On the Folkman-Lawrence Topological Representation Theorem for Oriented Matroids of Rank 3

Jürgen Bokowski
Technische Universität Darmstadt

Susanne Mock
Technische Universität Darmstadt

Ileana Streinu
Smith College, istreinu@smith.edu

Europ. J. Combinatorics (2001) **22**, 601–615
doi:10.1006/eujc.2000.0482
Available online at http://www.idealibrary.com on IDEAL®

On the Folkman–Lawrence Topological Representation Theorem for Oriented Matroids of Rank 3

JÜRGEN BOKOWSKI, SUSANNE MOCK AND ILEANA STREINU[†]

We present a new direct proof of the Folkman–Lawrence topological representation theorem for oriented matroids of rank 3.

1. INTRODUCTION

Oriented matroids capture combinatorial properties of finite vector configurations and oriented hyperplane arrangements. However, not all oriented matroids have a vector or hyperplane model. It is a remarkable result, due to Folkman and Lawrence [9], that each oriented matroid has a topological representation as an oriented pseudosphere arrangement, even a piecewise linear one, cf. Edmonds and Mandel [7]. Other authors [3, 15] have later simplified or complemented the original proof, but all use fundamentally the same approach: the face lattice (tope) formalism for oriented matroids and a shelling order to carry through the construction.

Finding a reasonably direct proof in rank 3, one that would rely on the structural simplicity of the planar case, has been posed as an open problem in the research monograph [3, Exercise 6.3]. In this article we provide such a proof. Unlike the previous ones, ours is based on *hyperline sequences*, an equivalent axiomatization for oriented matroids which is particularly natural in rank 3. We construct a piecewise linear pseudocircle arrangement on the S^2 sphere, compatible with a given rank 3 oriented matroid induced by hyperline sequences.

Hyperline sequences were first used in 1978 by Bokowski [1] (see [2] for an early reference and [4] for a more comprehensive exposition). Independently, Goodman and Pollack [12] introduced the rank 3 affine version known as *clusters of stars* or *local sequences* and Streinu [23] characterized them with a simple set of axioms. Hyperline sequences are a compact representation for oriented matroids and thus amenable to computer applications (see [5]). Their axioms allow for simpler proofs, a fact exploited in [21] for applications to visibility problems in computational geometry. Because the key facts about this formalism are scattered through the literature and have never been completely presented, in the format needed for our proof, as a unified axiomatic system for oriented matroids, we will devote a substantial part of this paper to them.

Our proof technique is based on a series of simple reductions and an inductive construction. We start with the most general setting (degeneracies included). The reductions transform the sequences from degenerate to uniform, from arbitrarily oriented and arbitrary labeled to a convenient normal form. The normalized sequences are then used to produce a piecewise linear affine pseudoline arrangement. To obtain the oriented, degenerate spherical arrangement, the reduction steps are now performed in reverse order: the pseudolines are oriented, relabeled, projected radially onto the sphere and then the arrangement is perturbed to put back the original degeneracies.

We use the oriented matroid axioms explicitly in performing the inductive construction, but in a rather unexpected way: they are essentially needed only for the proof of the base case ($n \leq 5$), on which the inductive step then relies.

[†]Research partially supported by NSF grant CCR-9731804.

0195–6698/01/050601 + 15 $35.00/0

We remark that there exist other constructions for pseudoline arrangements, such as those producing wiring diagrams from allowable sequences [12], sweeping [8] or starting from a matroid and questioning its orientability [6]. They are all based on the same idea: first finding a total ordering of the vertices of the arrangement (co-circuits). This is essentially a simplified version of a shelling order in rank 3. To the best of our knowledge, no other direct inductive proofs, where pseudolines are added one at a time, have been proposed.

This paper is organized as follows. Sections 2 and 3 are devoted to the hyperline sequence formalism. Section 4 contains the main result.

The following polar dual pairs of ordered geometric sets can be represented by a matrix: vector configurations and arrangements of oriented central planes, arrangements of oriented great circles and configurations of points on the 2-sphere, arrangements of oriented lines and configurations of signed points. In Section 2 we extract combinatorial properties of these geometrical objects as the oriented matroid induced by a set of hyperline sequences. In Section 3 we generalize this concept in two ways. The first one is topological: we define oriented matroids as a topological invariant of oriented great pseudocircle arrangements. The second is combinatorial: we define oriented matroids induced by abstract hyperline sequences satisfying a single axiom, the well-definedness of an alternating and anti-symmetric abstract sign of determinant function. The two concepts will turn out to be cryptomorphic. In proving this in Section 4 we provide the desired rank 3 version of the Folkman–Lawrence topological representation theorem for oriented matroids. We add a proof in Section 5 that oriented matroids induced by hyperline sequences are in one-to-one correspondence with oriented matroids defined by chirotopes.

Throughout this paper we will work only with rank 3 oriented matroids.

2. Hyperline Sequences from Linear Configurations and Arrangements

In this section we define hyperline sequences as combinatorial abstractions arising from diverse finite collections of geometric objects, such as vector configurations and arrangements of oriented central planes, arrangements of oriented great circles and configurations of points on the 2-sphere, arrangements of oriented lines and configurations of signed points.

2.1. Configurations and arrangements. We consider a non-degenerate *vector configuration* in R^3, i.e., a finite ordered set $V_n = \{v_1, \dots, v_n\} \subset R^3$, $n \geq 3$, $v_i \neq 0$, $i = 1, \dots, n$, such that the one-dimensional subspaces generated by v_i, $i = 1, \dots, n$, are pairwise different and such that the corresponding $n \times 3$ matrix M with v_i as its ith row vector has rank 3. The vector configuration will be viewed as a representative of the equivalence class of matrices $\mathrm{cl}_n(M) := \{M' \mid M' = D\,M,\ D = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n), \lambda_i > 0, i = 1, \dots, n\}$.

We introduce additional configurations and arrangements representing geometrically the class of matrices $\mathrm{cl}_n(M)$. It is useful to think simultaneously of all of them and to pick the most convenient model for a particular application.

A vector configuration V_n induces an *arrangement of oriented central planes* $H_n = \{h_1, \dots, h_n\}$, via the concept of polar duality. The unoriented plane of h_i is given as the zero space $\{\underline{x} = (x_1, x_2, x_3) \in R^3 \mid h_i(\underline{x}) = 0\}$ of a linear homogeneous function $h_i(\underline{x}) = v_{i_1}x_1 + v_{i_2}x_2 + v_{i_3}x_3$, $v_i = (v_{i_1}, v_{i_2}, v_{i_3}) \neq 0$. The positive and negative sides of an oriented central plane are the two induced half-spaces $h_i^+ : \{\underline{x} \mid h_i(\underline{x}) > 0\}$ and $h_i^- : \{\underline{x} \mid h_i(\underline{x}) < 0\}$.

