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Combinatorial Genericity and Minimal Rigidity

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ABSTRACT

A well studied geometric problem, with applications ranging from molecular structure determination to sensor networks, asks for the reconstruction of a set P of n unknown points from a finite set of pairwise distances (up to Euclidean isometries). We are concerned here with a related problem: which sets of distances are minimal with the property that they allow for the reconstruction of P, up to a finite set of possibilities? In the planar case, the answer is known **generically** via the landmark Maxwell-Laman Theorem from Rigidity Theory, and it leads to a combinatorial answer: the underlying structure of such a generic minimal collection of distances is a **minimally rigid** (or Laman) graph, for which very efficient combinatorial decision algorithms exist. For non-generic cases the situation appears to be dramatically different, with the best known algorithms relying on exponential-time Gröbner base methods, and some specific instances known to be NP-hard. Understanding what makes a point set **generic** emerges as an intriguing geometric question with practical algorithmic consequences.

Several definitions (some but not all equivalent) of genericity appear in the rigidity literature, and they have either a measure theoretic, topologic or algebraic-geometric flavor. Some generic point sets appear to be highly degenerate, and still turn out to be generic. All existing proofs of Laman's Theorem make use at some point of one or another of these **geometric** genericity assumptions.

The main result of this paper is the first purely combinatorial proof of Laman's theorem, together with some interesting consequences. Genericity is characterized in terms of a certain determinant being not identically-zero as a formal polynomial. We relate its monomial expansion to certain colorings and orientations of the graph and show that these terms cannot all cancel exactly when the underlying graph is Laman. As a surprising consequence, genericity emerges as a purely combinatorial concept.

Categories and Subject Descriptors: F.2.2 [Nonnumerical Algorithms and Problems]: Geometrical problems and computations; G.2.2 [Graph Theory]: Trees

General Terms: Theory

Keywords: Computational Geometry; Rigidity; Sparse Graphs

1. INTRODUCTION

Every computational geometer has encountered assumptions of *generic*, *general position* or *non-degenerate* for some algorithm's input data. We know that without these assumptions, one often has to plunge into complicated case analysis, and that in some cases, a comprehensive way of handling non-generic situations may not exist. Some problems become computationally hard without the genericity assumption. To make sure subtle cases don't pop up to ruin correctness claims, different authors may use different notions of what *generic* means, with some of these concepts appearing to be computationally intractable.

In this paper, we focus on what a *generic point set* is for a well-studied problem: two-dimensional point reconstruction from distances, or planar rigidity.

Our main theoretical result is a new proof of the fundamental theorem of planar rigidity which completely demystifies the genericity assumption by turning it into a purely combinatorial concept. Along the way, we generalize this fundamental theorem to handle other types of rigidity, and exhibit some very *degenerate*, yet still *generic* situations that would be very hard to sort out without the tools we develop in this paper. In particular, we establish the correctness of a (very simple and elegant) combinatorial algorithm for a natural *generic* rigidity-theoretic problem (*slider pinning*) that we have recently proposed [20], in a very degenerate situation (axis-parallel sliders).

The Point Reconstruction Problem: *Given a set of $m \le \binom{n}{2}$ pairwise distances among a set $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$ of n unknown points in Euclidean space $\mathbb{R}^d$, find a possible realization* $\mathbf{p}$. This problem arises naturally in many settings, including molecular structure determination [5] and sensor networks [30]. Implicit in the statement is the following relaxation: which sets of distances *allow* reconstruction of $\mathbf{p}$ up to a finite set of possibilities, modulo Euclidean isometries? This is the *bar-and-joint rigidity problem*, formally defined next.

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The Rigidity Problem: An abstract **bar-and-joint framework** is a pair $(G, \boldsymbol{\ell})$, where $G = (V, E)$ is a graph with $n = |V|$ vertices and $m = |E|$ edges, and $\boldsymbol{\ell} \in \mathbb{R}^m$ is a vector of non-negative numbers specifying *edge lengths*. A realization $G(\mathbf{p})$ (in some dimension d) of the abstract framework [1] is a mapping of the vertices of G onto a point set $\mathbf{p} \in (\mathbb{R}^d)^n$ achieving the given edge lengths: $||\mathbf{p}_i - \mathbf{p}_j|| = \boldsymbol{\ell}_{ij}, \forall ij \in E$. Intuitively, a bar-joint framework models a structure made of fixed-length bars connected by universal joints, allowing (in principle) full rotation of the bars around them. A bar-joint framework is **rigid** if it has only a discrete set of realizations, up to isometries (complete definitions are given in Section 2.1 below). It is *minimally rigid* if it is rigid, but ceases to be so if some bar is removed.

The purely geometric question of deciding rigidity of a framework seems to be intractable, even for small, fixed dimension d. The best known algorithms rely on exponential time Gröbner basis techniques, and specific cases are known to be NP-complete [29]. However, for **generic** frameworks in the plane, the landmark Maxwell-Laman theorem states that rigidity has a combinatorial characterization, for which several efficient algorithms are known (more about this later).

THEOREM 1.1 (**Maxwell-Laman [25, 16]**). *A generic bar-joint framework $G(\mathbf{p})$ is minimally rigid in $\mathbb{R}^2$ if and only if G has $2n - 3$ edges, and every non-empty subgraph induced by n' vertices spans at most $2n' - 3$ edges.*

A graph satisfying the counting condition of this theorem is called a *minimally rigid graph* or, abstractly (without any reference to rigidity), a **Laman graph**.

As an important consequence for Computational Geometry (one which is in fact paradigmatic), the Maxwell-Laman theorem allows *generic* rigidity questions to be formulated in terms of combinatorial objects (Laman graphs). But what does it mean for a framework to be *generic*?

