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Counting edge-Kempe-equivalence classes for 3-edge-colored cubic graphs

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Abstract

Two edge colorings of a graph are edge-Kempe equivalent if one can be obtained from the other by a series of edge-Kempe switches. This work gives some results for the number of edge-Kempe equivalence classes for cubic graphs. In particular we show every 2-connected planar bipartite cubic graph has exactly one edge-Kempe equivalence class. Additionally, we exhibit infinite families of nonplanar bipartite cubic graphs with a range of numbers of edge-Kempe equivalence classes. Techniques are developed that will be useful for analyzing other classes of graphs as well.

1 Introduction and Summary

Back in the frosts of time, Alfred Bray Kempe introduced the notion of changing colorings by switching maximal two-color chains of vertices (for vertex colorings) or edges (for edge colorings). The maximal two-color chains are now called Kempe chains and edge-Kempe chains respectively; switching the colors along such a chain is called a Kempe switch or edge-Kempe switch as appropriate. This process is of interest across the study of colorings. It is also of interest in statistical mechanics, where certain
dynamics in the antiferromagnetic $q$-state Potts model correspond to Kempe switches on vertex colorings \[8\], \[9\]. In some cases, these dynamics also correspond to edge-Kempe switches \[7\].

In the present work we are concerned with understanding when two edge-colorings are equivalent under a sequence of edge-Kempe switches and when not. We allow multiple edges on our (labeled) graphs; loops are prohibited (and will mostly be excluded by other constraints such as 3-edge colorability).

A single edge-Kempe switch is denoted by \(-\). That is, if coloring $c_i$ becomes coloring $c_j$ after a single edge-Kempe switch, then $c_i \sim c_j$. If coloring $c_j$ can be converted to coloring $c_k$ by a sequence of edge-Kempe switches, then $c_j \sim c_k$. Because $\sim$ is an equivalence relation, we may consider the equivalence classes on the set of colorings of a graph $G$ edge-colored with $n$ colors. In this paper we focus on the number of edge-Kempe equivalence classes and denote this quantity by $K'(G, n)$. (In other work this has been denoted $\text{Ke}(L(G), n) \[6\]$ and $\kappa_E(G, n) \[5\].)

Note that any global permutation of colors can be achieved by edge-Kempe switches because the symmetric group $S_n$ is generated by transpositions. Thus two colorings that differ only by a permutation of colors are edge-Kempe equivalent.

Recall that $\Delta(G)$ is the largest vertex degree in $G$ and that $\chi'(G)$ is the smallest number of colors needed to properly edge-color the graph, then there is but a single edge-Kempe equivalence class, i.e., when $n > \chi'(G) + 1$ then $K'(G, n) = 1 \[6\]$ Thm. 3.1]. More is known if $\Delta(G)$ is restricted; when $\Delta(G) \leq 4, K'(G, \Delta(G) + 2) = 1 \[5\]$ Thm. 2] and when $\Delta(G) \leq 3, K'(G, \Delta(G) + 1) = 1 \[3\]$ Thm. 3]. For bipartite graphs there is a stronger result: when $n > \Delta(G)$, $K'(G, n) = 1 \[6\]$ Thm. 3.3]. Little is known about $K'(G, \Delta(G))$.

This paper focuses on cubic graphs, particularly those that are 3-edge colorable. Mohar suggested classifying cubic bipartite graphs with $K'(G, 3) = 1 \[6\]; we provide a partial answer here. Mohar also points out in \[6\] that it follows from a result of Fisk in \[1\] that every planar 3-connected cubic bipartite graph $G$ has $K'(G, 3) = 1$. We show (in Section \[4\]) that for $G$ planar, bipartite, and cubic, $G$ has $K'(G, 3) = 1$.

The remainder of the paper proceeds as follows. Section \[2\] introduces decompositions of cubic graphs along 2- or 3-edge cuts that preserve planarity and bipartiteness. The theorems in Section \[3\] use the edge-cut decompositions
to combine and decompose 3-edge colorings. We also show that any edge-Kempe equivalence can avoid color changes at a particular vertex. Then, in Section 4 we compute $K'(G, 3)$ in terms of the edge-cut decomposition of $G$, and exhibit infinite families of simple nonplanar bipartite cubic graphs with a range of numbers of edge-Kempe equivalence classes.