An arrangement of oriented central planes H_n induces an *arrangement of oriented great circles* $C_n = \{c_1, \dots, c_n\}$ on the 2-sphere and vice versa. An oriented central plane cuts the unit sphere S^2 in R^3 along a great circle which we consider to be parameterized and oriented

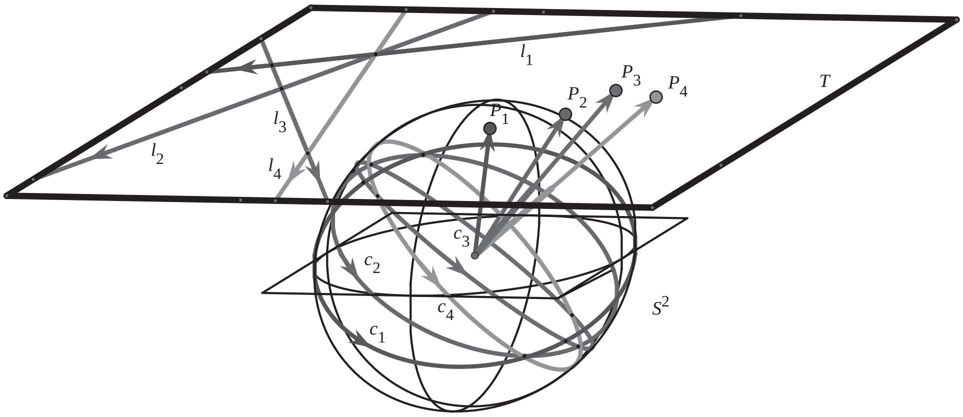

FIGURE 1. An equivalence class of matrices and geometric representatives.

such that, when looking from the outside, the positive half-space lies to its left when the parameter increases.

A vector $v \neq 0, v \in R^3$ induces a directed line $l_v : \{\alpha v | \alpha \in R\}$ through the origin, which intersects the sphere in two antipodal points s_v (in the direction of v) and $\overline{s}_v$ (in the opposite direction). A vector configuration V_n induces a *configuration of points on the sphere*, $S_n = \{s_1, s_2, \ldots, s_n\}$, where $s_i = s_{v_i}, i = 1, \ldots, n$. Each point p on the sphere has an associated antipodal point $\overline{p}$.

We carry over the previous polar dual pairs to the affine plane T, viewed as a plane tangent to the 2-sphere. We assume that $v_i, i \in \{1, \ldots, n\}$ is neither parallel nor orthogonal to the plane T.

The great circle parallel to T defines two open hemispheres. One of them, called the *upper hemisphere*, contains the tangent point of T. An oriented great circle c_i induces an oriented half-circle in this upper hemisphere which projects to an oriented straight line $l^T(c_i)$ in the plane T via radial projection, and vice versa, any oriented straight line in T defines an oriented great circle on S^2. An arrangement of oriented great circles induces an *arrangement of oriented lines* $L_n^T = \{l_1, \ldots, l_n\}$, where $l_i := l^T(c_i)$, in the affine plane.

The same transition from the sphere S^2 to the plane T leads from a point configuration on the sphere to a signed point configuration in the affine plane. We define $sp^T(s_i)$ to be a pair of a signed index and a point $p_i \in T$ obtained via radial projection from s_i, as follows. A point s_i on the upper hemisphere maps to the pair $sp^T(s_i) = (i, p_i), i \in \{1, \ldots, n\}$, and a point s_i on the lower hemisphere maps to a pair $(\overline{i}, p_{\overline{i}})$ and $p_{\overline{i}} := p_i \in T$. We obtain from $S_n = \{s_1, \ldots, s_n\}$ a *signed point configuration* $P_n^T = \{sp_1, \ldots, sp_n\}$, with $sp_i := sp^T(s_i)$, and vice versa.

We use $E_n = \{1, \ldots, n\}$, endowed with the natural order, to denote the index set of geometric objects such as vectors, planes, great circles and points on the sphere, lines and points in the Euclidian plane, or of a finite ordered set of abstract elements. The associated *signed index set* $\overline{E}_n = \{1, \ldots, n, \overline{1}, \ldots, \overline{n}\}$ makes it possible to denote orientations or signs of these elements. The $s \mapsto \overline{s}$ operator is required to be an involution: $\overline{\overline{s}} = s, \forall s \in \overline{E}_n$.

All ordered sets $V_n, H_n, C_n, S_n, L_n^T, P_n^T$ above can be viewed as geometric representations of the same equivalence class of matrices $\mathrm{cl}_n(M)$, see Figure 1. We can reorient the elements. The reorientation classes are the equivalence classes with respect to reorienting subsets such as vector configurations or central plane arrangements, great circle arrangements or pairs of

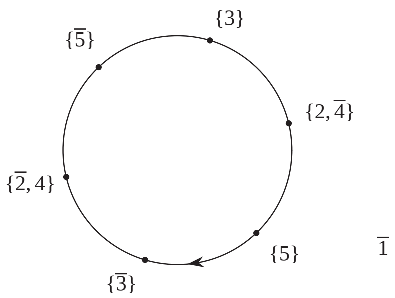

FIGURE 2. A hyperline sequence over $\overline{E}_5$.

antipodal points on the 2-sphere, line arrangements or point sets in the plane. These reorientation classes are obtained when the numbers $\lambda_i \neq 0$ can be negative as well. The *reorientation of a vector* v_i is the vector $v_{\bar{i}} = -v_i$ and the *reorientation of an oriented central plane* is the change of the sign of its normal vector. The *reorientation of an oriented great circle* or of an *oriented line* means replacing it by the same object with the reversed orientation. The *reorientation of a signed point* (i, p_i), $i \in \overline{E}_n$, is the signed point $(\bar{i}, p_{\bar{i}})$, $p_i = p_{\bar{i}}$. The *reorientation of an index* i is its replacement by $\bar{i}$. The *relabelling of an ordered set* is given by a permutation of its elements.

2.2. Hyperline sequences of configurations and arrangements. We now extract combinatorial information from all the geometric sets defined above. We will work *only* with signed subsets $q \subset \overline{E}_n$ which do not simultaneously contain both an element i and its negation $\bar{i}$. If $q \subset \overline{E}_n$, we define $\overline{q} = \{\bar{s} | s \in q\}$. The unsigned support supp$(q) \subset E_n$ of $q \subset \overline{E}_n$ is obtained by ignoring all the signs in q. A *signed partition* of E_n is a signed set $I = I^+ \cup I^-$ with $I^+, \overline{I^-} \in E_n$, $I^+ \cup \overline{I^-} = E_n$.

DEFINITION 2.1. A *hyperline sequence* hs_i *over* $\overline{E}_n$, $i \in \overline{E}_n$, with half-period length k_i is a pair $hs_i = (i, \pi_i)$, where π_i is a double infinite sequence $\pi_i = (q^i_j)_{j \in Z}$ with $q^i_j \subset \overline{E}_n \setminus \{i, \bar{i}\}$, $q^i_j = \overline{q^i_{j+k_i}}$, $\forall j \in Z$, $\text{supp}\big(\bigcup_{j \in Z} q^i_j\big) = E_n \setminus \text{supp}(\{i\})$, where the unsigned supports of $q^i_1, \ldots, q^i_{k_i}$ are mutually disjoint. We consider $hs_i = (i, \pi_i)$ and $hs_{\bar{i}} = (\bar{i}, \pi_{\bar{i}})$ to be equivalent when π_i is obtained from $\pi_{\bar{i}}$ by reversing the order.

The name hyperline for a subspace of codimension 2 is justified by the concept in higher dimensions. In the particular case when all the q^i_j's are one-element subsets, the sequence is said to be in *general position*, *simple* or *uniform*, and we replace the sets q^i_j with their elements. In this case, any half-period of π_i is a signed permutation of $E_n \setminus \text{supp}(\{i\})$. In general we have an additional ordered partition into pairwise disjoint subsets of the signed elements. An infinite sequence π_i in a hyperline sequence $hs_i = (i, \pi_i)$ can be represented by any half-period, i.e., by any k_i consecutive signed sets $q^i_{t+1}, \ldots, q^i_{t+k_i}, q^i_{t+j} \subset \overline{E}_n \setminus \{i, \bar{i}\}, t \in Z$.