An analogy. To make our point, we use an analogy with the best studied problem in Computational Geometry: constructing the convex hull of a planar point set. All known convex hull algorithms work in the purely combinatorial setting of an *order type*, relying on a simple primitive for deciding if an ordered triplet of points makes a *left* or *right* turn. To avoid cluttering the algorithm with case analyses, one assumes that the points are in *general position*: no three are collinear. We know that *most* of the point sets are in general position. We know that if a point set is *not* in general position, then a small perturbation of it must be so; if a point set is in general position, then so is a small perturbation of it. We have never seen a paper describing a convex hull algorithm that would assume much stronger *general position* assumptions such as, for instance, asking that the points be *algebraically independent*: such assumptions would make the algorithms useless in practice. We also know that such assumptions are not necessary for this problem.

Generic Rigidity. In contrast, various definitions of genericity that appear in the Rigidity Theory literature are not as clearly amenable to combinatorial descriptions, and some are not as geometrically apparent as general position is: whether a set of points is generic depends on the framework's underlying graph, and geometrically degenerate situations such as collinearities or coincident points may be generic enough for rigidity applications. Some authors [22, p. 92] define a *generic framework* as being one where the points $\mathbf{p}$ are algebraically independent. This definition certainly guarantees the correctness of all the known generic rigidity theorems, but, as we said, it is totally unsatisfactory from a practical point of view: it would certainly raise questions about the validity of any fixed-precision arithmetic implementation. Other frequent definitions used in rigidity theory require that generic properties hold *for most of* the point sets (measure-theoretical) [37, p. 1331] or focus on properties which, if they hold for a point set $\mathbf{p}$ (called generic for the property), then they hold for any point in some open neighborhood (topological) [10].

What the correct concept of genericity should be, seems to depend on the problem, and seems to often have a non-computational character, thus affecting clearness and simplicity in proofs and algorithms as well.

Main Contribution and Novelty: a preview of Combinatorial Genericity. The main contribution of this paper is to clarify, and turn into an entirely combinatorial object, the *genericity* concept for planar rigidity. Along the way, we give a new proof of Laman's Theorem in the more general setting of pinning rigidity. A disclaimer, though: we do *not* propose an efficient algorithm for deciding rigidity in non-generic situations; this seems to be a *much* harder problem.

Here is a preview of our approach. We start with the precise mathematical formulation of the minimum rigidity problem, in terms of the rank of the so-called *rigidity matrix*. We treat the point coordinates as unknowns, and formulate the rank in terms of a certain polynomial (arising from a formal determinant) not being identically zero. We remark that we use in fact the appropriate concept from algebraic geometry, where a property is called *generic* if it holds on the complement of an algebraic variety (zero-set of an algebraic system). In this case, the generic point sets would be those for which this determinant would not vanish. This is of course possible if and only if it is not identically zero, which implies that the set of non-generic points has measure zero.

The *main idea* is to associate a set of combinatorial objects to the formal determinant. Monomials in the Laplace expansion of the determinant give rise to *colorings* and *orientations* of the underlying graph of the framework. The colors arise from the two types of coordinates of the unknown points (x or y) and the orientations from the choice of x_i or x_j in the expansion of a product of terms of the form $(x_i - x_j)$. Monomials may appear multiple times and thus may cancel. To prove that a certain determinant is not identically zero *exactly when the graph is Laman*, it suffices to find a monomial which is not canceled; for instance, one which occurs only once. We reduce this problem to finding colorings and orientations of the underlying Laman graph, which satisfy specific, unambiguous degree sequences, which capture monomial power vectors.

Related work. Our techniques should be understood in the context of a wide range of previous work. Here are the most relevant references.

Proofs of Laman's theorem and other genericity condi-

[1] We abbreviate *bar-and-joint* to *bar-joint* and often refer to (G, ℓ) or $G(p)$ simply as *frameworks*.

tions. The observation that the $(2n-3)$-counts are necessary for minimal rigidity appears in Maxwell's landmark papers of 1864 [25]. Their sufficiency was proven over 100 years later by Laman [16], who employs what are now called Henneberg constructions [14] on minimally rigid graphs. Whiteley [36] simplified this argument with a very elegant, generic, yet *geometrically degenerate*, choice of vertex positions. Lovász and Yemini [22] give a different proof, assuming that the coordinates of $\mathbf{p}$ do not satisfy *any* polynomial relation with integer coordinates (i.e., they are algebraically independent over $\mathbb{Q}$). Whiteley's proof in [35] implicitly makes use of the same condition. Tay [33] gave a proof based on Crapo's [4] so-called 3T2 decompositons of Laman graphs. His approach is to start with a framework with many zero-length edges and then perturb the endpoints to produce the final realization, a generic point set in non-general position.

Another concept of genericity that appears in the rigidity literature [10] is that there is an open neighborhood N of $\mathbf{p}$ so that $\mathbf{q} \in N$ implies that $G(\mathbf{q})$ is a realization of $G(\mathbf{p})$. This paper's definition of genericity implies this condition.

Pinning frameworks. The problem of pinning bar-joint frameworks in the plane (completely immobilizing, removing all motions, including trivial rigid ones) appeared in Lovász [24, 23] and, more recently, Fekete [7]. In their model, a framework is immobilized by fixing both coordinates of a vertex or neither of them. This model for pinning is different from the one we use in our paper [20] and here, in which coordinates are fixed separately. Though the problems are related, they induce different underlying combinatorial structures, and algorithmic solutions.

Combinatorial related work. Laman graphs, and their generalizations to sparse graphs have been very well-studied combinatorially. Our papers [32, 13, 17] provide an introduction to the combinatorial study of sparsity, and the references given there serve as a guide to the large combinatorial literature that we build upon [36, 28, 35, 26, 34, 12]. A specialization of Crapo's [4] 3T2 decomposition of Laman graphs appears as *induced-cut* 2-*forest* later in this paper.

