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Non-Stretchable Pseudo-Visibility Graphs

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Abstract

We exhibit a family of graphs which can be realized as pseudo-visibility graphs of pseudo-polygons, but not of straight-line polygons. The construction is based on the characterization of vertex-edge pseudo-visibility graphs of O'Rourke and Streinu [ORS96] and extends recent results on non-stretchable vertex-edge visibility graphs of Streinu [Str99]. We show that there is a pseudo-visibility graphs for which there exists only one of vertex-edge visibility graph compatible with it, which is then shown to be non-stretchable. The construction is then extended to an infinite family.

1 Introduction

Characterizing visibility graphs is a problem with a distinguished history (Ghosh [Gho88], Everett [Ev90], Abello and Kumar [AK95]), but so far several attempts to give good sets of conditions have been proved insufficient.

A different approach, introduced by O'Rourke and Streinu [ORS96] is to separate the combinatorial aspects of the problem from the question of stretchability (known to be notoriously hard for pseudo-line arrangements, cf. Mnëv [Mn91] and Shor [Sh91]). They have introduced two new concepts: vertex-edge visibility graphs [ORS98] and pseudo-visibility [ORS96], and gave a complete combinatorial characterization of vertex-edge pseudo-visibility graphs. However, it is not clear a priori that the new class is any larger than just the class of straight-line vertex-edge visibility graphs, since it is conceivable that all such graphs can be realized with straight line edges (just as the class of planar graphs is realizable in this manner). In [Str99] a whole class of non-stretchable vertex-edge pseudo-visibility graphs is exhibited, thus settling this question. Moreover, it is shown that the stretchability question can in fact be decided efficiently for a class of vertex-edge pseudo-visibility graphs which includes these examples.

The original question was for visibility graphs, not vertex-edge visibility graphs. Since vertex-edge pseudo-visibility graphs contain more information than pseudo-visibility graphs, it is possible to have several vertex-edge pseudo-visibility graphs compatible with a given pseudo-visibility graph: some may be stretchable, some not. We are interested in the question: are there any pseudo-visibility graphs for which none of the compatible vertex-edge pseudo-visibility graph is stretchable? The visibility graphs of the non-stretchable pseudo-polygons from [Str99] are in fact compatible with straight line polygons, so the same family of examples does not work directly.

In this paper we exhibit a slightly more involved example of a pseudo-polygon with the property that its pseudo-visibility graph uniquely induces a vertex-edge visibility graph, which is then shown to be non-stretchable. This provides a strong separation between straight-line and pseudo visibility graphs. The example is then extended to an infinite family.

2 Preliminaries

Abbreviations: We may abbreviate the prefix pseudo by p- (as in p-line for pseudo-line), vertex-edge pseudo-visibility graph by ve-graph, pseudo-visibility graph by v-graph and generalized configuration of points by gcp. We use ccw for counter-clockwise.

An arrangement of pseudolines $\mathcal{L}$ is a collection of simple curves, each of which separates the plane, such that each pair of p-lines of $\mathcal{L}$ meet in exactly one point, where they cross.

Definition 2.1 Let $V = \{v_0, v_1, \ldots, v_{n-1}\}$ be a set of points in the Euclidean plane $\mathbb{R}^2$, and let $\mathcal{L}$ be an arrangement of $\binom{n}{2}$ pseudolines such that every pair of points $v_i$ and $v_j$ be on exactly one pseudoline $l_{ij} \in \mathcal{L}$.
and each pseudoline in \( L \) contains exactly two points of \( V \). Then the pair \((V, L)\) is a generalized configuration of points in general position.

Two points \( a \) and \( b \) on a pseudoline \( l \in L \) determine a unique (closed) segment \( ab \) consisting of those points on \( l \) that lie between the two points. For \( 0 \leq i \leq n-1 \), let \( e_i = v_i v_{i+1} \) be the segment determined by \( v_i \) and \( v_{i+1} \) on \( l_{i,i+1} \).

**Definition 2.2** The segments \( e_i = v_i v_{i+1} \) form a pseudo-polygon iff:

1. The intersection of each pair of segments adjacent in the cyclic ordering is the single point shared between them: \( e_i \cap e_{i+1} = v_{i+1} \), for all \( i = 0, 1, \ldots, n-1 \).
2. Nonadjacent segments do not intersect: \( e_i \cap e_j = \emptyset \), for all \( j \neq i+1 \).