EXAMPLE 2.1. $(\bar{1}, \pi_{\bar{1}}) = (\bar{1}, (\ldots, \{\bar{3}\}, \{\bar{2}, 4\}, \{\bar{5}\}, \{3\}, \{2, \bar{4}\}, \{5\}, \ldots))$ is a hyperline sequence over $\overline{E}_5$, $E_5 = \{1, \ldots, 5\}$, with half-period length $k_1 = 3$, see Figure 2.

We obtain the *normalized representation* $hs_r = (r, \pi_r)$ of a hyperline sequence $hs_i = (i, \pi_i)$ by first choosing $(r, \pi_r) := (i, \pi_i)$ if $i \in E_n$ or $(r, \pi_r) := (\bar{i}, \text{reverse } (\pi_i))$ if $\bar{i} \in E_n$, and afterwards choosing the half-period of π_r starting with the set $q^r_j \subset \overline{E}_n$ containing the smallest positive element.

EXAMPLE 2.2. The normalized representation of the hyperline sequence in the previous example is $(1, (\{2, \overline{4}\}, \{3\}, \{\overline{5}\}))$. From now on, we will use the more convenient notation $(1 : \{2, \overline{4}\}, \{3\}, \{\overline{5}\})$.

To a *signed point configuration* $P_n^T = \{(i, p_i) \mid i \in I\}$ (obtained from a vector configuration as described above) we associate a set $HS(P_n^T) = \{hs_1, \dots, hs_n\}$ of n hyperline sequences $hs_i = (i, \pi_i)$ over $\overline{E}_n$. The sequence π_i, denoted by a half-period $q_1^i, q_2^i, \dots, q_{k_i}^i$, with $q_j^i \subset \overline{E}_n \setminus \{i, \overline{i}\}$, corresponds to the signed point $(i, p_i) \in P_n^T$. It is obtained by rotating an oriented line in counterclockwise (ccw), or in clockwise (cw), order around p_i if $i \in E_n$, or if $\overline{i} \in E_n$, respectively, and looking at the successive positions where it coincides with lines defined by pairs of points (p_i, p_j) with $p_j \neq p_i$. When P_n^T is not in general position, several points may become simultaneously collinear with the rotating line, and they are recorded as a set q_k^i. If the point p_j of the signed point (j, p_j) is encountered by the rotating line in positive direction from p_i, it will be recorded as the index j, otherwise as the negated index $\overline{j}$. The whole sequence is recorded in the order induced by the rotating line, and an arbitrary half-period is chosen to represent it.

DEFINITION 2.2. The rank 3 *oriented matroid induced by hyperline sequences associated to a signed point configuration* $P_n^T = \{(i, p_i) | i \in I\}$, where I is a signed partition of E_n, is $HS(P_n^T) = \{hs_i = (i, \pi_i) \mid i \in I\}$ as described above. We identify $HS(P_n^T)$ with $\{(\overline{i}, \pi_i) \mid i \in I\}$.

Note that if the orientation of the plane T is reversed, all the sequences are reversed. The identification in the previous definition makes the notion of hyperline sequences independent of the chosen orientation of the plane T.

REMARK. When we start with a set of vectors V_n and two admissible tangent planes T and T', by radial projection we obtain two sets of signed planar points P_n^T and $P_n^{T'}$. The reader can verify that our definition ensures that the resulting hyperline sequences $HS(P_n^T)$ and $HS(P_n^{T'})$ will coincide. This allows for a definition of hyperline sequences associated to any of the previously considered geometric ordered sets: vectors, oriented central planes, etc.

Consider an arrangement $C = \{c_1, \dots, c_n\}$ of n oriented great circles on the sphere S^2. To each circle c_i associate a hyperline sequence by recording the points of intersection (ordered according to the orientation of the circle c_i) with the remaining oriented circles. An index j is recorded as positive (resp. negative) when the circle c_j crosses c_i from left to right (resp., right to left).

An arrangement of oriented lines $L_n^T = \{l_1, \dots, l_n\}$ induces a set of n hyperline sequences $HS(L_n^T)$: for each line l_i, record the points of intersection with the other lines (ordered according to the orientation of the line). Each element j is signed: positive if line l_j crosses l_i from left to right, negative otherwise.

EXAMPLE 2.3. For the arrangement of oriented great circles in Figure 3, we have the following induced set of normalized representations of hyperline sequences $HS(C_5)$. We obtain the same set of normalized representations $HS(M)$ of hyperline sequences for M,

$$HS(C_5) = HS(M) = \begin{pmatrix} 1: & \{2\}, & \{\overline{3}\}, & \{5\}, & \{\overline{4}\} \\ 2: & \{1\}, & \{3, 4\}, & \{\overline{5}\} & \\ 3: & \{1\}, & \{5\}, & \{\overline{2}, \overline{4}\} & \\ 4: & \{1\}, & \{\overline{2}, 3\}, & \{\overline{5}\} & \\ 5: & \{1\}, & \{4\}, & \{2\}, & \{\overline{3}\} \end{pmatrix} \qquad M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -4 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

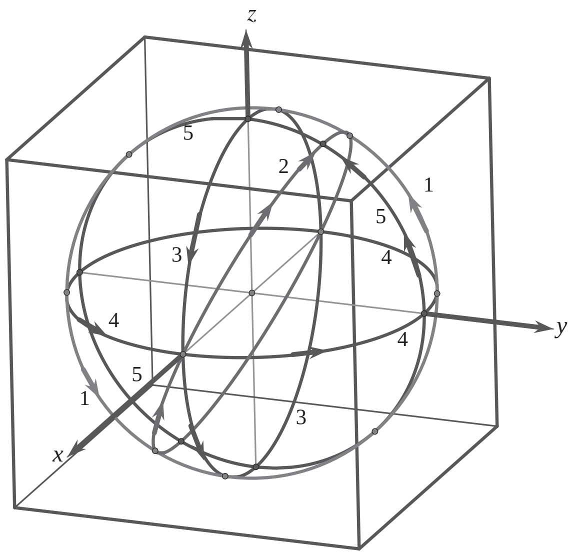

FIGURE 3. Arrangement C_5 of oriented great circles on the 2-sphere.

3. HYPERLINE SEQUENCES AND PSEUDOLINE ARRANGEMENTS

We start by describing a useful representation of great circle and affine line arrangements, which smooths the transition from lines to pseudolines.

Standard representation for great circle and line arrangements. Choose an oriented great circle c_i of the oriented great circle arrangement on the sphere S^2. Orient the plane spanned by c_i so that its positive side lies to the left when walking around the circle in the given direction and looking from the outside. Let A be the oriented plane and A^+ and A^- its two induced open half-spaces. Do an orthogonal projection from the closed hemisphere $S^2 \cap (A^+ \cup A)$ onto A. The resulting planar picture (an oriented circle with oriented arcs inside) will be called the *standard representation* of the oriented great circle arrangement C_n with equator c_i. From the standard representation we can always recover the whole oriented great circle arrangement on the sphere: do the orthogonal projection in reverse onto the closure of the hemisphere $S^2 \cap A^+$, to obtain oriented half-circles, then by taking antipodal points, complete them to great circles.

According to Section 2 the standard representation of the oriented great circle arrangement $C_n = \{c_1, \dots, c_n\}$ with equator c_i can also be viewed as a representation for an oriented line arrangement $L^T_{n-1} = \{l_1, \dots, l_n\} \setminus \{l_i\}$ with $n-1$ elements. If we forget all orientations and extend T to its projective plane, the standard representation also corresponds to an arrangement of n projective lines, the ith element being the *line at infinity* of T. The standard representation with antipodal points on the circle c_i identified defines a cell decomposition of the projective plane induced by the n lines. Note that in the projective setting any pair of lines cross exactly once. We will use the standard representation in two ways: as the projective model, for the cell decomposition properties and incidence properties of its lines, and as the sphere model (as the double covering of the projective plane), for oriented objects.