Algorithmic rigidity. Although the Laman counts seem to require exponential time to check, all the questions about them, and thus about the generic rigidity of a graph are algorithmically tractable. For the **Decision** question, which asks whether the input is a Laman graph, the best known algorithm, which runs in time $O(n\sqrt{n \log n})$ is due to Gabow and Westermann [8]. The other major algorithmic questions of interest involve extracting a maximum-size Laman subgraph for the input, or finding the inclusion-wise maximal rigid subgraphs of the input. *All the known algorithms for these questions require $O(n^2)$ time, even for inputs with $O(n)$ edges.* See [17] for a more complete discussion of algorithmic rigidity problems and references. The most practical family of algorithms for (various problems about) Laman graphs are based on the elegant **pebble games** of Hendrickson and Jacobs [15], which we have generalized and adapted to other rigidity and combinatorial problems in [17, 32, 19]. We make use of these generalizations here and in [20].

From algebra to combinatorics. Combinatorial objects appear naturally in connection with other algebraic-geometric problems. Examples include perfect matchings, arising from the Pfaffian [21, p. 318], and the combinatorics of Newton polytopes [9]. Here, we will interpret determinants of pinned-rigidity matrices via graph colorings and orientations.

Overview of the paper. This extended abstract presents the main result, and is largely self-contained in its presentation of rigidity, infinitesimal rigidity, and combinatorial rigidity (but see also [31] for further technical details). We give the necessary background in rigidity and the related theory of sparse graphs in Section 2. Section 3 develops the combinatorial results on unambiguous degree sequences associated with certain colorings of Laman graphs, and connects them to the more general concept of (combinatorial) pinned rigidity. Finally, in Section 4, we develop the (geometric) rigidity theory for pinned Laman graphs and give the new proof of the Maxwell-Laman theorem, by relating the unambiguous degree sequences to the monomial expansion of a not-identically-zero determinant.

2. PRELIMINARIES

We refer the reader to [11, 37] for an introduction to rigidity theory. For a self-contained presentation, we briefly introduce now the most relevant results.

2.1 Rigidity background

Planar bar-and-joint rigidity relies on three fundamental concepts, built upon one another: continuous, infinitesimal and combinatorial rigidity. We will follow the same paradigm in Section 4, for our new model of pinned rigidity.

Notation. We will identify $\left(\mathbb{R}^2\right)^n$ with $\mathbb{R}^{2n}$ and consider a point set $\mathbf{p} \in \left(\mathbb{R}^2\right)^n$ as either a vector of n points, with $\mathbf{p}_i = (a_i, b_i)$, or as a flattened vector $(a_1, b_1, \ldots, a_n, b_n) \in \mathbb{R}^{2n}$.

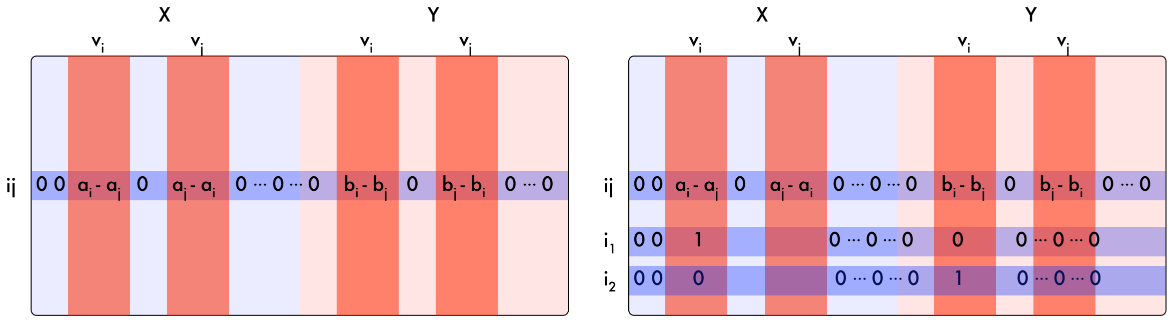
Frameworks and continuous rigidity. We are interested in analyzing the rigidity properties of a particular framework $G(\mathbf{p})$. Its **configuration space** $\mathcal{C}(G(\mathbf{p}))$ (shortly $\mathcal{C}$) is the set of all other r ealizations of the edge lengths of $G(p)$: $\mathcal{C} = \{\mathbf{q} \in \mathbb{R}^{2n} : G(\mathbf{q}) \text{ realizes } G(\mathbf{p})\}$.

Applying a Euclidean isometry to $\mathbf{p}$ results in a new realization of $G(\mathbf{p})$. To factor out these equivalent realizations, we take the quotient of $\mathcal{C}$ by the group Γ of Euclidean isometries. We say that a framework $G(\mathbf{p})$ is **rigid** if $\mathbf{p}$ is isolated in the quotient topology of $\mathcal{C}/\Gamma$.

Infinitesimal rigidity. Rigidity of $G(\mathbf{p})$ is a difficult property to establish. Instead, one uses the linearization of the problem. Taking the differential at $\mathbf{p}$ gives rise to the **rigidity matrix**: an $m \times 2n$ matrix with its rows indexed by the edges of G and two columns for each vertex, one for each coordinate. We order the columns so that they form two blocks: the first n correspond to x-coordinates and the second n correspond to y-coordinates. The row for edge ij has $a_i - a_j$ in the column for vertex i's x-coordinate and $a_j - a_i$ in the column for vertex j's x-coordinate; the y-coordinates similarly contain $b_i - b_j$ and $b_j - b_i$; and all the other entries are zero. Figure 1 (a) shows the pattern.

A framework is **infinitesimally rigid** if its **rigidity matrix** $\mathbf{M}(G(\mathbf{p}))$ has corank 3. It is well known [2] that if $G(\mathbf{p})$ is infinitesimally rigid, then it is rigid.