## 2 Decompositions of Cubic Graphs

Any 3-edge cut of a cubic graph may be used to decompose a cubic graph $G$ into two cubic graphs $G_1, G_2$ as follows. For 3-edge cut $E_C = \{(s_{11}s_{21}), (s_{12}s_{22}), (s_{13}s_{23})\}$ where vertices $s_{1j}$ are on one side of the cut and $s_{2j}$ on the other, let the induced subgraphs of $G \setminus E_C$ separated by $E_C$ be $G'_1, G'_2$. Then for $i = 1, 2$ define $G_i$ by $V(G_i) = V(G'_i) \cup v_i$ and $E(G_i) = E(G'_i) \cup E_{C_i}$ where $E_{C_i} = \{(v_is_{ij}) \mid j = 1, 2, 3\}$, as is shown in Figure 1. This decomposition will be written as $G = G_1 \uplus G_2$.

A similar decomposition is defined analogously for a 2-edge cut of a cubic graph. Here $G$ has 2-edge cut $E_C = \{(s_{11}s_{21}), (s_{12}s_{22})\}$ and for $i = 1, 2$ we define $G_i$ by $V(G_i) = V(G'_i)$ and $E(G_i) = E(G'_i) \cup e_i$ where $e_i = \{(s_{1i}s_{i2})\}$. This decomposition will be written as $G = G_1 \uplus G_2$.

For both of these decompositions, we say the edge cut is nontrivial if both $G_1$ and $G_2$ have fewer vertices than $G$. Using nontrivial edge cuts, we may decompose a cubic graph $G$ into a set of smaller graphs $\{G_i\}$ where each $G_i$ has no nontrivial edge cuts (but may have additional multiple edges).

Notice that these decompositions are reversible, though not uniquely so. Consider two cubic graphs $G_1, G_2$. Form $G_1 \uplus G_2$ by distinguishing a vertex on each $(v_1, v_2$ respectively) and identifying the edges incident to $v_1$ with the edges incident to $v_2$. \textit{A priori}, there are many ways to choose $v_1, v_2$ and many
ways to identify their incident edges. We will abuse the notation $G_1 \circ G_2$ by using it to denote a particular one of these many choices. Similarly, $G_1 \diamond G_2$ can be formed by choosing an edge $e_i = (s_{i1}s_{i2})$ from each $G_i$, deleting $e_i$, and then adding the edges $\{(s_{11}s_{21}), (s_{12}s_{22})\}$. Note that constructing $G_1 \diamond G_2$ is equivalent to cutting an edge of $G_2$ and inserting it into a single edge of $G_1$.

**Lemma 2.1.** Let $G$ be a cubic graph. If $G = G_1 \circ G_2$ or $G = G_1 \diamond G_2$, then $G$ is planar if and only if $G_1$ and $G_2$ are planar.

**Proof.** Suppose that $G$ has a cellular embedding on the sphere. Then the removal of an edge cut $E_C$ separates $G$ into two subgraphs, $G'_1, G'_2$ embedded on the sphere, each of which is contained in one of two disjoint discs $D_1, D_2$. Note that the resulting degree-1 and degree-2 vertices of each subgraph are on its outer face (relative to $D_i$) as in Figure 2. If $E_C$ was a 2-edge cut, edges may be added on the outside face that join these vertices to create planar $G_i$. If $E_C$ was a 3-edge cut, add vertices $v_1, v_2$ on the outside faces of discs $D_1, D_2$ respectively, and join $v_i$ to the degree-1 and degree-2 vertices in $D_i$ to create planar $G_i$.

Conversely, spherical embeddings of $G_1$ and $G_2$ may be converted to planar drawings with distinguished vertices $v_1, v_2$ or edges $e_1, e_2$ on the outside faces of discs $D_1, D_2$ respectively. Removing $v_1, v_2$ (resp. $e_1, e_2$) produces $G$ with three edges (resp. two edges) of a cut missing. Any desired pairing of the vertices may be completed on a sphere without edges crossing by using judicious placement of $D_i$ (and perhaps flipping one over). This will result in $G_1 \circ G_2$ (resp. $G_1 \diamond G_2$).
Lemma 2.2. Let $G$ be a cubic graph. If $G = G_1 \uplus G_2$, or $G = G_1 \vee G_2$, then $G$ is bipartite if and only if $G_1$ and $G_2$ are bipartite.

Proof. If $G$ is a cubic bipartite graph with nontrivial 2-edge cut, then let there be $m_j$ vertices from part $j$ on side 1; if both cut edges emanate from part 1 then $3m_1 - 2 = 3m_2$ which is impossible. Thus each cut edge must emanate from a different part on side $i$ of the cut, so both removing the edge cut and placing edges on each side maintains bipartition.