A p-polygon is a simple closed Jordan curve and separates the plane into two regions. We assume without loss of generality that the vertices of the p-polygon are numbered in ccw order, i.e., that the interior of the polygon lies to the left as the boundary is traversed in this order.

Pseudo-visibility is determined by the underlying arrangement \( L \): lines-of-sight are along pseudolines in \( L \).

**Definition 2.3** Vertex \( v_i \) sees vertex \( v_j \) \((v_i \leftrightarrow v_j)\) iff either \( v_i = v_j \), or they lie on a line \( l_{ij} \in L \) and the segment \( v_iv_j \) is nowhere exterior to \( P \).

We will often abbreviate \( G_V(P) \) to \( G_V \). Note that \( G_V \) is Hamiltonian: the arcs corresponding to the polygon boundary form a Hamiltonian circuit \((v_0, \ldots, v_{n-1})\).

And also note that since \( G_V \) is labeled by \( V \), which we assumed was labeled in a ccw boundary traversal order, the Hamiltonian circuit is provided by the labeling of the graph.

To define vertex-edge pseudo-visibility we need to define when a vertex sees an edge. This is based on the notion of a “witness” for a visible pair. Let \( r_{ij} \subseteq l_{ij} \) be the ray directed from \( v_i \) not including \( v_i \) closed at \( v_j \).

**Definition 2.5** Vertex \( v_j \) is a witness for the vertex-edge pair \((v_i, e)\) (and we say that \( v_i \) sees edge \( e \)) iff either

1. \( v_i \) is an endpoint of \( e \), and \( v_j \) is also (here we permit \( v_j = v_i \)); or
2. \( v_j \) is not an endpoint of \( e \), and
   
   a. \( v_i \) sees \( v_j \); and
   b. the ray \( r_{ij} \) intersects \( e \) at a point \( p \),
   c. either \( v_j = p \), or the segment \( v_j p \) is nowhere exterior.

We will refer to the line \( l_{ij} \) in the above definition as the witness line.

**Definition 2.6** The vertex-edge pseudo-visibility graph \( G_{VE} \) of a polygon is a labeled bipartite graph with node node set \( V \cup E \), and an arc between \( v \in V \) and \( e \in E \) if \( v \) can see \( e \) (according to Def. 2.5).

**Notation:** Let \( P(i, j) \) be the open boundary interval containing all vertices and edges of \( P \) encountered in a ccw traversal of the boundary of \( P \) from \( v_i \) to \( v_j \). Similarly we define \( P[i, j], P[i, j] \) and \( P[i, j] \) to include one or both endpoints of the interval.

The following lemma has been proved in [ORS96]:

**Lemma 2.7** If \( v_k \) sees non-adjacent edges \( e_i \) and \( e_j \) and no edge between, \( v_k \in P[j+1, i] \), then exactly one of Case A or B holds:

![Figure 1: A non-stretchable generalized configuration of points.](image1)

![Figure 2: A non-stretchable pseudo-polygon.](image2)
A 1. \( v_k \) sees \( v_{k+1} \) but not \( v_j \).
2. \( v_{k+1} \) is the right-witness for \( (v_k, e_j) \).
3. \( v_{k+1} \) sees \( e_j \) but \( v_j \) does not see \( e_j \).

B 1. \( v_k \) sees \( v_j \) but not \( v_{k+1} \).
2. \( v_j \) is the left-witness for \( (v_k, e_i) \).
3. \( v_j \) sees \( e_i \) but \( v_{k+1} \) does not see \( e_j \).

One more concept needed is that of a “pocket”:

**Definition 2.8** If \( v_i \) sees \( e_j \) and \( v_j \), and \( v_i \) are the right and left witnesses respectively, then \( P[j, r] \) and \( P[l, i] \) are the right and left near pockets, and \( P(r, j) \) and \( P[j + 1, l] \) are the right and left far pockets of \( v_i \rightarrow e_j \) respectively.

The following lemma has been proved in [ORS96].

**Lemma 2.9** If \( v_i \) sees \( e_j \) and \( v_j \), and \( v_i \) are the right and left witnesses respectively, then
1. No vertex in the right near pocket sees an edge in the right far pocket.
2. No vertex in the right far pocket sees an edge in the right near pocket.