A *pseudoline* in the projective plane is the image of a projective line under a homeomorphic transformation of the projective plane. A *pseudoline arrangement* $\mathcal{A}$ in the projective plane is a finite ordered set of pseudolines, each pair of which crosses exactly once. We exclude the case when all pseudolines have one point in common. This concept goes back to Levi [20].

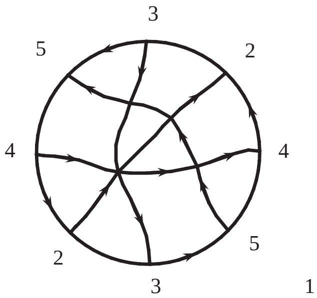

FIGURE 4. Homeomorphic image of the front half-sphere of Figure 2.

Let G^T be the group of homeomorphic transformations of the projective plane of T. For an arrangement $\mathcal{A}$ we have the equivalence class of arrangements $\mathrm{cl}(\mathcal{A}) := \{\mathcal{A}' | \mathcal{A}' = t\mathcal{A}, t \in G^T\}$. We always consider pseudoline arrangements $\mathcal{A}$ as representatives of their equivalence class $\mathrm{cl}(\mathcal{A})$.

We now come back to the sphere S^2 as a double covering of the projective plane. As we use the transition from the standard representation back to the sphere as before, the pseudolines become centrally symmetric simple closed curves on the sphere which we call *pseudocircles*. Any pair of pseudocircles crosses in a pair of antipodal points on the sphere (i.e., exactly once in the projective sense). But now we can introduce orientations for all elements to obtain an *arrangement of oriented pseudocircles*. By abuse of terminology, we will refer to our object as an *arrangement of oriented pseudolines* when we want to emphasize the incidence properties inherited from the projective setting and the orientations from the spherical setting. The standard representation has the advantage of showing both of these properties.

DEFINITION 3.1. The *oriented matroid associated to an arrangement of n oriented pseudolines* is its equivalence class with respect to homeomorphic transformations of the projective plane.

The oriented pseudoline arrangement is called *simple*, *uniform* or *in general position*, if no more than two pseudolines cross at a point.

EXAMPLE 3.1. The class of uniform oriented pseudoline arrangements with five elements contains exactly one element, up to reorientation and up to relabeling. Start with line 1 as the line at infinity. The next three pseudolines form an interior triangle. The insertion of the last pseudoline at pseudoline 1 is unique up to symmetry. The remaining possibilities all lead to the same equivalence class, that of the arrangement of five lines extending the sides of a regular 5-gon.

EXAMPLE 3.2. In Figure 4 we have depicted a standard representation of a non-uniform example, a homeomorphic image of the front half-sphere of Figure 3.

The rule to create a set of hyperline sequences $HS(L_n)$ from an arrangement of oriented lines $L_n = \{l_1, \dots, l_n\}$ can be carried over in the same way to any arrangement $PL_n = \{pl_1, \dots, pl_n\}$ of oriented pseudolines. Since there are oriented pseudoline arrangements for which there is no oriented line arrangement within the class of homeomorphic transformations for $n \geq 9$, we obtain in the pseudoline case a strictly more general concept, $|\{HS(L_n)|\forall L_n\}| < |\{HS(PL_n)|\forall PL_n\}|$ for $n \geq 9$.

We extend the concept of oriented matroids induced by hyperline sequences in another way. Hyperline sequences of configurations and arrangements of the last section store the signs of determinants of 3×3 submatrices of the matrix M of a corresponding vector configuration $V_n = \{v_1, \ldots, v_n\} \subset R^3$, $n \geq 3$, $v_i \neq 0$, $i = 1, \ldots, n$. This is an invariant for all matrices $M' \in \text{cl}_n(M)$. Let i, j, k be three distinct signed indices in $\overline{E}_n$. Let $[i, j, k]$ be the determinant of the submatrix of M with row vectors v_i, v_j, v_k. If j and k appear within the same set q_k^i of π_i, we have sign $[i, j, k] = 0$. If j and k occur in this order in some half-period of π_i, we have sign $[i, j, k] = +1$, and sign $[i, j, k] = -1$ otherwise. The sign of the determinant $\chi(ijk) :=$ sign $[i, j, k]$ is independent of the chosen half-periods and compatible by alternation $\chi(ijk) = \chi(jki) = \chi(kij) = -\chi(ikj) = -\chi(kji) = -\chi(jik)$ and anti-symmetry $\chi(\bar{i}jk) = -\chi(ijk)$.

Given an abstract set of hyperline sequences, let us choose its corresponding normalized form and define $\chi : \overline{E}_n^3 \to \{-1, 0, +1\}$, (partially) by: $\chi(ijk) := 0$, if j and k appear within the same set q_s of π_i, for i in E_n, j, k in $\overline{E}_n$, $j \neq k$, $\chi(ijk) := +1$, if j and k occur in this order in π_i, and $\chi(ijk) := -1$, if j and k occur in the reversed order in π_i.

Extending this partial definition of χ by alternation and anti-symmetry, the value of $\chi(ijk)$ for $0 < i < j < k$ is obtained either directly, by the above rule applied to each of the three hyperline sequences, or via alternation and anti-symmetry. When these three values for $\chi(ijk)$ are compatible in all cases, we say that *the set of hyperline sequences admit an abstract sign of determinant function*.

DEFINITION 3.2. A *rank* 3 *oriented matroid with n elements given by hyperline sequences* is a set of hyperline sequences $\{(i, \pi_i) \mid i \in I\}$ over $\overline{E}_n$ which admit an abstract sign of determinant function. The oriented matroid is uniform when all hyperline sequences are uniform.

THEOREM 3.3. *The hyperline sequences $HS(PL_n)$ of an oriented pseudoline arrangement PL_n admit an abstract sign of determinant function.*

PROOF. When we restrict the oriented pseudoline arrangement to three elements, the corresponding oriented pseudoline arrangement can be represented by three oriented lines. Hyperline sequences of oriented lines admit an abstract sign of determinant function. □

4. THE TOPOLOGICAL REPRESENTATION THEOREM

The following theorem shows that rank 3 oriented matroids given by hyperline sequences constitute a topological invariant with respect to the group of homeomorphic transformations of the projective plane. For each rank 3 oriented matroid given by hyperline sequences HS we can find an oriented pseudoline arrangement PL_n which induces it, $HS = HS(PL_n)$.

It will be clear from our construction that the oriented pseudoline arrangement represents a whole equivalence class with respect to homeomorphic images, each element of which leads back to the given hyperline sequences.

Similarly, it can be seen easily that mapping the oriented pseudoline arrangement via Theorem 3.3 to its hyperline sequences can be carried over to the whole equivalence class with respect to homeomorphic images.

The resulting map followed by our construction of the next theorem leads back to a representative of the equivalence class we started with. This can be seen via induction on the number of elements.

When we start in the following construction with a polygon as a representative for the projective plane, we see that we can carry on with our construction having convex polygons as cells all the time when some of which are subdivided by straight line segments.