Suppose $G$ is a bipartite cubic graph with nontrivial 3-edge cut $E_C$ and $G'_1, G'_2$ the induced subgraphs of $G \setminus E_C$. For a bipartition of $G$ to descend naturally to bipartitions of $G_1, G_2$, the edges of $E_C$ must be incident only to vertices in $G'_i$ that are in the same part of $G$. Therefore, assume this is not the case and (without loss of generality) that two of the edges of $E_C$ are incident to one part of $G'_i$ and the remaining edge of $E_C$ is incident to the other part of $G'_i$. Let $G'_1$ have $m_j$ vertices belonging to part $j$ of $G$. There are $3m_1 - 1$ edges emanating from part 1 of $G'_1$ that must be incident to vertices of part 2 of $G'_1$. On the other hand, there are $3m_2 - 2$ edges emanating from part 2 of $G'_1$ that must be incident to vertices in part 1. Thus $3m_1 - 1 = 3m_2 - 2$, which is impossible.

Conversely, if $G_1, G_2$ are bipartite, with distinguished $e_1 = s_{11}s_{12}, e_2 = s_{21}s_{22}$ for the purpose of forming $G_1 \uplus G_2$, then the bipartition of $G_1$ extends to $G_1 \uplus G_2$ by assigning $s_{12}$ (resp. $s_{22}$) to the opposite part as $s_{11}$ (resp. $s_{21}$). Similarly, if $G_1, G_2$ are bipartite, with distinguished $v_1, v_2$ for the purpose of forming $G_1 \vee G_2$, then use the bipartition of $G_1$ and assign $v_2$ to the opposite part as $v_1$ to induce a bipartition of $G_1 \vee G_2$.

Theorem 2.3. A cubic graph $H$ that is 2-connected but not 3-connected may be decomposed via $\uplus$ into a set of cubic loopless graphs $\{H_i\}$ where each $H_i$ is 3-connected.

Proof. The proof is inductive on the number of vertices of $H$. Because $H$ is 2-connected but not 3-connected, there exists a 2-vertex separating set. Figure 3 shows the three possible edge configurations for a 2-vertex separating set of a cubic graph, along with (at top) associated 2-edge cuts. Each 2-edge cut can be used to form $H = H_1 \uplus H_2$, and $|H_j| < |H|$ so the inductive hypothesis holds for $H_j$.

It is worth noting that while the decomposition can create multiple edges, any multiple edge in a cubic graph will be associated with a 2-edge cut. Thus
the final set of $H_j$ will be composed of theta graphs, and graphs with no multiple edges.

**Corollary 2.4.** The $\gamma$ decomposition of 2-connected cubic graphs given by Theorem 2.3 preserves both planarity and bipartiteness.

**Proof.** This follows from Lemmas 2.1 and 2.2.

An alternative decomposition using the $\gamma$ product can also be found. This is because every 2 vertex separating set is also associated with a 3-edge cut as seen in Figure 3 (bottom). This decomposition also preserves planarity and bipartiteness.

### 3-Manipulating and Composing Colorings

We begin by showing that we can fix the colors on the edges incident to a given vertex, and accomplish any sequence of edge-Kempe switches without changing the fixed colors. As a result, representatives of all edge-Kempe equivalence classes will be present in the set of colorings with fixed colors at a vertex. The following theorem holds for all base graphs $G$, not just cubic graphs, and all $n \geq \chi'(G)$.

**Theorem 3.1.** If $c \sim d$ are two proper edge colorings of a loopless graph $G$, and there exists a vertex $v$ such that $c(e_i) = d(e_i)$ for all $e_i$ incident to $v$,
then there exists a sequence of edge-Kempe switches from \(c\) to \(d\) that never change the colors on the edges incident to \(v\).

Recall that \(o_i - o_{i+1}\) is the notation for two colorings that differ by exactly one edge-Kempe switch. It will be useful to have a further notation for the switch itself. Let \(s_i = (\{p_{i_1}, p_{i_2}\}, t_i)\) where \(\{p_{i_1}, p_{i_2}\}\) is the pair of colors to be switched on the chain \(t_i\) of \(G\). Then write \(o_i - s_i o_{i+1}\), if \(o_{i+1}\) is obtained from \(o_i\) by switching colors \(\{p_{i_1}, p_{i_2}\}\) on chain \(t_i\). Considering \(S_n\) as acting on the set of colors \(\{1, \ldots, n\}\), let \(\pi_i \in S_n\) be the transposition \(\pi_i(p_{i_1}) = p_{i_2}, \pi_i(p_{i_2}) = p_{i_1}\).

The idea of the proof is as follows. Each time a switch \(s_i = (\{p_{i_1}, p_{i_2}\}, t_i)\) affects an edge incident to \(v\), replace it by making all other \(\{p_{i_1}, p_{i_2}\}\) switches in the graph. This results in a coloring of the graph that is equivalent to the original, at the same stage, via a global color permutation. Therefore we need to track the colors to be switched on \(t_k\), for \(k > i\). Each switch \(s_k\) that does not affect an edge incident to vertex \(v\) will be replaced by a switch, on the same chain \(t_k\), of the colors that are currently on that chain. Our proof gives this precisely as an algorithm.