Generic combinatorial rigidity. A framework $G(\mathbf{p})$ is **generic** if the rank of the rigidity matrix $\mathbf{M}(G(\mathbf{p}))$ is maximum over all choices of $\mathbf{p}$. *Combinatorial rigidity* is concerned with finding good characterizations of the *graphs* of generically rigid frameworks. In dimension $d \geq 3$, no combinatorial characterizations are known, but dimension two is fully understood due to Maxwell-Laman's Theorem.



(a) (b)

Figure 1: The pattern of the rigidity matrix: (a) the matrix $\mathbf{M}(G)$; (b) the pinned rigidity matrix $\mathbf{M}^\star(G)$; the two rows i_1 and i_2 correspond to fixing both coordinates of the vertex i.

2.2 Combinatorial rigidity and sparse graphs

We summarize now the relevant combinatorial properties of Laman graphs.

Sparse graphs. Let $G = (V, E)$ be a graph with $n = |V|$ vertices and $m = |E|$ edges; in this paper we will not encounter multiple edges, but we do allow self-loops (shortly loops). A graph G is **(k, ℓ)-sparse** if every non-empty subgraph induced by n' vertices spans $m' \leq kn - \ell$. If, additionally, G has $kn - \ell$ edges, then G is **(k, ℓ)-tight**.

In particular, the Laman graphs of the Maxwell-Laman theorem are $(2, 3)$-tight. We observe that the sparsity parameters for Laman graphs imply that they are simple (no parallel edges), and that they do not contain self-loops. We will also employ a characterization of Laman graphs in terms of a special decomposition into forests. A 2-coloring of the edges of a graph is an **induced-cut 2-forest** if each color forms a forest and any induced subgraph contains a monochromatic cut; graphs admitting such a coloring are exactly Laman graphs [2].

Proposition 2.1 (**Induced-cut 2-forests [32]**). *Let G be a graph with n vertices and $2n - 3$ edges. Then G is a Laman graph if and only if it can be colored by an induced-cut 2-forest.*

Proposition 2.1 is related to another characterization of Laman graphs used in this paper.

Proposition 2.2 (**Adding one edge [22, 28]**). *Let G be a graph with n vertices and $2n - 3$ edges. Then G is a Laman graph if and only if adding any edge to it results in a graph that decomposes into two edge-disjoint spanning trees.*

Haas [12] has generalized his result to all sparse graphs with $k < \ell < 2k$. A further generalization, needed for modeling the slider-pinning problem described in Section 3, is in terms of **map-graphs**. A map-graph is an undirected graph which admits an orientation with out-degree exactly one. Equivalently [27] a map-graph has exactly one cycle per connected component, counting loops as cycles.[3] Map-graphs are known to coincide with $(1, 0)$-tight graphs, and graphs which decompose into k edge-disjoint map-graphs (k-map-graphs) coincide with $(k, 0)$-tight graphs. In a previous paper we proved the following characterization of sparse graphs in terms of map-graphs.

Proposition 2.3 (**Adding edges and loops [13]**). *Let G be a graph with n vertices and $kn - \ell$ edges. Then G is (k, ℓ)-tight if and only if adding any ℓ edges (including loops) to it results in a k-map-graph.*

Here the added edges come from $K_n^\star$, the complete graph on n vertices with k loops on every vertex and $2k$ copies of each edge. In particular, adding any three loops (not all on one vertex) to a Laman graph results in a 2-map. Our paper [32] gives a more algorithmic treatment of this topic.

3. COMBINATORIAL PINNED RIGIDITY

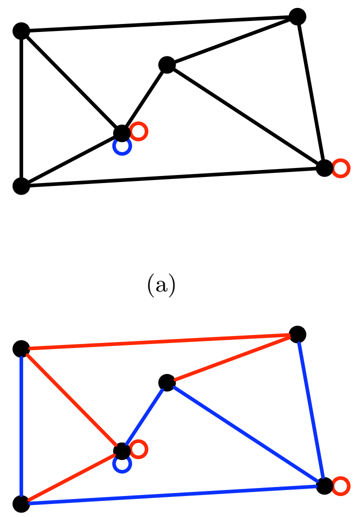
This section describes the *combinatorial essence* of our results. As previewed in the Introduction, we aim at studying the maximum rank of a generic rigidity matrix derived from a Laman graph. A slight generalization will come up handy: *pinned Laman graphs*. The monomial expansion of the formal rigidity matrix of a pinned Laman graph will be expressed as a sum of terms that are in one-to-one correspondence with the labeled, colored in-degree sequences of induced-cut 2-forest colorings compatible with the colored loops, defined in this section.

Pinned Laman graphs and mechanisms. Let G be a graph with n vertices, $2n - c$ edges and c loops with specified colors (blue or red). We say that G is an **axis-parallel slider-pinned Laman mechanism** (shortly, a pinned Laman mechanism) if the edges of G can be colored as an induced-cut 2-forest so that each monochromatic tree contains exactly one loop of its color; such a coloring is said to be **valid** for G. We observe that this implies that G is a 2-map-graph and that G without the loops is $(2, 3)$-sparse. Figure 2 shows an example. We call a pinned Laman mechanism with $2n - 3$ edges an **axis-parallel slider-pinned**

[2] We remind the reader that this concept is a specialization of the 3T2 characterization of Laman graphs given by Crapo [4], and it appeared under that name in our paper [32]. We have changed our terminology to highlight the additional condition that the three trees in a Laman graph must form two forests, which Crapo does not require.

[3] Map-graphs are also known in the matroid literature as pseudotrees, pseudoforests, functional graphs, and bases of the bicycle matroid.

Laman graph (shortly, a pinned Laman graph). Adding any three loops, not all of the same color, to a Laman graph results in a pinned Laman graph. As above, the added loops come from $K_n^\star$, with the additional restriction that each vertex has exactly one red loop and one blue one.