THEOREM 4.1 (FOLKMAN–LAWRENCE TOPOLOGICAL REPRESENTATION THEOREM). *There is a one-to-one correspondence between rank* 3 *oriented matroids given by hyperline sequences and the equivalence classes of oriented pseudoline arrangements.*

PROOF. We start with a rank 3 oriented matroid with n elements given by hyperline sequences HS in normalized form. We are going to construct an arrangement of oriented pseudolines with line 1 being the line at infinity. We first prove the uniform case by induction, showing that if an arrangement of $n-1$ oriented pseudolines has been constructed, it is possible to insert the nth oriented pseudoline in a manner compatible with the given hyperline sequences.

We start the induction for $n \leq 5$. Let HS be a uniform rank 3 oriented matroid with five elements given by hyperline sequences. How many such different oriented matroids can we find? After relabeling and reorientation (in order to obtain 1: 2, 3, 4, 5 as the first hyperline sequence), we can assume that the abstract sign of determinant function yields positive values for the following 3-tuples 123, 124, 125, 134, 135, 145.

All possible extensions admitting an abstract sign of determinant function turn out to be the following:

$$\begin{pmatrix} 1: & 2, & 3, & 4, & 5 \\ 2: & 1, & \bar{3}, & \bar{4}, & \bar{5} \\ 3: & 1, & 2, & \bar{4}, & \bar{5} \\ 4: & 1, & 2, & 3, & \bar{5} \\ 5: & 1, & 2, & 3, & 4 \end{pmatrix} \begin{pmatrix} 1: & 2, & 3, & 4, & 5 \\ 2: & 1, & \bar{3}, & \bar{4}, & \bar{5} \\ 3: & 1, & 2, & \bar{5}, & \bar{4} \\ 4: & 1, & 2, & \bar{5}, & 3 \\ 5: & 1, & 2, & 4, & 3 \end{pmatrix} \begin{pmatrix} 1: & 2, & 3, & 4, & 5 \\ 2: & 1, & \bar{4}, & \bar{3}, & \bar{5} \\ 3: & 1, & \bar{4}, & 2, & \bar{5} \\ 4: & 1, & 3, & 2, & \bar{5} \\ 5: & 1, & 2, & 3, & 4 \end{pmatrix} \begin{pmatrix} 1: & 2, & 3, & 4, & 5 \\ 2: & 1, & \bar{3}, & \bar{5}, & \bar{4} \\ 3: & 1, & 2, & \bar{5}, & \bar{4} \\ 4: & 1, & \bar{5}, & 2, & 3 \\ 5: & 1, & 4, & 2, & 3 \end{pmatrix}$$

relabeling: (1)(2, 5, 4, 3) (1)(2, 3, 4, 5) (1)(2, 4)(3, 5)

reorientation 2, 3, 4 3, 4, 5 4, 5

$$\begin{pmatrix} 1: & 2, & 3, & 4, & 5 \\ 2: & 1, & \bar{5}, & \bar{4}, & \bar{3} \\ 3: & 1, & \bar{5}, & \bar{4}, & 2 \\ 4: & 1, & \bar{5}, & 3, & 2 \\ 5: & 1, & 4, & 3, & 2 \end{pmatrix} \begin{pmatrix} 1: & 2, & 3, & 4, & 5 \\ 2: & 1, & \bar{5}, & \bar{4}, & \bar{3} \\ 3: & 1, & \bar{4}, & \bar{5}, & 2 \\ 4: & 1, & 3, & \bar{5}, & 2 \\ 5: & 1, & 3, & 4, & 2 \end{pmatrix} \begin{pmatrix} 1: & 2, & 3, & 4, & 5 \\ 2: & 1, & \bar{5}, & \bar{3}, & \bar{4} \\ 3: & 1, & \bar{5}, & 2, & \bar{4} \\ 4: & 1, & \bar{5}, & 2, & 3 \\ 5: & 1, & 4, & 3, & 2 \end{pmatrix} \begin{pmatrix} 1: & 2, & 3, & 4, & 5 \\ 2: & 1, & \bar{4}, & \bar{5}, & \bar{3} \\ 3: & 1, & \bar{4}, & \bar{5}, & 2 \\ 4: & 1, & 3, & 2, & \bar{5} \\ 5: & 1, & 3, & 2, & 4 \end{pmatrix}.$$

(1)(2, 5, 4, 3) (1)(2, 3, 4, 5) (1)(2, 4)(3, 5)

2, 3, 4, 5 5 2 2, 3

The last seven cases are equal to the first one when applying first the given relabelings and afterwards the given reorientations. In particular, we have that each of the last four cases differs from the corresponding upper one just by a suitable reorientation. This implies that up to reorientations and up to relabelings, we have just one example which matches the corresponding oriented pseudoline arrangement, compare Example 3.1. The theorem is true for $n \leq 5$. □

REMARK. It is noteworthy that the axiom concerning the abstract sign of determinant function is used later only in the case when we come back to this assertion.

We apply induction to obtain an arrangement PL_{n-1} of $n-1$ oriented pseudolines with pseudoline 1 as the line at infinity, whose set of normalized hyperline sequences is obtained by removing the element n from each sequence and deleting the nth sequence. Using the position of element n in each of the original hyperline sequences, mark $n-1$ points, labelled with unordered pairs of indices (i, n), $i = 1, \ldots, n-1$ (denoted for simplicity as ni) on the existing pseudolines $1, \ldots, n-1$ (see Figure 5). The nth hyperline sequence defines an ordering of these points $n : 1, k_1, k_2, \ldots, k_{n-2}$. Here and in what follows we understandthe

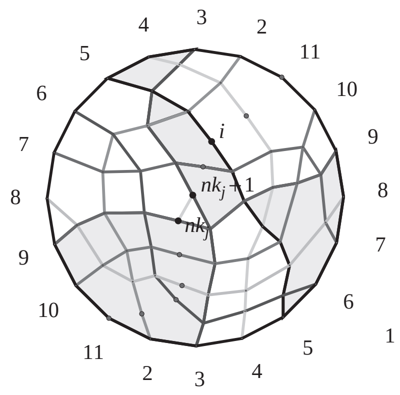

FIGURE 5. Essential step of the induction.

index j modulo $(n-1)$ and $k_0 = 1$. To prove the inductive step, it suffices to show that the following three conditions hold.

(1) We can join any two consecutive points nk_j and nk_{j+1}, $j = 0, \ldots, n-2$ with a pseudoline segment such that the open segment does not meet any of the already existing pseudolines.
(2) The resulting curve $\mathcal{K}$ obtained from all these segments together with the already existing pseudolines form a pseudoline arrangement PL_n, i.e., $\mathcal{K}$ is a simple closed curve which crosses each pseudoline j, $j \leq n-1$ at nj, and nowhere else.
(3) The nth pseudoline has a unique orientation.

PROOF OF (1). We show that any two consecutive points nk_j, nk_{j+1}, $j = 0, \ldots, n-2$ belong to the same cell of the arrangement PL_{n-1}, i.e., they are not separated by any of the existing pseudolines $i, i \neq 1, k_j, k_{j+1}$.

Consider two consecutive points nk_j, nk_{j+1} for an index $j = 0, \ldots, n-2$ and a pseudoline $i \neq 1, k_j, k_{j+1}$. Applying the induction hypothesis to the restriction of HS to the set of five, resp. four, elements $\{1, k_j, k_{j+1}, i, n\}$ (extensions up to four and five elements are even unique) implies a unique corresponding oriented pseudoline arrangement PL_5, resp. PL_4, up to a homeomorphic transformation of the projective plane. We have four elements, e.g., in the special case $j = 0$, because $k_0 = 1$. The ith pseudoline does not separate the points nk_j, nk_{j+1}. Therefore any two consecutive points nk_j, nk_{j+1}, $j = 0, \ldots, n-2$ of the arrangement PL_{n-1} are not separated by any of the existing pseudolines $i, i \neq 1, k_j, k_{j+1}$. This implies that we can connect two consecutive points nk_j, nk_{j+1} by a pseudoline segment without crossing any of the existing pseudolines.