**Proof.** Suppose that \(c = o_0 - s_0 o_1 - s_1 \ldots - s_{n-1} o_m = d\), and there is at least one \(i\) such that \(v \in t_i\). Let \(\sigma_0\) be the identity permutation. For \(0 \leq i \leq m-1\), replace \(s_i\) with a set of edge-Kempe switches \(\hat{s}_i\) as follows. Set \(\hat{\pi}_i = \sigma_i \pi_i \sigma_i^{-1}\) so that \(\hat{\pi}_i(\sigma_i(p_{i_1})) = \sigma(p_{i_2})\).

If \(v \notin t_i\) then set \(\hat{s}_i = \{(\{p_{i_1}, p_{i_2}\}, t_i)\}\) and \(\sigma_{i+1} = \sigma_i\).

If \(v \in t_i\) then for \(t_j\) the edge-Kempe chains of \(o_i\) in colors \(\{p_{i_1}, p_{i_2}\}\), set \(\hat{s}_i = \{(\{\sigma_i(p_{i_1}), \sigma_i(p_{i_2})\}, t_j)\} t_j \neq t_i\) and \(\sigma_{i+1} = \sigma_i \pi_i\). Note that the set \(\hat{s}_i\) may be empty if \(t_i\) is the only \(\{p_{i_1}, p_{i_2}\}\) chain in \(o_i\).

Define \(\hat{o}_{i+1}\) to be the result of performing the sets of switches \(\hat{s}_1, \ldots, \hat{s}_i\) to \(c\). We show that \(\hat{o}_{i+1}\) and \(o_i\) are equivalent up to a global color permutation by \(\sigma_i\). Recall that \(o_i(e)\) is the color assigned to edge \(e\) by \(o_i\). We must show that on each edge \(e\), \(\hat{o}_{i+1}(e) = \sigma_{i+1} o_{i+1}(e)\). We proceed by induction and so assume that for \(k < i\), \(\hat{o}_k(e) = \sigma_k o_k(e)\).

There are 5 cases.

First suppose \(v \notin t_i\).

Case 1a. If \(e \in t_i\) then \(\hat{o}_{i+1}(e) = \hat{\pi}_i \hat{o}_i(e)\) because \(\hat{\pi}_i\) is the action of switch \(\hat{s}_i\). By definition of \(\hat{\pi}_i\) and using the inductive hypothesis for \(\hat{o}_i\), \(\hat{\pi}_i \hat{o}_i(e) = (\sigma_i \pi_i \sigma_i^{-1})(\sigma_i o_i(e))\). Simplifying, we have \(\sigma_i \pi_i o_i(e) = \sigma_i o_{i+1}(e)\) (by action of \(s_i\) on \(o_i\)), which, by definition of \(\sigma_{i+1}\) in this case, equals \(\sigma_{i+1} o_{i+1}(e)\).
as desired. Similar reasoning justifies the remaining cases so we present them in an abbreviated fashion.

Case 1b. If \( e \not\in t_i \) then \( \hat{o}_{i+1}(e) = \hat{o}_i(e) = \sigma_i o_i(e) = \sigma_{i+1} o_{i+1}(e) \).

Now suppose \( v \in t_i \).

Case 2a. If \( o_i(e) \not\in \{p_{i_1}, p_{i_2}\} \) then \( \hat{o}_{i+1}(e) = \hat{o}_i(e) = \sigma_i o_i(e) = (\sigma_i \pi_i) o_i(e) = \sigma_{i+1} o_{i+1}(e) \).

Case 2b. If \( o_i(e) \in \{p_{i_1}, p_{i_2}\} \) and \( e \in t_i \), then the color on \( e \) does not change from \( \hat{o}_i \) to \( \hat{o}_{i+1} \) while it did change from \( o_i \) to \( o_{i+1} \). Thus, \( \hat{o}_{i+1}(e) = \hat{o}_i(e) = \sigma_i o_i(e) = \sigma_i \pi_i \pi_i o_i(e) = \sigma_{i+1} o_{i+1}(e) \).

Case 2c. If \( o_i(e) \in \{p_{i_1}, p_{i_2}\} \) and \( e \not\in t_i \), then the color on \( e \) does change from \( \hat{o}_i \) to \( \hat{o}_{i+1} \) while it did not change from \( o_i \) to \( o_{i+1} \). Thus, \( \hat{o}_{i+1}(e) = \hat{o}_i(e) = (\sigma_i \pi_i \pi_i^{-1}) (\sigma_i o_i(e)) = (\sigma_i \pi_i) o_i(e) = \sigma_{i+1} o_{i+1}(e) \).