(a)

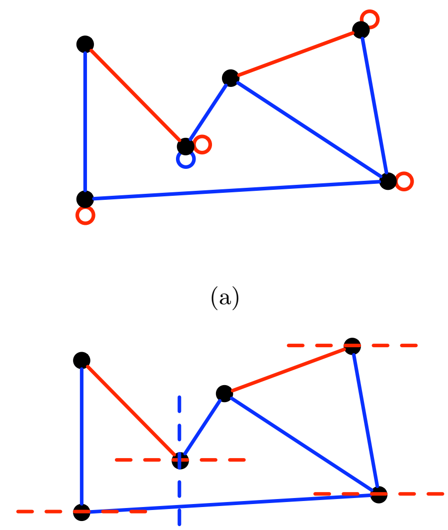
(b)

Figure 2: Pinned Laman graphs: (a) with pre-colored loops; (b) the same pinned graph with a compatible induced-cut 2-forest.

Lemma 3.1 (**Pinned Laman graphs**). *Let G be a graph with $2n-3$ edges. Then G is a Laman graph if and only if adding any three colored loops to G, not all of the same color, results in a pinned Laman mechanism.*

Proof. If G can be extended by three loops to a pinned Laman graph, then Proposition 2.1 implies immediately that G is a Laman graph. For the other direction, we suppose that G is a Laman graph. Any coloring of G into two forests is an induced-cut 2-forest, since no subgraph has enough edges to induce two edge-disjoint spanning trees. Now suppose that there are two red loops on necessarily distinct vertices i and j. Add the edge ij, which may be a copy of an existing edge, to G. By Proposition 2.2 the resulting graph decomposes into two edge-disjoint spanning trees. We may assume that the added edge ij is red. Removing it and keeping the coloring of all the other edges gives the induced-cut 2-forest we need: what is left is a blue spanning tree (and thus incident with the third, blue loop) and two disjoint red trees each containing exactly one red loop. □

Slider pinning. The terminology of pinned Laman mechanisms comes from our previous work on immobilizing bar-joint frameworks by adding **sliders**, which force a vertex to move on a given line [20]. For the specific case of axis-parallel sliders, this amounts to adding an equation to pin down one of the coordinates of the vertex. We introduced pinned Laman mechanisms in [20] as a combinatorial model for these axis-parallel bar-slider structures and studied (a generalization of) their combinatorics in [19]. It is important to note that Lemma 3.1 does *not* hold for arbitrary pinned Laman mechanisms; the allowed locations and colors for completing a $(2,3)$-sparse graph to a pinned mechanism depend in a strong way on where the edges are. However, the pinned mechanisms do form the bases of a matroid for which we have developed the combinatorial and algorithmic theory [20, 19]. Corollary 4.3 below provides a Laman-type theorem for bar-slider frameworks. Figure 3 illustrates the relationship between pinned Laman mechanisms and bar-slider frameworks.



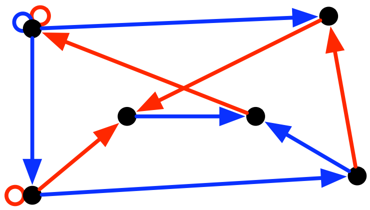
(a)

(b)

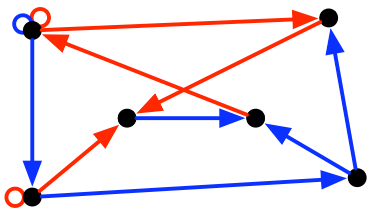
Figure 3: Pinned Laman mechanisms and slider pinnning: (a) the combinatorial object; (b) the associated axis-parallel bar-slider framework.

Colored in-degree sequences of pinned Laman mechanisms. Let G be a pinned Laman mechanism with n vertices and fix an induced-cut 2-forest coloring of the edges of G that certifies to this (i.e., it is compatible with the colors of the loops). We call such colorings of the edges **valid**. For example, the coloring of the edges in Figure 2(b) is a valid induced-cut 2-forest coloring for the pinned Laman mechanism in Figure 2(a). Note that not every induced-cut 2-forest coloring of the edges will be valid for a specific pinned Laman mechanism, as is the case in Fig. 4(b).

Given a pinned Laman mechanism and a valid coloring of its edges, fix an orientation of the edges of G; we use the convention that an oriented loop points both into and out of the vertex it is on. This leads to a **labeled colored in-degree sequence** $(\mathbf{r}, \mathbf{b})$, with $\mathbf{r} = (r_1, \cdots, r_n)$, $\mathbf{b} = (b_1, \cdots, b_n)$ and r_i, respectively b_i, being the number of red and blue edges pointing into vertex i. In the next section, these will be given an algebraic interpretation as monomials in the expansion of a determinant. The oriented induced-cut



(a)



(b)

Figure 4: Two oriented induced-cut 2-forest colorings of pinned $K_{3,3}$ with the same colored in-degree sequence: (a) valid and unambiguous; (b) same in-degree sequence, but invalid, because red is not a spanning map-graph.

2-forest is said to **realize** its in-degree sequence; note that a given degree sequence may have more than one realizer. Fig. 4 shows some examples.

Our main combinatorial result is the following theorem for colored in-degree sequences of Laman mechanisms.

THEOREM 3.2 (**Unambiguous degree sequences**). *Let G be a pinned Laman mechanism. Then there exists a compatible induced-cut 2-forest and an orientation of the edges of G so that the resulting colored in-degree sequence allows for an unambiguous reconstruction of the original orientation and induced-cut 2-forest coloring of G.*

The difficult part of the proof (which is deferred to the full paper) is characterizing the space of valid induced-cut 2-forest colorings of a pinned Laman mechanism. The unambiguity of a particular degree sequence depends in a strong way on the assumption that is has only one *valid* realizer; invalid realizers of the same sequence may exist, and are typically easy to find in small examples. In Figure 4(a), the oriented induced-cut 2-forest realizes an unambiguous colored in-degree sequence. Although the coloring and orientation in Figure 4(b) is an induced-cut 2-forest, it is not valid, and thus does not contradict the unambiguity of the degree sequence from Figure 4(a).