PROOF OF (2). We consider two consecutive points nk_j, nk_{j+1}, $j = 0, \ldots, n-2$ and we pick a point np_j on the open pseudoline segment nk_j, nk_{j+1}, as constructed above. We show that all points nk_i, $i \in \{j+2, \ldots, n-2\}$ are separated from point np_j by the pseudoline k_{j+1} and in a similar way that all points nk_i, $i \in \{1, \ldots, j-1\}$ are separated from point np_j by pseudoline k_j. The argument in both cases is the same, so we prove only the first case (see Figure 5). We restrict the hyperline sequences to the set $\{1, k_j, k_{j+1}, i, n\}$ (the cases $1 = k_0$ and $k_{n-1} = 1$ are included) in which we find a unique corresponding pseudoline arrangement which uses the open pseudoline segment from nk_j to nk_{j+1}. The separation property holds, and it carries over to the arrangement with $n-1$ elements. This implies that the closed curve $\mathcal{K}$

above consisting of all pseudoline segments has no self-intersections, and it crosses all other $n-1$ pseudolines just once.

PROOF OF (3). On each oriented pseudoline $i \in \{1, \ldots, n-1\}$, we put an arrow A_{ni} at point ni pointing to the right side, or to the left side, of pseudoline i if the sign of element n in the ith hyperline sequence is positive, or negative, respectively. We show that for any two consecutive points nk_j, nk_{j+1}, $j = 0, \ldots, n-2$, the corresponding arrows $A_{nk_j}, A_{nk_{j+1}}$ are compatible, i.e., the induced orientation of pseudoline n by A_{nk_j} coincides with that of $A_{nk_{j+1}}$. We restrict the hyperline sequences to the set $\{1, k_j, k_{j+1}, n\}$. Applying the induction hypothesis for each j shows that we have in each case a unique corresponding oriented pseudoline arrangement with the desired property. This implies a unique orientation of the nth pseudoline in the globally constructed pseudoline arrangement. This concludes the proof by induction that a new pseudoline can be inserted in the uniform case. $\square$

The non-uniform case is also proven inductively, by eliminating a degeneracy at a time until we obtain the uniform case. Denote by $q^i(t)$ the set containing element t in the ith hyperline sequence of HS, and $q^i(s) < q^i(t)$ says that $q^i(s)$ lies in the chosen half-period left of $q^i(t)$.

We start with a set of hyperline sequences HS in normalized form and relabel and reorient them so that line 1 is the line at infinity and the first degeneracy contains the elements $2, 3, \ldots, k$. In the nth hyperline sequence the sign of each element $i \in \{1, \ldots, k\}$ is positive and the sets $q^n(i)$, $i \in \{1, \ldots, k\}$ are pairwise different. Relabel the elements $i \in \{2, \ldots, k\}$ such that $q^n(1) < q^n(2) < \cdots < q^n(k)$. We construct from the given hyperline sequence HS a new one HS' with a reduced degeneracy. We change the position of element 2 in the first hyperline sequence such that $q'^1(2) < q'^1(k)$, $q'^1(2) := \{2\}$, and $q'^1(k) := \{3, 4, \ldots, k\}$. The remaining changes in HS' compared with HS are consequences.

$$
\begin{pmatrix}
1: & \{2 \cdots i \cdots k\} & \cdots & & q^1(n) \\
2: & \{1, \overline{3} \cdots \overline{k}\} & \cdots & q^2(\overline{n}) & \cdots \\
 & & \vdots & & \\
i: & \{1 \cdots i-1, \overline{i+1} \cdots \overline{k}\} & \cdots & q^i(\overline{n}) & \cdots \\
 & & \vdots & & \\
k: & \{1, 2, 3 \cdots k-1\} & \cdots & q^k(\overline{n}) & \cdots \\
 & & \vdots & & \\
n: & q^n(1) \cdots q^n(2) \cdots q^n(k) & \cdots & &
\end{pmatrix}
\mapsto
\begin{pmatrix}
1: & \{2\}, q^1(2) \setminus \{2\} & \cdots & & q^1(n) \\
2: & \{1\}, \{\overline{k}\}, \{\overline{k-1}\} \cdots \{\overline{3}\} & \cdots & q^2(\overline{n}) & \cdots \\
 & & \vdots & & \\
i: & q^i(1) \setminus \{2\}, \{2\} & \cdots & q^i(\overline{n}) & \cdots \\
 & & \vdots & & \\
k: & q^k(1) \setminus \{2\}, \{2\} & \cdots & q^k(\overline{n}) & \cdots \\
 & & \vdots & & \\
n: & q^n(1) \cdots q^n(2) \cdots q^n(k) & \cdots & &
\end{pmatrix}.
$$

After a finite sequence of changes $HS_1, HS_2, \ldots, HS_i := HS, HS_{i+1} := HS', \ldots, HS_z$ we end up with a uniform oriented matroid with hyperline sequences HS_z. In the uniform case we find a corresponding oriented pseudoline arrangement. We go back all the steps from HS_{i+1} to HS_i, $i \in \{z-1, \ldots, 1\}$. The corresponding changes of the oriented pseudoline arrangements are evident (see Figure 6). Of course, we have to perform in reverse order all the previous reorientations and relabelings.

COROLLARY 4.2. *There is a one-to-one correspondence between the reorientation classes of rank* 3 *oriented matroids given by hyperline sequences and those of oriented pseudoline arrangements.*

4.1. Reorientation class invariant based on hyperline sequences. The unoriented pseudoline arrangement $\mathcal{A}(\mathcal{M})$ characterizes the reorientation class of an oriented matroid $\mathcal{M}$. We also provide a corresponding characterization in the hyperline sequence terminology. By identifying each unsigned hyperline sequence with its reversed one, we see that such an induced

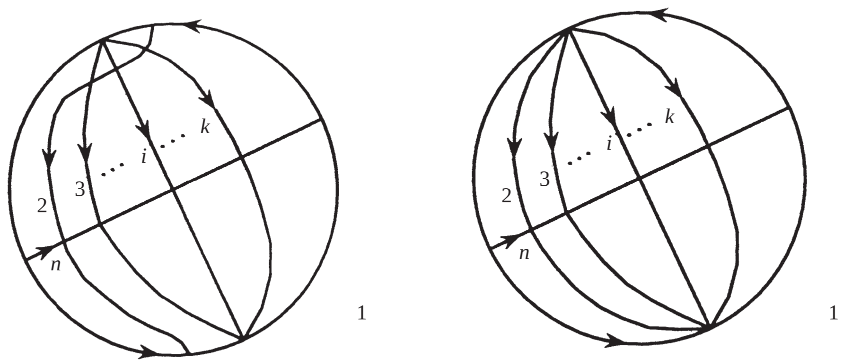

FIGURE 6. Inductive step for the pseudoline arrangements.

set of unsigned hyperline sequences $\mathcal{I}(\mathcal{M})$ of a rank 3 oriented matroid $\mathcal{M}$ is an invariant of its reorientation class. The unique construction in Theorem 4.1 of the unoriented pseudoline arrangement uses only the ordering of the unsigned sequences.

COROLLARY 4.3. *The reorientation class of a rank* 3 *oriented matroid* $\mathcal{M}$ *is characterized by both the unoriented pseudoline arrangement* $\mathcal{A}(\mathcal{M})$ *and the invariant of a set of hyperline sequences* $\mathcal{I}(\mathcal{M})$.