Finally, we consider \( \hat{o}_m \) and compare it to \( d \). Note \( c \) and \( d \) have the same colors on \( v \) by hypothesis, and the total number of colors used in \( d \) is \( n \). If \( n \leq \deg(v) + 1 \), then at most one color is not represented at \( v \) and \( \sigma_m \) must be the identity permutation; thus \( \hat{o}_m = o_m = d \). If \( n > \deg(v) + 1 \), then it is possible that some colors that do not occur at \( v \) are globally permuted between \( o_m \) and \( \hat{o}_n \). In this case, additional edge-Kempe switches that globally permute colors can be applied to \( \hat{o}_m \) so that the coloring now matches \( d \).

\( \square \)

This result shows when counting the number of edge-Kempe equivalence classes it is sufficient to consider only colorings of \( G \) that are different up to global color permutation. To make this observation precise requires careful definition of an edge-Kempe-equivalence graph of a graph. This will be done in [2].

Returning to cubic graphs, we next consider how combining graphs affects \( K'(G, n) \). Let \( G_1, G_2 \) be two 3-edge-colorable cubic graphs and distinguish a vertex on each \((v_1, v_2)\) for the purpose of forming \( G_1 \circ G_2 \). Recall that in addition to the choice of \( v_1, v_2 \), there are multiple ways their incident edges may be identified; by \( G_1 \circ G_2 \) we mean some particular set of these choices. Let \( \{x_1, x_2, x_3\} \) and \( \{y_1, y_2, y_3\} \) be the ordered sets of edges in \( G_1 \) and \( G_2 \) that will be identified in \( G_1 \circ G_2 \). Similarly, choose a distinguished edge in each graph \((x \in G_1, y \in G_2)\) for the purpose of forming \( G_1 \underset{\perp}{\circ} G_2 \). The following several results relate 3-edge colorings of \( G_1 \) and \( G_2 \) to those of \( G_1 \circ G_2 \) and \( G_1 \underset{\perp}{\circ} G_2 \).
Definition 3.2. Let $c, d$ be proper edge colorings of $G_1, G_2$ respectively. There exists a proper coloring $\hat{d}$ of $G_2$ such that $c(x_i) = \hat{d}(y_i)$ for $i = 1, 2, 3$, and such that $d, \hat{d}$ are the same up to a permutation of the colors ($d \sim \hat{d}$). Define $(c \triangleright d)$ to be the proper coloring of $G_1 \triangleright G_2$ given by

\[(c \triangleright d)(e) = \begin{cases} 
  c(e) & \text{if } e \in G_1 \\
  \hat{d}(e) & \text{if } e \in G_2 \\
  c(e) = \hat{d}(e) & \text{if } e \text{ is the edge resulting from identifying } x_i \text{ and } y_i.
\end{cases}\]

Similarly, there exists a proper coloring $\tilde{d}$ of $G_2$ such that $c(x) = \tilde{d}(y)$ and such that $d, \tilde{d}$ are the same up to a global permutation of the colors. Define $(c \triangleright \tilde{d})$ to be the proper coloring of $G_1 \triangleright G_2$ given by

\[(c \triangleright \tilde{d})(e) = \begin{cases} 
  c(e) & \text{if } e \in G_1 \\
  \tilde{d}(e) & \text{if } e \in G_2 \\
  c(e) = \tilde{d}(e) & \text{if } e \text{ is one of the edges added after deleting } x \text{ and } y.
\end{cases}\]

Two cases of the Parity Lemma ([3]) will be useful.

Lemma 3.3. Let $E_C$ be an edge cut of a of a 3-edge-colorable cubic graph $G$ and $c$ be any proper 3-edge coloring of $G$. Then

(a) if $E_C$ is a 2-edge cut, then $c(E_C)$ uses exactly one color, and

(b) if $E_C$ is a 3-edge cut, then $c(E_C)$ uses all three colors.

Theorem 3.4. Every 3-edge coloring $f$ of $G = G_1 \triangleright G_2$ (resp. $G = G_1 \triangleright G_2$) can be written as $c_1 \triangleright d_1$ (resp. $c_1 \triangleright d_1$) where $c_1$ is some 3-edge coloring of $G_1$ and $d_1$ is some 3-edge coloring of $G_2$.

Proof. Consider a 3-edge coloring $f$ of $G = G_1 \triangleright G_2$. There is a 3-edge cut $E_C$ corresponding to the decomposition $G_1 \triangleright G_2$. By Lemma 3.3(b), each $e_i \in E_C$ must be a different color in $c$. Therefore considering $f$ on the edges of $G_1$ (and particularly at $v_1$), it is still a proper coloring $c_1$, and likewise $f$ considered on $G_2$ is a proper coloring $d_1$. The result for $\triangleright$ is similarly an immediate corollary of Lemma 3.3.\[\square\]

Implicit in the preceding results is the following.

