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# Singularity Locus for the Endpoint Map of Serial Manipulators with Revolute Joints

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**Abstract**
We present a theoretical and algorithmic method for describing the singularity locus for the endpoint map of any serial manipulator with revolute joints. As a surface of revolution around the first joint, the singularity locus is determined by its intersection with a fixed plane through the first joint. The resulting plane curve is part of an algebraic curve called the singularity curve. Its degree can be computed from the specialized case of all pairs of consecutive joints coplanar, when the singularity curve is a union of circles, counted with multiplicity two. Knowledge of the degree and a simple iterative procedure for obtaining sample points on the singularity curve lead to the precise equation of the curve.

**Key words:** serial manipulator, revolute joints , endpoint map singularity.

## 1 Introduction

We consider a serial manipulator with an arbitrary number $n \geq 2$ of revolute joints. The end-effector or hand is abstracted to a single point $T$ on the last link. The $n$ joints, also called joint axes or hinges, are envisaged as full lines and labeled in order $A_1, ..., A_n$. For theoretical purposes, we assume full rotational capability around each joint and allow all geometrical configurations, without regard for possible self-collisions. Thus, the configuration space is parametrized by the $n$-dimensional torus $(S^1)^n$. The *endpoint map* $e : (S^1)^n \to R^3$ takes a configuration $\theta = (\theta_1, ..., \theta_n) \in (S^1)^n$ to the corresponding position of the endpoint $T(\theta) \in R^3$. When the differential $De(\theta)$ has rank strictly less than three, we have a *singular configuration*. The locus



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of $T$ for all singular configurations is called the *singularity locus* for the endpoint map.

It is fairly well known that the singularity locus is a *surface of revolution*, with the first joint as symmetry axis. When sectioned with a plane passing through this axis, the singularity locus yields a *plane curve* which is part of an *algebraic curve*, called here the *singularity curve*. In this paper we describe a complete and rigorous procedure for obtaining the *equation of the singularity curve*.

The singularity locus is of fundamental importance not only for path planning and avoidance of singular configurations, but also for positional *workspace determination*. The *workspace boundary* is necessarily included in the singularity locus [4, 20].

Previous attempts for describing either the singularity locus or the workspace boundary have usually addressed cases with a very small number of joints often relying on numerical procedures of uncertain accuracy [1, 2, 3, 14, 15, 16, 18, 19, 21]. The general recursion proposed in [11] seems difficult to work out for larger $n$ and has been explicitly used only in a few instances [12, 17].

The main elements of novelty of the solution presented here reside in the method itself, centered on obtaining an *explicit degree formula* for the singularity curve, the recognition of the algebraic and geometric advantage of using for this purpose the specialized case of manipulators with any two consecutive joints coplanar and the possibility of producing the necessary amount of sample points in the general case, based on the geometric characterization of singular configurations.

More precisely, our determination of the singularity curve is based on the following principles: (i) the *degree* of the curve does not change when the manipulator is continuously altered until any two consecutive joints become coplanar, (ii) for a manipulator with any two consecutive joints coplanar, the singularity curve is made of circles, counted with multiplicity two, (iii) recursion on $n$ yields the general *degree formula*, (iv) sample points on the singularity curve can be produced in arbitrary numbers, (v) with known degree and sufficiently many sample points computed, the equation of the curve is obtained from solving a linear system.

This summary leaves aside some technical details. It will be seen in due course that, for full mathematical rigor, one has to work over the algebraically closed field of *complex numbers*. The 'continuity principle' used in (i), while intuitively persuasive, is actually justified through a more elaborate argument [13]. However, when assuming a certain background in algebraic geometry, these aspects take lesser roles. Thus, the key elements of our approach rely on (a) the fact that manipulators with coplanar pairs of consecutive joints allow the computation of the degree formula, in combination with (b) the possibility of producing sample points based on a simple geometrical characterization of singular configurations.

Part (a) follows from our work on extremal reaches and workspace determination for manipulators with coplanar pairs of consecutive joints, also called *panel-and-hinge chains* : [6, 7]. This special class of manipulators is adequate for computing the degree $\delta_n$ of the singularity curve since the latter decomposes into irreducible components of degree two and multiplicity two. The degree formula is obtained in Section 3 from a linear recurrence relation which gives:

$$\delta_n = \frac{1}{\sqrt{3}}[(1+\sqrt{3})^{n+1} - (1-\sqrt{3})^{n+1}] - 2^{n+1}$$

with $\delta_2 = 4,\ \delta_3 = 16,\ \delta_4 = 56,\ \delta_5 = 176,\ \delta_6 = 528\ \ldots.$

The *exponential complexity* of the workspace boundary described in [8] is aptly reflected in this degree formula.