For the following invariant $\mathcal{I}(\mathcal{M})$ we reconstruct a representative of its reorientation class:

$$\mathcal{I}(\mathcal{M}) = \begin{pmatrix} 1: & \{2,3\}, & \{4,5\}, & \{6,7\} & \\ 2: & \{1,3\}, & \{5,7\}, & \{4,6\} & \\ 3: & \{1,2\}, & \{7\}, & \{5,6\}, & \{4\} \\ 4: & \{1,5\}, & \{7\}, & \{6,2\}, & \{3\} \\ 5: & \{1,4\}, & \{3,6\}, & \{2,7\} & \\ 6: & \{1,7\}, & \{3,5\}, & \{2,4\} & \\ 7: & \{1,6\}, & \{4\}, & \{2,5\}, & \{3\} \end{pmatrix} \begin{pmatrix} 1: & \{2,3\}, & \{4,5\}, & \{6,7\} & \\ 2: & \{1,3\}, & \{\bar{5},\bar{7}\}, & \{\bar{4},\bar{6}\} & \\ 3: & \{1,2\}, & \{\bar{7}\}, & \{\bar{5},\bar{6}\}, & \{\bar{4}\} \\ \hline 4: & \{1,5\}, & \{\bar{7}\}, & \{\bar{6},2\}, & \{3\} \\ 5: & \{1,4\}, & \{3,\bar{6}\}, & \{2,\bar{7}\} & \\ 6: & \{1,7\}, & \{3,5\}, & \{2,4\} & \\ 7: & \{1,6\}, & \{4\}, & \{2,5\}, & \{3\} \end{pmatrix}.$$

We keep the half-periods in the first two hyperlines and we insert the signs (written as overbars) from hyperline 1.

The half-periods of hyperlines 3, 4, 5, 6, 7 are not necessarily kept. We consider hyperline 3 later since the set {2, 3} in hyperline 1 plays a special role. The signs (written on the right-hand side) determine whether to keep, or to reverse, the order in hyperlines 4, 5, 6, 7, respectively. Compare signs in hyperline 2 with those of the actual hyperline.

$$\begin{pmatrix} 1: & \{2,3\}, & \{4,5\}, & \{6,7\} & \\ 2: & \{1,3\}, & \{\bar{5},\bar{7}\}, & \{\bar{4},\bar{6}\} & \\ 3: & \{1,2\}, & \{\bar{7}\}, & \{\bar{5},\bar{6}\}, & \{\bar{4}\} \\ \hline 4: & \{1,5\}, & \{\bar{7}\}, & \{\bar{6},2\}, & \{3\} \\ 5: & \{1,4\}, & \{3,\bar{6}\}, & \{2,\bar{7}\} & \\ 6: & \{1,7\}, & \{3,5\}, & \{2,4\} & \\ 7: & \{1,6\}, & \{4\}, & \{2,5\}, & \{3\} \end{pmatrix} \quad \begin{array}{l} \chi(2,4,7) = -1,\ \chi(2,5,6) = +1 \\ \\ \\ \\ \chi(2,4,7) = -1,\ \text{order kept} \\ \chi(2,5,6) = -1,\ \text{reversed order} \\ \chi(2,5,6) = -1,\ \text{reversed order} \\ \chi(2,4,7) = -1,\ \text{order kept.} \end{array}$$

We next determine the signs within the first sets in hyperlines 4, 5, 6, 7 from hyperline 2.

$$\left(\begin{array}{lllll} 1: & \{2,3\}, & \{4,5\}, & \{6,7\} & \\ 2: & \{1,3\}, & \{\bar{5},\bar{7}\}, & \{\bar{4},\bar{6}\} & \\ 3: & \{1,2\}, & \{\bar{7}\}, & \{\bar{5},\bar{6}\}, & \{\bar{4}\} \\ \hline 4: & \{1,\bar{5}\}, & \{\bar{7}\}, & \{\bar{6},2\}, & \{3\} \\ 5: & \{1,4\}, & \{2,\bar{7}\}, & \{3,\bar{6}\} & \\ 6: & \{1,\bar{7}\}, & \{2,4\}, & \{3,5\} & \\ 7: & \{1,6\}, & \{4\}, & \{2,5\}, & \{3\} \end{array}\right).$$

We determine the signs within the first set in hyperline 2 from hyperline 4. $\chi(3,4,5) = -1$ from hyperline 4 implies that the order of the half-period of hyperline 3 is kept. Finally, we obtain the signs within the first set of hyperline 3 from hyperline 2.

$$\left(\begin{array}{lllll} 1: & \{2,3\}, & \{4,5\}, & \{6,7\} & \\ 2: & \{1,\bar{3}\}, & \{\bar{5},\bar{7}\}, & \{\bar{4},\bar{6}\} & \\ 3: & \{1,2\}, & \{\bar{7}\}, & \{\bar{5},\bar{6}\}, & \{\bar{4}\} \\ 4: & \{1,\bar{5}\}, & \{\bar{7}\}, & \{\bar{6},2\}, & \{3\} \\ 5: & \{1,4\}, & \{2,\bar{7}\}, & \{3,\bar{6}\} & \\ 6: & \{1,\bar{7}\}, & \{2,4\}, & \{3,5\} & \\ 7: & \{1,6\}, & \{4\}, & \{2,5\}, & \{3\} \end{array}\right).$$

5. Chirotopes and Hyperline Sequences

Oriented matroids can be introduced via other axiomatic systems, such as chirotopes. In this section we show that the sets of hyperline sequences admiting an abstract sign of determinant function and the chirotopes define the same class of objects, thus establishing the equivalence (not formally proven elsewhere) between these two systems of axioms.

Definition 5.1. A chirotope of rank 3 with n elements is an alternating and anti-symmetric map $\chi : \overline{E}_n{}^3 \to \{-1, 0, +1\}$ such that for pairwise different elements i, j, k, l, m $M := \{\chi(i, j, k)\cdot\chi(i, l, m), -\chi(i, j, l)\cdot\chi(i, k, m), \chi(i, j, m)\cdot\chi(i, k, l)\} = \{0\}$ or $\{-1, +1\} \subset M$. The map $-\chi$ for which all signs are negated is identified with χ, and we require that for each element i, there is at least one pair (j, k) with $\chi(i, j, k) \neq 0$.

Theorem 5.2. *There is a one-to-one correspondence between rank* 3 *oriented matroids given by hyperline sequences and those defined by chirotopes.*

Proof. Let the rank 3 oriented matroid with n elements be given by hyperline sequences. We will show that the abstract sign of determinant function fulfills the chirotope condition: $\forall i, j, k, l, m$, pairwise different,

$$M := \{\chi(i, j, k)\cdot\chi(i, l, m), -\chi(i, j, l)\cdot\chi(i, k, m), \chi(i, j, m)\cdot\chi(i, k, l)\} = \{0\}$$
$$\text{or } \{-1, +1\} \subset M,$$

which is invariant under permuting the elements j, k, l, m and reorienting all its five elements i, j, k, l, m. When considering the ith hyperline sequence, we can assume that the elements j, k, l, m occur in that order. When the elements belong pairwise to different q_s, this implies $\chi(i, j, k) = \chi(i, l, m) = \chi(i, j, l) = \chi(i, k, m) = \chi(i, j, m) = \chi(i, k, l) = 1$. When all the elements belong to the same q_s, we have $\chi(i, j, k) = \chi(i, l, m) = \chi(i, j, l) = \chi(i, k, m) = \chi(i, j, m) = \chi(i, k, l) = 0$. We use $q(t)$ for the set containing element t, and $q(s) < q(t)$ says that $q(s)$ lies in the chosen half-period left of $q(t)$. The remaining cases are now $q(j) < q(k) < q(l) = q(m)$, $q(j) < q(k) = q(l) < q(m)$, $q(j) = q(k) < q(l) < q(m)$, $q(j) < q(k) = q(l) = q(m)$, $q(j) = q(k) = q(l) < q(m)$, and $q(j) = q(k) < q(l) = q(m)$. We easily see that the chirotope condition holds in all these cases.