Corollary 3.5. If $G = G_1 \triangleright G_2$ or $G = G_1 \triangleright G_2$, then $G$ is 3-edge colorable if and only if $G_1$ and $G_2$ are 3-edge colorable.

Next we note how edge-Kempe equivalences on the colorings of $G_1$ and $G_2$ transfer to edge-Kempe equivalences in combinations of these graphs.
Lemma 3.6. Let 3-edge colorings $c_1 \sim c_2$ in $G_1$ and $d_1 \sim d_2$ in $G_2$. Then $(c_1 \cdot d_1) \sim (c_2 \cdot d_2)$ in $G_1 \cdot G_2$ and $(c_1 \times d_1) \sim (c_2 \times d_2)$ in $G_1 \times G_2$.

Proof. Using the notation from Definition 3.2 let $c'_2 \sim c_2$ by global color permutation such that $c'_2(x_i) = c_1(x_i)$ for $i = 1, 2, 3$. By Theorem 3.1 there exists a sequence of edge-Kempe switches in $G_1$ that exhibits $c_1 \sim c'_2$ and that never changes the color of any edge incident to $v_1$. Similarly, define $d'_2 \sim d_2 \sim d_2$ such that there is a sequence of edge-Kempe switches in $G_2$ that exhibits $d'_1 \sim d_2$ and that never changes the color of any edge incident to $v_2$. Then $(c_1 \cdot d_1) = (c_1 \cdot d_1) \sim (c'_2 \cdot d_1) \sim (c'_2 \cdot d_2) \sim (c_2 \cdot d_2) = (c_2 \cdot d_2)$.

For the $\times$ composition, assume without loss of generality that $c_1(x) = d_1(y)$. Let $c'_2 \sim c_2$ by global color permutation such that $c'_2(x) = c_1(x)$ and $d'_2 \sim d_2$ by global color permutation such that $d'_2(y) = d_1(y)$. By Lemma 3.3 the two edges created after deleting $x, y$ will be assigned the same color in any proper 3-coloring of $G_1 \times G_2$, so fixing the color on one will also fix the color on the other. Hence, $(c_1 \times d_1) \sim (c'_2 \times d_1) \sim (c'_2 \times d'_2) \sim (c_2 \times d_2)$.

Lemma 3.7. Let $G_1, G_2$ be 3-edge colorable cubic graphs with $G_1 \cdot G_2$ and $G_1 \times G_2$ particular compositions of the two. If $(c_1 \cdot d_1) \sim (c_2 \cdot d_2)$ in $G_1 \cdot G_2$ (resp. $(c_1 \times d_1) \sim (c_2 \times d_2)$ in $G_1 \times G_2$) then $c_1 \sim c_2$ in $G_1$ and $d_1 \sim d_2$ in $G_2$.

Proof. It is sufficient to show this when $(c_1 \cdot d_1) \sim_s (c_2 \cdot d_2)$ and $(c_1 \times d_1) \sim_s (c_2 \times d_2)$, where $s = (p, t)$ with $p$ a pair of colors and $t$ an edge-Kempe chain. If $t \subset G_1$ or $t \subset G_2$, then the lemma holds. Otherwise, $t \cap E_C \neq \emptyset$, and $t$ must use exactly 2 edges of $E_C$ because every edge-Kempe chain of a proper 3-edge coloring of a cubic graph is a cycle. The decomposition $G_1 \cdot G_2$ (resp. $G_1 \times G_2$) over $E_C$ will decompose $t$ into an edge-Kempe chain $t_1$ of $G_1$ and $t_2$ of $G_2$. Then $c_1 \cdot t_2 \sim (c_1 \times t_1)$ in $G_1$ and $d_1 \sim (c_2 \times t_2)$ in $G_2$.

Theorem 3.8. Let $G_1, G_2$ be cubic graphs. If $K'(G_1, 3) = a$ and $K'(G_2, 3) = b$, then $K'(G_1 \cdot G_2, 3) = K'(G_1 \times G_2, 3) = ab$.