The geometric characterization of singular configurations used in (b) is the following **singularity criterion**: *the nR manipulator is in a singular configuration $\theta$ i.e. rank$(De(\theta)) \leq 2$ when there is a line through $T$ projectively incident with all joint axes* [5, 9, 10]. A line of this type will be called a $T$-transversal. This criterion and its sibling for the end-to-end (squared) distance from a marked start point $S$ on the first link are already implicated in (a), leading to the important notions of *fold point* and *pivoting* [7, 8]. For (b) the criterion serves in the following way.

We choose an arbitrary line through the terminus point $T$ and the last hinge $A_n$ and then find the two solutions given by the intersection of this line with the hyperboloid generated by rotating $A_{n-1}$ around $A_n$. Then, with any one of these two solutions in place, we look for the two solutions given by intersecting the line with the hyperboloid generated by rotating $A_{n-2}$ around $A_{n-1}$ and so on. This procedure produces $2^{n-1}$ singular configurations i.e. $2^{n-1}$ points of the singularity curve. With sufficiently many sample points determined in this manner all coefficients of the curve can be determined (up to proportionality) by solving the resulting homogeneous linear system. Actually, $\binom{\delta_n+2}{2} - 1$ points imposing independent conditions suffice.

# 2 Fold points and pivoting

In this section we review the argument showing that a manipulator with all pairs of consecutive joints coplanar has a singularity curve made of irreducible components of degree two, which have to be counted with multiplicity two. The key notions are those of *fold point* and *pivoting* at a first fold point introduced in [6, 7, 8].

Let $p_{i,i+1} = A_i \cap A_{i+1}$ denote the intersection of a pair of consecutive joint axes. The plane containing this pair of joint axes is called a panel. Our *nR* manipulator can thereby be conceived as a *panel-and-hinge chain* since one panel is joint to the next by their common joint axis or hinge. The fist panel is taken as a fixed plane through $A_1$ and the last panel is the plane given by $A_n$ and $T$. As recalled in the introduction, a singular configuration must allow a $T$-transversal for all hinges. As long as the $T$-transversal avoids intersection points $p_{i,i+1}$, consecutive panels must remain in one and the same plane. Thus, singular configurations are either *flat*, with all panels in the same plane, or *non-flat*, with at least one point $p_{f,f+1}$ on the $T$-transversal such that the three consecutive panels incident to $p_{f,f+1}$ are not coplanar. Such a point $p_{f,f+1}$ is called a **fold point** of the singular configuration.

For the notion of *pivoting* it is useful to review first the case of a 2R manipulator with incident joints, illustrated in Figure 1. Note that, by definition, all configurations are singular, since $n = 2$ and the rank of $De(\theta)$ cannot be 3. Thus, as a set, the singularity locus is given by all possible positions of $T$. These positions cover a ring-shaped region of a sphere centered at $p_{1,2} = A_1 \cap A_2$. Algebraically, one should remain alert to several aspects: the actual singularity locus is only part of an algebraic surface, but this 'inconvenience' is removed when reformulating the problem over the field of *complex numbers*. Then, 'complex configurations' would map to the complex quadric whose real points are seen as the sphere, and in fact cover it twice. This double covering can be seen on the real scenario over the ring-shaped region and becomes intuitive also when imagining the manipulator with incident joint axes as a limit of manipulators with two skew joint axes. In the latter case, the locus of $T$ is a torus and when the joint axes intersect, the torus degenerates to a 'doubled' spherical region.

In short, all matters algebraic become simpler over the *complex field* $C$ and this must be the adopted setting in general for properly speaking about irreducible components and degree for the singularity curve.

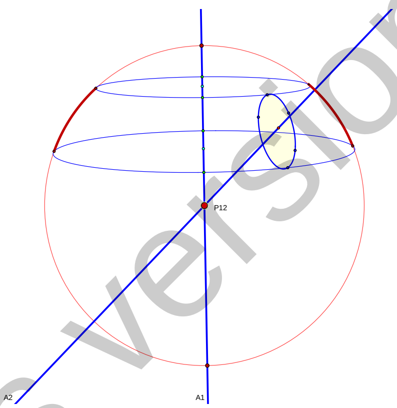


**Fig. 1** The locus of $T$ for a manipulator with two intersecting hinges is a ring-shaped spherical region (covered twice). In the reference plane through $A_1$ given by the red grand circle it traces two arcs along this circle.