Symmetric claims hold for the left pockets.

**Lemma 2.10** If \( v_i \) sees \( e_j \) and \( v_j \), and \( v_i \) are the right and left witnesses respectively, then \( v_k \) is an articulation point of the subgraph of \( GV_E \), induced by \( P[l, j] \), and symmetrically \( v_l \) is an articulation point of the subgraph induced by \( P[j + 1, l] \).

**Theorem 2.11** If \( GV_E \) is the vertex-edge visibility graph of a pseudo-polygon \( P \), then it satisfies these two properties:
1. If \( v_k \) sees non-adjacent edges \( e_j \) and \( e_j \) and no edge between, \( v_k \in P[j + 1, i] \), then exactly one of these holds:
   A. \((v_{k+1}, e_j) \in GV_E \), or
   B. \((v_j, e_i) \in GV_E \).
2. In the two cases above, additionally:
   A. \( v_{k+1} \) is an articulation point of the subgraph of \( GV_E \) induced by \( P[k, j] \).
   B. \( v_j \) is an articulation point of the subgraph of \( GV_E \) induced by \( P[j + 1, k] \).

It has been shown in [ORS96] that these properties provide a complete characterization of vertex-edge pseudo-visibility graphs.

We now turn to stretchability questions. The basis of our construction comes from a classical example of a non-realizable allowable sequence, the so-called non-realizable pentagon (see [GP93]). The five central points in Figure 1 are pairwise connected by pseudolines. We force these pseudolines to cross such that we can then place the other exterior five points. We then draw extra pseudolines connecting all these pairs of points. This can be achieved in several ways, but each is a non-stretchable configuration of points.

To get an example of a non-stretchable pseudo-polygon, as in [Str90], we place a pseudo-polygon on top of this generalized configuration of points, as in Figure 2. No matter how the other pseudolines in the configuration meet, the basic internal structure remains the same: they all have the same vertex-edge visibility graph. Also, any pseudo-polygon with this ve-graph has to contain the non-realizable pentagon as an underlying rearrangement of its configuration of points.

### 3 An unstretchable vertex-vertex pseudo-visibility graph

Our goal is to extend this example to a non-realizable pseudo ve-graph. We would like to get a ve-graph with only one compatible ve-graph - the realizable one. But unfortunately the ve-graph underlying the unstretchable ve-graph (Figure ??) has both realizable and unrealizable compatible ve-graphs.

So we'll have to work harder to get our example. Notice that in general a ve-graph can have many compatible ve-graphs (exponentially many, as shown in the full paper).

![Figure 3: The gadget used in constructing the non-realizable ve-graph.](image-url)
The argument is based on the properties of \( \text{ve/-graphs} \) listed in the Preliminaries. Let's denote the relevant vertices of the gadget as in Figure 4: \( v_1, v_2, b_2, w_2, t, w_1 \) and \( b_1 \).

![Figure 4: The labeled gadget.](image)

Then since \( v_1 \) sees \( w_2 \), \( v_1 \) sees \( b_1 \) and no other vertex between these two (in the ccw order induced by the boundary of the polygon), it follows that either \( b_1 \) or \( w_2 \) is an articulation point for \( t \), when visibility from \( v_1 \) is considered. But \( w_2 \) can't be, since \( b_2 \) sees \( t \) (otherwise we would get a contradiction of the articulation point property of ve-graphs for \( b_2 \) in the near pocket, \( t \) in the far pocket, with articulation point \( w_2 \) and visibility from \( v_1 \)).

So it follows that \( b_1 \) has to be articulation point for \( t \) with visibility from \( v_1 \). But then it follows that the pseudo-line through \( v_1b_1 \) extends to intersect the edge \( tw_2 \) (as in Figure 4).

A similar argument holds for the pseudo-line through \( v_2b_2 \). When we repeat this for all the five gadgets, we realize that we forced the unrealizable pentagon from Figure 1 as part of the underlying configuration of points for the ve-graph!

This completes the un-stretchable v-graph example. In the full paper we also show that this example can be extended to an infinite family.

4 Conclusion

We have shown that the class of pseudo v-graphs is strictly larger than the class of straight-line v-graphs. This result, together with the main characterization from [ORS96], yields a number of open problems for further research, which are described in the full paper.

References