Proof. Choose colorings $c_1, \ldots, c_a$, one from each of the $a$ edge-Kempe-equivalence classes of $G_1$, and likewise choose colorings $d_1, \ldots, d_b$, one from each of the $b$ edge-Kempe-equivalence classes of $G_2$. Every 3-edge coloring $f$ of $G_1 \cdot G_2$ can be written as $f = \tilde{c} \cdot \tilde{d}$ by Theorem 3.4 $\tilde{c} \sim c_i$ for some $c_i \in \{c_1, \ldots, c_a\}$, and $\tilde{d} \sim d_j$ for some $d_j \in \{d_1, \ldots, d_b\}$, so by Lemma 3.6 $f \sim c_i \cdot d_j$ for some $c_i \in \{c_1, \ldots, c_a\}, d_j \in \{d_1, \ldots, d_b\}$. Further by Lemma 3.7 $c_{i_1} \cdot d_{j_1} \sim c_{i_2} \cdot d_{j_2}$ only when $i_1 = i_2, j_1 = j_2$. Therefore there are $ab$ edge-Kempe-equivalence classes of $G_1 \cdot G_2$. The proof for $G_1 \times G_2$ is identical.
4 Results on $K'(G, 3)$

Theorem 3.8 can be extended to compose several graphs, or alternatively to decompose a graph into many smaller pieces. We will use the theorem below in both contexts to get results about possible numbers of edge-Kempe equivalence classes for cubic graphs.

**Theorem 4.1.** Let $G$ be a 3-edge colorable cubic graph. Then $K'(G, 3) = \prod_i K'(G_i, 3)$ where $\{G_i\}$ is a decomposition of $G$ along nontrivial 2-edge cuts or 3-edge cuts.

*Proof.* This follows from multiple applications of Theorem 3.8.

4.1 Planar, cubic, bipartite graphs

The following theorem answers a question from [6, Section 3].

**Theorem 4.2.** Let $H$ be a 2-connected, but not 3-connected, planar bipartite cubic graph. Then $K'(H, 3) = 1$.

*Proof.* By Theorem 2.3, $H$ may be decomposed into $\{H_i\}$ where all $H_i$ are 3-connected. By Lemmas 2.1 and 2.2, all $H_i$ are planar and bipartite. As pointed out in [6], it follows from [11] that all 3-connected planar bipartite cubic graphs $G$ have $K'(G, 3) = 1$ so for all $H_i, K'(H_i, 3) = 1$. It then follows from Theorem 4.1 that $K'(H, 3) = 1$.

Recall that if $G$ is cubic and bipartite then it must be bridgeless. Thus we get the following result.

**Corollary 4.3.** Let $H$ be a planar bipartite cubic graph. Then $K'(H, 3) = 1$.

4.2 Nonplanar, cubic, bipartite graphs

Matters are quite different for nonplanar bipartite cubic graphs. It is well known that $K_{3,3}$ has two different edge-colorings (shown in Figure 4). In each of these colorings, each color-pair forms a Hamilton cycle. Therefore, any edge-Kempe switch results in a permutation of the colors and neither coloring of Figure 4 can be obtained from the other. Thus, there are two edge-Kempe equivalence classes, i.e. $K'(K_{3,3}, 3) = 2$. 

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Lemma 4.4. Every simple bipartite nonplanar cubic graph $B$ with $n \leq 10$ has $K'(B, 3) > 1$.

Proof. Every simple bipartite nonplanar cubic graph is a subdivision of $K_{3,3}$. To maintain the bipartition and avoid multiple edges, $K_{3,3}$ must be subdivided with at least 4 vertices, two on each of two edges. These edges may be independent or may be incident.

Figure 5: The two possible colorings around subdivided independent or incident edges.

Any coloring of the original graph extends to either one or two new (edge-Kempe equivalent) colorings, as is shown in Figure 5. If a coloring had three Hamilton cycles before subdivision (as is true for both colorings of $K_{3,3}$), at most it gains an isolated edge-Kempe cycle after subdivision of this sort. Thus when subdividing $K_{3,3}$ with a single 4-vertex subdivision, there still exist two colorings that are not edge-Kempe-equivalent.

Further examples of nonplanar cubic bipartite graphs with $K'(G, 3) > 1$ will be given in Section 4.3. In contrast, Figure 6 shows a bipartite nonplanar cubic graph $U$ with 12 vertices and $K'(U, 3) = 1$. $K'(U, 3)$ was computed manually and verified using custom Mathematica code. We can use $U$ to produce an interesting infinite class of graphs.
Theorem 4.5. There exists an infinite family of simple nonplanar 3-connected bipartite cubic graphs $U_k$ with $2 + 10k$ vertices and $K'(U_k, 3) = 1$.

Proof. Let $U_k = U \gamma \cdots (k \, \text{copies}) \cdots \gamma U$. By Theorem 3.8 $K'(U_k, 3) = 1$. Graphs $U_2, U_3,$ and $U_4$ are shown in Figure 7.

By composition of $U$ with a planar cubic bipartite graph with $n - 10$ vertices we get the following more general result.

Theorem 4.6. For any $n \geq 18$ there is a simple, nonplanar, bipartite, 3-connected, cubic graph $G$ with $n$ vertices and $K'(G, 3) = 1$.