Returning now to arbitrary $n$ and a given configuration, we define **pivoting** at $p_{k,k+1}$ to mean 'locking' all joints except $A_k$ and $A_{k+1}$ and using only these two degrees of freedom. Thus, the $nR$ manipulator becomes a $2R$ manipulator. When we have a singular configuration with first fold point at $p_{f,f+1}$, we take as reference plane the common plane of the first $f$ panels and pivoting at $p_{f,f+1}$, together with the singularity criterion, show that all configurations with $T$ in this reference plane are singular configurations for the $nR$ manipulator. When we start with a *flat configuration*, pivoting at any $p_{k,k+1}$ immediately provides a degree two irreducible component of the complex singularity curve passing through the corresponding $T$.

Thus, over $C$, any point of the singularity curve belongs to some degree two irreducible component. In other words: *the complex singularity curve of a manipulator with all pairs of consecutive joints coplanar decomposes into irreducible components of degree two (which must be counted with multiplicity two).*

Before engaging the degree computation, we recall the analogous case of the end-to-end squared distance function for a panel-and-hinge chain [6]. The first panel is fixed and has a marked point $S$ (start). The end-to-end function $f : (S^1)^n \to R$ gives the squared distance $f(\theta)$ between $S$ and the endpoint $T(\theta) = e(\theta)$. When the differential $df(\theta) = 0$, we have a *critical configuration*. The *geometrical criterion for critical configurations* says that the end-to-end line $ST$ must intersect (projectively) all hinges. Thus, all critical configurations of an end-to-end function are singular configurations for the endpoint map.

# 3 Counting circles and the degree formula

We consider **critical configurations** for a panel and hinge chain with $k$ hinges and marked $S$ (start) and $T$ (terminus) points. The first panel is fixed. We denote by $c_k$ the number of (real or complex) configurations with line $ST$ intersecting all hinges (i.e. critical configurations).

There are $2^k$ flat configurations.

For non-flat configurations **we look at the first fold point** $p_{f,f+1}$.

From this point to $T$ we have a chain with $k-(f+1)$ hinges (and $k-f$ panels). There are $c_{k-(f+1)}$ critical configurations (with line $p_{f,f+1}T$ intersecting all $k-(f+1)$ hinges.

The first $f$ panels have $2^{f-1}$ flat configurations.

By *pivoting* at $p_{f,f+1}$ there are **two** (real or complex) alignments of $Sp_{f,f+1}$ with $p_{f,f+1}T$, hence:

$$c_k = 2^k + 2\sum_{f=1}^{k-1} 2^{f-1} c_{k-(f+1)} \tag{1}$$

with $c_0 = 1$. If we put $c_{-1} = 1$, we have:

$$c_k = \sum_{f=1}^{k} 2^f c_{k-(f+1)}, \;\; c_{-1} = c_0 = 1 \tag{2}$$

This gives the linear recurrence relation:

$$c_{k+1} = 2(c_k + c_{k-1}), \;\; c_0 = c_1 = 1, \; k \geq 1 \tag{3}$$

For the **number of (complex) circles** $v_n$ traced in the plane of the first panel by the (complex) singularity locus of the endpoint map, we look again at the first fold point $p_{k,k+1}$. (Note that the circle is traced by pivoting at this point.) We have:

$$v_n = \sum_{k=1}^{n-1} 2^{k-1} c_{n-(k+1)}, \quad n \geq 2 \tag{4}$$

It follows that:

$$v_{n+1} = 2v_n + c_{n-1}, \quad n \geq 2 \tag{5}$$

which yields by (3) the linear recurrence relation:

$$v_{n+3} = 2(2v_{n+2} - v_{n+1} - 2v_n), \;\; n \geq 2 \tag{6}$$

with $v_2 = 1, \; v_3 = 4, \; v_4 = 14$.

**Remark:** When passing from counting circles to degrees, the circles have degree two and have to be counted twice since each 'sphere' is covered doubly. Two skew hinges give a torus (for the real picture) i.e. a degree four surface.