While unambiguous colored in-degree sequences exist for every pinned Laman mechanism, not every valid induced-cut 2-forest coloring of a pinned Laman mechanism has an orientation which gives rise to one. Figure 5 shows an example of such a coloring; for it, we have verified by exhaustive enumeration that *every* orientation of the valid induced-cut 2-forest coloring of the mechanism in Figure 5 yields a degree sequence that can be obtained from a different realizer.

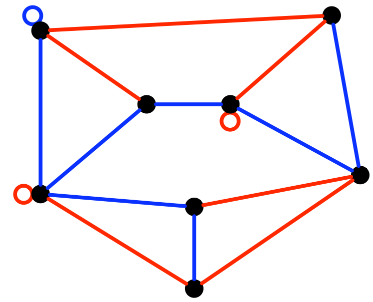


Figure 5: A valid edge coloring of a pinned Laman mechanism for which no orientation yields an unambiguous in-degree sequence.

We now have the tools we will need for our combinatorial proof of Laman's theorem, which relies on an algebraic interpretation of colored in-degree sequences arising from valid induced-cut 2-forest coloring of pinned Laman mechanisms.

4. MAIN RESULT: LAMAN'S THEOREM VIA PINNED RIGIDITY

We are ready for our main result: a purely combinatorial approach to generic bar-joint rigidity in the plane. More precisely, we develop, formally, a rigidity theory for structures made from bars, joints and axis-parallel slider-pins (which also serves as the formal setting, not developed anywhere so far, for the more general slider pinning model underlying our algorithms from [20]). The structure of this section echoes our presentation of (unpinned) bar-joint rigidity in Section 2.1, in its development of the three concepts of continuous, infinitesimal and combinatorial rigidity.

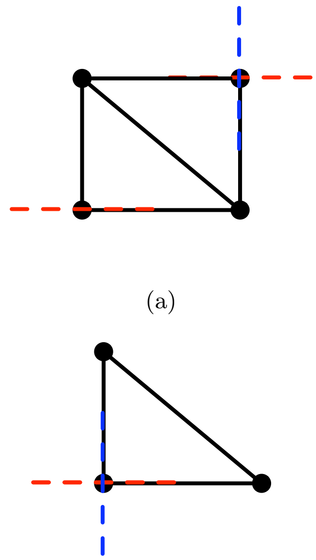
Pinned rigidity. Let $G(\mathbf{p})$ be a framework in the plane. We pin down an edge ij by fixing the coordinates of its endpoints. Let $G^{\star}_{ij}$ be the pinned Laman graph obtained from G by adding three loops (not all of the same color) on i, j. Define the ij-**pinned configuration space** of a pinned framework $G^{\star}_{ij}(\mathbf{p})$ as

$$\sigma_{ij}(\mathcal{C}) = \{\mathbf{q} \in \mathbb{R}^{2n} : \mathbf{q}_i = \mathbf{p}_i,\, \mathbf{q}_j = \mathbf{p}_j, \text{ and } G(\mathbf{q}) \text{ realizes } G(\mathbf{p})\}$$

Note that one of the four equations added to pin i and j is made redundant by the equation fixing the distance between i and j. In what follows, we assume that it is omitted.

A pinned framework is **pinned rigid** when $\mathbf{p}$ is an isolated point of $\sigma_e(\mathcal{C})$ and flexible otherwise. We remark that pinned rigidity, unlike rigidity, is defined in terms of an algebraic condition. However, it has an apparent dependency on the choice of edge to pin. We can remove this in the simple case where only the endpoints of an edge are pinned. The next Lemma justifies studying Laman rigidity properties via pinned frameworks and dropping the subscript ij for pinned frameworks.

LEMMA 4.1 (**Pinned rigidity and rigidity**). *A framework $G(\mathbf{p})$ is rigid if and only $G^{\star}_{ij}(\mathbf{p})$ is pinned rigid for* any *choice of edge ij to pin.*



(a)

(b)

Figure 6: Two embeddings of the same axis-parallel bar-slider framework: (a) generic; (b) non-generic. In (b) one triangle is reflected along the diagonal edge, resulting in coincident vertices and overlapping edges.

PROOF. (sketch) The proof strategy is to show that the quotient space $\mathcal{C}/\Gamma$ (of the configuration space by the group of isometries) is homeomorphic to $\sigma_{ij}(\mathcal{C})$. $\square$

Extension to slider-pinning. Pointing to some more subtle aspects, not fully addressed in this extended abstract, we note that we can describe the pinned configuration space for any graph with edges and colored loops that are not necessarily on the endpoints of an edge in a similar manner. This corresponds to the continuous theory for slider-pinning. But then, generic point sets for these more general slider-pinned frameworks are only a subset of those for unpinned frameworks. For example, both of the embeddings in Figure 6 are generic for the underlying bar-joint framework, but Figure 6 (b) is not pinned (it can rotate around the bottom left vertices, which are coincident). This situation does not arise when we pin the endpoints of an edge.

We next develop the infinitesimal theory, which is the same in both cases.

Pinned infinitesimal rigidity. Let $G^\star(\mathbf{p})$ be a pinned framework. The **pinned rigidity matrix** $\mathbf{M}^\star(G(\mathbf{p}))$ is an $(m+3)\times 2n$ matrix that has one row for each edge $ij \in E$ and three additional rows corresponding to the pinned vertices i and j. (We add only three additional rows, since we can just drop one of the equations pinning the coordinates of i and j.) The rows corresponding to edges have the same form as those of the rigidity matrix. The row corresponding to pinning the x-coordinate of vertex i has a 1 in the x-coordinate column associated with vertex i and zeros elsewere. Rows for pinning the y-coordinates of vertices i and j are defined similarly. Figure 1 (b) shows the pattern of the pinned rigidity matrix. Like the rigidity matrix, $\mathbf{M}^\star(G(\mathbf{p}))$ arises from the differential of $\sigma(\mathcal{C})$ at $\mathbf{p}$.