To prove the other direction, let the chirotope be given. We have to show that we can construct an oriented matroid induced by a set of hyperline sequences. The abstract sign of determinant function in the sense of Section 3 will be the function χ of the chirotope. During the construction process, we confirm that all images of χ are compatible with the hyperline sequence structure. For the given element i we have at least one pair $(a, b) \in \overline{E}_n^{\,2}$ with $\chi(i, a, b) = 1$. We start to construct a half-period of the ith hyperline by sorting these two elements in the correct order beginning with $a \in \overline{E}_n$ followed by $b \in \overline{E}_n$. Now we use induction. Using for the next element $k \notin q(a)$ k or $\overline{k}$ depends on $\chi(i, k, a)$. Using for the next element $k \in q(a)$ k or $\overline{k}$ depends on $\chi(i, k, b)$. The first element k not belonging to $q(a)$ and $q(b)$ gets its unique position by $\chi(i, k, b)$. For additional insertions we have to show compatibility. Assume that $k - 1$ elements have been sorted already in the correct way, forming the ordered sets $q_1 < q_2 < \cdots < q_t$, $t \geq 3$. We consider all signs $\chi(i, k, x)$. We insert the kth element which was not used so far. We observe first that $\chi(i, k, x)$ is the same for all $x \in q_s$ for some s. We show this for $q_s \neq q(a)$. Otherwise $q(b)$ can be used instead. For $x_1, x_2 \in q_s$ we have $\chi(i, x_1, x_2) = 0$ and $\chi(i, a, x_1) = \chi(i, a, x_2) = 1$. The chirotope property gives us either $\chi(i, k, x_1) = \chi(i, k, x_2) = 0$, i.e., $k \in q_s$ or $\chi(i, k, x_1) = \chi(i, k, x_2) \neq 0$. The sorting of the elements $x \in \overline{E}_n$ in the ith half-period is a sorting of its classes $q(x)$. Assuming $q(a) < q(x) < q(y)$ and $q(a) < q(y) < q(z)$, $a, x, y, z \in \overline{E}_n$, we have $\chi(i, a, x) = 1$, $\chi(i, y, z) = 1$, $\chi(i, a, y) = 1$, $\chi(i, a, z) = 1$, $\chi(i, x, y) = 1$. The chirotope property implies $q(x) < q(z)$, i.e., the ordering is always compatible: for $k = x$ we find that there is a smallest upper bound or no upper bound, for $k = z$ we find that there is a largest lower bound and thus insert k in the sequence. For all i we can construct the corresponding half-period in accordance with the chirotope function which serves as the abstract sign of determinant function. □

6. Conclusion

In higher dimensions, the existing proofs of the Folkman–Lawrence representation theorem can also be greatly simplified using our inductive approach based on hyperline sequences. But even in the uniform case, the proof has to use topological results that would go beyond the elementary character of this paper, and the non-uniform case is even more involved. The hyperline sequences, as a model for oriented matroids, need a slightly lengthier description in the general non-uniform case. To keep the results in this note as elementary as possible, we defer the higher dimensional case to another article.

References

1. A. Altshuler, J. Bokowski and S. Steinberg, The classification of simplicial 3-spheres with nine vertices into polytopes and non-polytopes, *Discrete Math.*, **31** (1980), 115–124.
2. C. Antonin, Ein Algorithmusansatz für Realisierungsfragen im E^d getestet an kombinatorischen 3-Sphären, Staatsexamensarbeit, Bochum, 1982.
3. A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. Ziegler, *Oriented Matroids*, Cambridge University Press, Cambridge, U.K., 1993.

4. J. Bokowski, *Oriented Matroids*, in [13], 1993, pp. 555–602.
5. J. Bokowski and A. Guedes de Oliveira, On the generation of oriented matroids, *Discrete Comput. Geom.*, **24** (2000), 197–208.
6. R. Cordovil, Sur les matroïdes orientés de rang 3 et les arrangements de pseudodroites dans le plan projectif réel, *Europ. J. Combinatorics*, **3** 307–318.
7. J. Edmonds and A. Mandel, *Topology of Oriented Matroids*, Abstract 758-05-9, *Notices of the American Mathematical Society*, **25**, 1978, p. A-410; also Ph.D. Thesis of A. Mandel, University of Waterloo, 1982.
8. S. Felsner and H. Weil, Sweeps, arrangements and signotopes, preprint, 1998.
9. J. Folkman and J. Lawrence, Oriented matroids, *J. Comb. Theory, Ser. B*, **25** (1978), 199–236.
10. J. E. Goodman, *Pseudoline Arrangements*, in [11], 1997, pp. 83–110.
11. J. E. Goodman and J. O'Rourke (eds), *Handbook of Combinatorial and Computational Geometry*, CRC Press, Boca Raton, FL, 1997.
12. J. E. Goodman and R. Pollack, Semispaces of configurations, cell complexes of arrangements, *J. Comb. Theory, Ser. A*, **37** (1984), 257–293.
13. P. M. Gruber and J. M. Wills (eds), *Handbook of Convex Geometry*, Vol. A and B, North-Holland, Amsterdam, 1993.
14. B. Grünbaum, *Arrangements and Spreads, Regional Conf. Ser. Math., American Mathematical Society*, **10**, Providence, RI, 1972.
15. W. Hochstättler, Seitenflächenverbande orientierter matroide, Ph.D. Thesis, Köln, 1992.
16. D. E. Knuth, *Axioms and Hulls*, Springer Verlag, New York, 1992.
17. M. Las Vergnas, Matroïdes orientables, *C.R. Acad. Sci., Paris, Ser. A*, **280** (1975), 61–64.
18. M. Las Vergnas, Bases in oriented matroids, *J. Comb. Theory, Ser. B*, **25** (1978), 283–289.
19. J. Lawrence, Oriented matroids and multiply ordered sets, *Linear Algebr. Appl.*, **48** (1982), 1–12.
20. F. Levi, Die Teilung der projektiven Ebene durch Gerade oder Pseudogerade, *Ber. Verh. Sächs. Ges. Wiss. Leipzig, Math. Phys. Kl.*, **78** (1926), 256–267.
21. J. O'Rourke and I. Streinu, Pseudo-visibility graphs: characterization and recognition, *Proc. Symp. Comp. Geom., Nice*, 1997, pp. 119–128.
22. J. Richter-Gebert and G. Ziegler, *Oriented Matroids*, in [12], 1997, pp. 111–132.
23. I. Streinu, Clusters of stars, *Proc. 13th ACM Symp. Comput. Geometry, Nice*, 1997, pp. 439–441.

Received 31 January 2000 and accepted 1 December 2000

JÜRGEN BOKOWSKI AND SUSANNE MOCK
Department of Mathematics,
Darmstadt University of Technology,
Schlossgartenstr. 7,
D-64289 Darmstadt,
Germany
E-mails: bokowski@mathematik.tu-darmstadt.de; mock@mathematik.tu-darmstadt.de

AND

ILEANA STREINU
Department of Computer Science,
Smith College,
Northampton,
MA 01063, U.S.A.
E-mail: streinu@cs.smith.edu