x, 0, 0)$...	$(\mathbf{p} : 1) \vee$	$(c^y, -c^x, 0, 0)$
...	$(\mathbf{p} : 1) \vee$	$(0, c^z, -c^y, 0)$...	$(\mathbf{p} : 1) \vee$	$(0, c^z, -c^y, 0)$

4. COMBINATORICS

Now we address the question of combinatorially characterizing when a body-and-cad rigidity matrix is generically independent, i.e., the rank function drops only on a measure-zero set of possible entries. We derive a necessary condition called *nested sparsity* and prove by a counterexample that it is insufficient. We observe that finding a complete combinatorial characterization may require overcoming well-known obstacles such as detecting dependences in 3D bar-and-joint, 2D points-and-angles, 2D circles-and-angles and 2D point-line incidence constraint systems.

Nested sparsity. A graph on n vertices is (k, ℓ) -sparse if every subset of n' vertices spans at most $kn' - \ell$ edges; it is *tight* if, in addition, it spans $kn - \ell$ total edges.

Let $G = (V, RUB)$ be a graph with its edge set colored into red and black edges, corresponding to R and B , respectively. We define $G = (V, R \cup B)$ to be $(k_1, \ell_1, k_2, \ell_2)$ -nested sparse if G is (k_1, ℓ_1) -sparse and $G_1 = (V, R)$ is (k_2, ℓ_2) -sparse; the graph is $(k_1, \ell_1, k_2, \ell_2)$ -tight if G is (k_1, ℓ_1) -tight. Note that nested sparsity only makes sense when (k_2, ℓ_2) -sparsity is more restrictive than (k_1, ℓ_1) -sparsity.

Given a body-and-cad structure, let $G = (V, RUB)$ be the graph obtained by assigning vertices to bodies and primitive constraints to disjoint edge sets R and B , corresponding respectively to primitive angular and blind constraints. For each body-and-cad constraint, associate primitive angular constraints with edges in R and primitive blind constraints with edges in B .

THEOREM 1. *Let $G = (V, R \cup B)$ be the graph associated to a body-and-cad structure, where R and B correspond to primitive angular and blind constraints, respectively. Then $(6, 6, 3, 3)$ -nested sparsity is a necessary condition for generic minimal rigidity of the structure.*

Counterexample. We now show that $(6, 6, 3, 3)$ -nested sparsity is not sufficient. The example in Figure 4 depicts a *flexible* structure whose associated graph is $(6, 6, 3, 3)$ -nested tight. It is composed of 3 bodies A, B and C ; Figure 4b colors the constraints. A and B have 2 point-point distance constraints (cyan and purple) and a line-line coincidence constraint (pink); A and C have a line-line angle constraint (orange) and a plane-plane coincidence constraint (yellow); B and C have a plane-line coincidence constraint (green).

5. CONCLUSIONS AND FUTURE DIRECTIONS

Motivated by CAD applications, we have initiated the study of body-and-cad rigidity. Constraint-based CAD software contains a rich set of geometric constraints. As a first step towards understanding these, we have identified a class of constraints amenable to rigidity-theoretical investigation. We

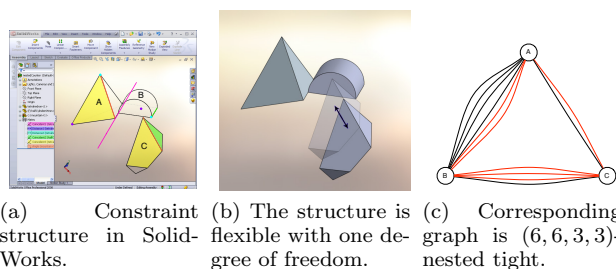


Figure 4: Counterexample shows nested sparsity condition is not sufficient.

are hopeful that the study of all or some of the body-and-cad constraints introduced here will prove to be more tractable than classical 3D bar-and-joint rigidity. A possible difficulty arises from projective incidence theorem, which are combinatorially undetectable.

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Appendix

Notation. Superscripts x, y, z, w denote the components of a vector in \mathbb{R}^4 . The minor of a 3×4 matrix A determined by columns i, j and k is denoted $|A_{ijk}|$.

Proof of Fact 1.

If \mathbf{t} is a decomposable 2-tensor (a 2-extensor), then its components are the minors of a 2×4 matrix M . Let A be the 3×4 matrix obtained by appending $(\mathbf{p} : 1)$ to the bottom of M . Then

$$\mathbf{t} \vee (\mathbf{p} : 1) \vee (\mathbf{q} : q^w) = \begin{vmatrix} & & M & \\ p^x & p^y & p^z & 1 \\ q^x & q^y & q^z & q^w \end{vmatrix}$$

Performing a Laplace expansion along the 4th row of the matrix yields $q^x |A_{234}| - q^y |A_{134}| + q^z |A_{124}| - q^w |A_{123}| = q^x (\mathbf{t} \vee (\mathbf{p} : 1))^x + q^y (\mathbf{t} \vee (\mathbf{p} : 1))^y + q^z (\mathbf{t} \vee (\mathbf{p} : 1))^z + q^w (\mathbf{t} \vee (\mathbf{p} : 1))^w$.

Crapo and Whiteley [1] derived that $\mathbf{t} \vee (\mathbf{p} : 1) = (\mathbf{p}', -\langle \mathbf{p}, \mathbf{p}' \rangle)$. Applying it, we obtain our desired result. The derivation when \mathbf{t} is indecomposable (the sum of two 2-extensors) is a simple extension.