A pinned framework is **infinitesimally rigid** if the rank of $\mathbf{M}^\star(G(\mathbf{p}))$ is $2n$ and flexible otherwise. With the observation above we have the following lemma relating rigidity and pinned rigidity (we skip the proof in the extended abstract, but see [31]).

LEMMA 4.2 (**Infinitesimal rigidity and rigidity**). *Let $G^\star(\mathbf{p})$ be a pinned Laman graph. If $G^\star(\mathbf{p})$ is infinitesimally pinned rigid, then it is pinned rigid. Moreover, the underlying unpinned framework $G(\mathbf{p})$ is rigid.*

Generic combinatorial pinned rigidity. As in the unpinned case, a pinned framework $G^\star(\mathbf{p})$ is **generic** when the rank of $\mathbf{M}^\star(G(\mathbf{p}))$ is maximum over all choices of $\mathbf{p} \in \mathbb{R}^{2n}$.

Equivalently, we can consider the **generic pinned rigidity matrix** $\mathbf{M}^\star(G)$. This has the same form as $\mathbf{M}^\star(G(\mathbf{p}))$, but has indeterminate entries of the form $a_i - a_j$ and $b_i - b_j$ instead of concrete numbers. The rank of the generic matrix $\mathbf{M}^\star(G)$ is then given by the size of a maximum minor which is not zero as a formal polynomial.

We observe that the generic matrix $\mathbf{M}^\star(G)$ depends only on the graph G and the choice of edge to pin. Thus we will use the pinned Laman mechanisms of the previous section as the combinatorial model of pinned frameworks. Recall that a pinned Laman graph has two red loops and one blue one. We interpret the color of the loops as indicating which coordinate of that vertex to pin, putting the loops in correspondence with the rows of the pinned rigidity matrix associated with pinning.

Comparison to other genericity concepts. At this point, for emphasis, we remind the reader of the stronger concept of genericity that appears in the rigidity literature (to contrast it with ours): the requirement that the coordinates of the vertices be algebraically independent over $\mathbb{Q}$. Frameworks on a variety of *degenerate* point sets, including those having small integer coordinates, would never be generic in this model, making the theory unsuitable for algorithmic purposes.

Proof of Laman's theorem. We now have all the ingredients for a combinatorial proof of Laman's theorem. Here we concentrate on the difficult ("Laman") direction: we will prove that every Laman graph can be realized as a generic, minimally rigid framework. In what follows, we will use the notation $\mathbf{A}[I, J]$ for the submatrix of an $m \times n$ matrix $\mathbf{A}$ induced by the set of rows $I \subset [m]$ and columns $J \subset [n]$.

PROOF. Let G be a Laman graph. By Lemma 3.1, G can be extended to a pinned Laman graph $G^\star$ by adding three loops to the endpoints of any edge. We now consider the generic pinned rigidity matrix $\mathbf{M}^\star$ of $G^\star$. This is a $2n \times 2n$ matrix. We will show that its determinant is non-zero as a formal polynomial, implying that a generic framework with the underlying graph G is minimally rigid by Lemma 4.2.

Let $X = [n]$, $Y = [2n] - [n]$, and since $\mathbf{M}^\star$ is square, identify its rows $[m]$ with $[2n]$. Using the generalized Laplace expansion (see [1, p. 76]) for the determinant around X we obtain $\det(\mathbf{M}^\star)$ as plus or minus

$$\sum_{B\subset[2n],|B|=n} \pm \det(\mathbf{M}^\star[B, X]) \det(\mathbf{M}^\star[R, Y])$$

$$
\begin{pmatrix}
a_1-a_2 & a_2-a_1 & 0 & 0 & 0 & 0 & b_1-b_2 & b_2-b_1 & 0 & 0 & 0 & 0 \\
a_1-a_4 & 0 & 0 & a_4-a_1 & 0 & 0 & b_1-b_4 & 0 & 0 & b_4-b_1 & 0 & 0 \\
a_1-a_6 & 0 & 0 & 0 & 0 & a_6-a_1 & b_1-b_6 & 0 & 0 & 0 & 0 & b_6-b_1 \\
0 & a_2-a_3 & a_3-a_2 & 0 & 0 & 0 & 0 & b_2-b_3 & b_3-b_2 & 0 & 0 & 0 \\
0 & a_2-a_5 & 0 & 0 & a_5-a_2 & 0 & 0 & b_2-b_5 & 0 & 0 & b_5-b_2 & 0 \\
0 & 0 & a_3-a_4 & a_4-a_3 & 0 & 0 & 0 & 0 & b_3-b_4 & b_4-b_3 & 0 & 0 \\
0 & 0 & a_3-a_5 & 0 & a_5-a_3 & 0 & 0 & 0 & b_3-b_5 & 0 & b_5-b_3 & 0 \\
0 & 0 & 0 & a_4-a_6 & 0 & a_6-a_4 & 0 & 0 & 0 & b_4-b_6 & 0 & b_6-b_4 \\
0 & 0 & 0 & 0 & a_5-a_6 & a_6-a_5 & 0 & 0 & 0 & 0 & b_5-b_6 & b_6-b_5 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

Figure 7: The pinned rigidity matrix of the pinned Laman graph in Figure 8(a).

where the sum is taken over sets of blue edges B and red edges $R = [2n] - B$.