From (2) and (4) we obtain:

$$c_n = 2v_n + 2^n, \quad n \geq 2 \tag{7}$$

hence

$$v_n = 2v_{n-1} + c_{n-2} = c_{n-1} + c_{n-2} - 2^{n-1} = \frac{1}{2}c_n - 2^{n-1} \tag{8}$$

The **degree** $\delta_n$ of the curve made of $v_n$ double circles is $4v_n$ and we have therefore:

$$\delta_n = 2c_n - 2^{n+1} = \frac{1}{\sqrt{3}}[(1+\sqrt{3})^{n+1} - (1-\sqrt{3})^{n+1}] - 2^{n+1} \tag{9}$$

where the last part of the formula follows from solving the linear recurrence (3). The first few terms of the degree sequence are:

$$\delta_2 = 4, \;\; \delta_3 = 16, \;\; \delta_4 = 56, \;\; \delta_5 = 176, \;\; \delta_6 = 528 \;\; \ldots$$

## 4 Generating sample points

With the degree $\delta_n$ determined from $nR$ manipulators with all pairs of consecutive joints coplanar, we return to the general case when consecutive joints would be skew lines in space. In order to determine the equation of the singularity curve, we need $\binom{\delta_n+2}{2} - 1$ points of the curve which impose independent conditions. As already described in the introduction, we may produce any number of sample points on the curve since we may execute the procedure with arbitrary positions of the $T$-transversal in the last link. In fact, the needed number of independent sample points is roughly halved by virtue of the reflection symmetry of the singularity curve in the $A_1$ point axis.

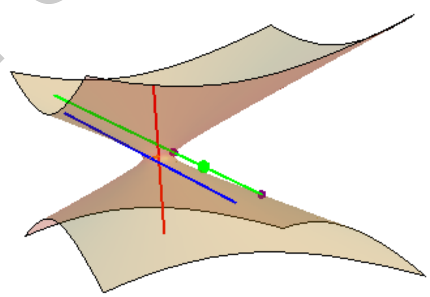

**Fig. 2** The main iterative step for generating sample points. The green line is the intended $T$-transversal, with $T$ marked larger in green.

We illustrate in Figure 2 the main step of the iterative procedure which starts with a chosen line through $T$ and some point on $A_n$, our ‘designated’ $T$-transversal.

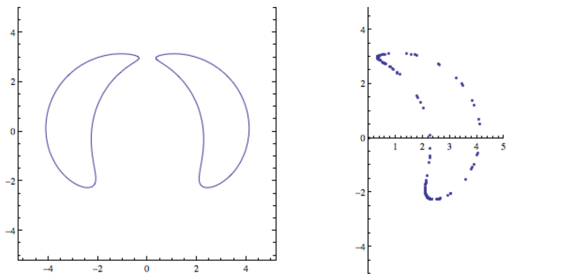

**Fig. 3** The singularity curve for a 2R manipulator, obtained after computing all coefficients of the degree four equation from sample points illustrated nearby.

After successively placing $A_{n-1}$, ..., $A_{n-k}$ in contact with this line, we have the situation depicted in the figure, with the designated $T$-transversal in green, the joint axis $A_{n-k}$ in red and $A_{n-k-1}$ in blue. When all the remaining part of the manipulator is rotated around $A_{n-k}$, the blue line sweeps the shown hyperboloid and the two specific rotations which position $A_{n-k-1}$ in contact with the designated $T$-transversal are determined from simple quadratic conditions. Thus, using one or the other rotation we have one more joint on the designated $T$-transversal. At the final step, $A_1$ is positioned in contact with the green line, making it a genuine $T$-transversal.

Figure 3 shows sample points for a 2R manipulator next to a full plot of the (real points of the) singularity curve. The degree is $\delta_2 = 4$ in this case and one needs $\binom{6}{2} - 1 = 14$ independent points for the determination (up to a proportionality factor) of the 15 coefficients implicated in the general equation of a plane curve of degree four.

# 5 Conclusions

We addressed and solved the fundamental problem of obtaining the equation of the singularity curve of a serial manipulator with an arbitrary number $n$ of revolute joints. The key elements of our solution are the *degree formula* (9) , derived from the specialized case of manipulators with any two consecutive joints coplanar and the general possibility of obtaining *sample points* on the curve by an iterative procedure. The full singularity surface for the endpoint map of the manipulator is generated by rotating the singularity curve around the fixed first joint axis. Computational designs for effective implementations of this solution will be detailed in separate publications.

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