We now interpret each term in the sum combinatorially. If the set of edges corresponding to B does not correspond to a map-graph where the blue loop forms the only cycle, then $\det(\mathbf{M}^\star[B, X])$ is zero since any cycle of edges leads to a dependency and any row corresponding to a red loop induces a row of all zeros. A similar argument applies to $\det(\mathbf{M}^\star[R, Y])$. It follows that combinatorially, the non-zero terms in the expansion correspond to the induced-cut 2-forests of G that are valid with respect to the added loops. Each of these non-zero terms is of the form

$$\det(\mathbf{M}^\star[B, X]) \det(\mathbf{M}^\star[R, Y]) = \pm\left(\prod_{ij\in B}(a_i - a_j)\right)\left(\prod_{ij\in R}(b_i - b_j)\right)$$

where the edge ij is identified with its corresponding row in $\mathbf{M}^\star$. To complete the proof, we show that they do not cancel out. The critical observation is that each monomial obtained by multiplying out this product corresponds to picking an orientation of a valid induced-cut 2-forest coloring of the edges and using the blue in-degree as the power of a_i and the red in-degree as the power of b_i. Theorem 3.2 implies that there is a monomial which cannot be canceled symbolically. □

Example: To illustrate the correspondence between oriented induced-cut 2-forest colorings and the monomials in the expansion of the determinant of the pinning matrix, we consider the pinned Laman mechanism shown in Figure 8(a). Its pinned rigidity matrix is 12×12 matrix in Figure 7.

Taking the Laplace expansion as in our proof of Laman's theorem, we find the non-zero term

$$(a_1 - a_2)(a_2 - a_3)(a_4 - a_1)(a_5 - a_3)(a_6 - a_1)\cdot (b_3 - b_4)(b_5 - b_2)(b_4 - b_6)(b_6 - b_5)$$

which corresponds to the valid coloring in Figure 8(b). The monomial $a_1^2 a_3^2 a_4 b_4 b_5^2 b_6$ appears exactly once in the expansion of the determinant, and it corresponds to the unambiguous orientation of Figure 8(b) shown in Figure 8(c); similarly the monomial $a_1^3 a_3^2 b_2 b_4^2 b_5$ corresponds to the orientation shown in Figure 8(d).

Remark: The rigidity matrix of a Laman graph is not square, and thus it may have many non-singular $(2n-3)\times(2n-3)$ minors. In light of Lemma 3.1, we can interpret all the possible ways of slider-pinning a Laman graph as picking out a particular minor to test. Since there are only a finite number of these, we have shown that, for a given Laman graph G almost all point sets $\mathbf{p}$ are generic for every extension of G to a pinned Laman graph. The same argument applied to a pinned Laman mechanism establishes a Laman-type theorem for bar-slider frameworks, completing the proof of correctness of our slider pinning algorithms from [20].

COROLLARY 4.3 (**Generic bar-slider rigidity**). *Let G be a graph with n vertices, $2n - c$ edges, and c colored loops. G is realizable as a generic slider-pinned axis-parallel bar-slider framework if and only if G is a pinned Laman mechanism.*

The case for sliders that are generic lines or even differentiable curves follows from this. Here we skip the details.

5. CONCLUSIONS AND OPEN QUESTIONS

We gave a new, completely combinatorial, proof of Laman's landmark characterization of planar generic rigidity. Along the way, we introduced a new approach to rigidity and genericity which reduces the problem to elementary combinatorics, completely avoiding complicated geometric arguments.

Although Laman graphs have been well-studied over the past 30 years, our work here introduces oriented colorings of their induced-cut 2-forests as interesting objects of study. In particular, given the close connection between induced-cut 2-forests and the rigidity matrix, enumerating them (and understanding their cancelation patterns) would be interesting.

Some prominent remaining open questions include: (1) finding efficient (combinatorial) algorithms for deciding *rigidity* (as opposed to infinitesimal rigidity, which can be decided by Gaussian elimination) in non-generic cases (alternatively, prove NP-hardness); (2) *extracting* a spanning Laman subgraph from a dense graph in time $o(n^2 + m)$ [18]. In constrast, the problem of *deciding* whether a graph is Laman is known to be $o(n^2)$. This last problem has recently received renewed attention [6, 3], and simplifications for a part of an older $O(n\sqrt{n \log n})$ algorithm of [8] have been proposed, within the same overall asymptotic complexity.

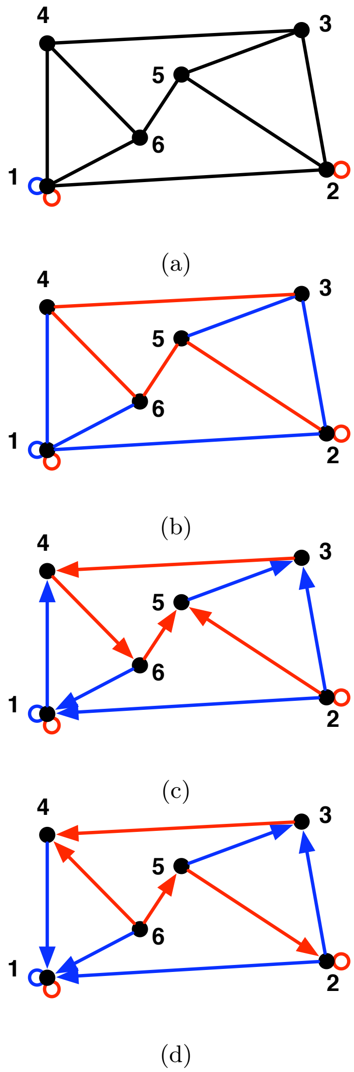


Figure 8: The correspondence between orientations of valid colorings and monomials: (a) a pinned Laman mechanism; (b) a valid coloring; (c) unambiguous orientation for $a_1^2 a_3^2 a_4 b_4 b_5^2 b_6$; (c) unambiguous orientation for $a_1^3 a_3^2 b_2 b_4^2 b_5$.

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