Notice that similar results can be obtained for graphs that are only 2-connected as well by using the $\pm$ composition.
4.3 Cubic graphs with $K'(G, 3) > 1$

We can form $K_{3,3} \gamma G$ with any 3-connected cubic graph $G$ to obtain a 3-connected nonplanar cubic graph. By Theorem 3.8

$$K'(K_{3,3} \gamma G, 3) = K'(K_{3,3}, 3)K'(G, 3) = 2K'(G, 3).$$

**Theorem 4.7.** For every even $n \geq 8$, there exists a 3-connected nonplanar cubic graph $G$ with $n$ vertices and exactly 2 edge-Kempe equivalence classes.

**Proof.** Form $K_{3,3} \gamma G$ with any 3-connected planar cubic graph $G$ on $n - 4$ vertices to obtain a 3-connected nonplanar cubic graph with $n$ vertices and $K'(K_{3,3} \gamma G, 3) = 2$. \hfill $\square$

**Corollary 4.8.** For every even $n \geq 12$, there exists a 3-connected nonplanar bipartite cubic graph $G$ with $n$ vertices and exactly 2 edge-Kempe equivalence classes.

**Proof.** Form $K_{3,3} \gamma G$ with any 3-connected planar cubic bipartite graph $G$ on $n - 4$ vertices. The smallest 3-connected planar cubic bipartite graph has 8 vertices. \hfill $\square$

More generally, once we have one example with $k$ edge-Kempe equivalence classes then there will be an infinite family of them with the same number of classes.

**Theorem 4.9.** If $\hat{G}$ is a cubic graph on $\hat{n}$ vertices with $k$ edge-Kempe equivalence classes then for every even $n \geq \hat{n} + 6$, there exists a cubic graph on $n$ vertices with exactly $k$ edge-Kempe equivalence classes. Further, if $\hat{G}$ is planar then a planar family exists, if $\hat{G}$ is bipartite then a bipartite family exists and if $\hat{G}$ is 3-connected then a 3-connected family exists.

**Proof.** Compose $\hat{G}$ with any cubic planar bipartite graph on $n+2-\hat{n}$ vertices using the $\gamma$ operation. The result follows from Theorem 3.8 \hfill $\square$

We can make graphs with increasingly large numbers of edge-Kempe equivalence classes this way as well.

**Theorem 4.10.** For every $k \geq 1$, there exists a 3-connected nonplanar bipartite cubic graph $G$ with $4k + 2$ vertices and $2^k$ edge-Kempe equivalence classes.
Proof. For $k \geq 1$, take $K_{3,3} \cdot \cdots \cdot (k \text{ copies}) \cdots \cdot K_{3,3}$, which has $2 + 4k$ vertices. By Theorem 3.8 it has $2^k$ edge-Kempe equivalence classes. This produces the desired graph. 

Theorem 4.11. For every simple nonplanar (bipartite) cubic graph $G$ with $n$ vertices, there exists an infinite family of nonplanar (bipartite) cubic graphs $G_k$ such that $G_k$ has $6k + n$ vertices and $2^k K'(G, 3)$ edge-Kempe equivalence classes.

Proof. Take $G \sqcup K_{3,3} \sqcup \cdots \sqcup K_{3,3}$. 

5 Computations of $K'(G, 3)$

Computing $K'(G, 3)$ for particular $G$, or for families of graphs, is surprisingly difficult. A single computation can be done by brute force by computer, but constructing a proof is another matter. As examples of the kinds of arguments needed to determine $K'(G, 3)$, we analyze Möbius ladder graphs, prism graphs, and crossed prism graphs.

Theorem 5.1. Let $ML_k$ be the Möbius ladder graph on $2k$ vertices, let $Pr_k$ be the prism graph on $2k$ vertices, and let $CPr_k$ be the crossed prism graph on $4k$ vertices.

1. $K'(ML_k, 3) = 1$ when $k$ is even and $K'(ML_k, 3) = 2$ when $k$ is odd.
2. $K'(Pr_k, 3) = 1$.
3. $K'(CPr_k, 3) = 1$.

Note that $Pr_k$ is planar, and bipartite exactly when $k$ is even; $ML_k$ is toroidal.

Proof. Our arguments are inductive.

First, consider the edge coloring of $ML_k$ given at left in Figure 8, and note that it only exists for $k$ odd. Every edge-Kempe chain in this coloring is a Hamilton circuit, so this coloring represents a edge-Kempe-equivalence class of of $ML_k$. Now consider any other 3-edge coloring of $ML_k$. If it has a square colored as shown at right in Figure 8, then the square may be removed (and the remaining half-edges glued together) to produce a 3-edge coloring.
Figure 8: A tri-Hamiltonian edge coloring of $ML_k$ for $k$ odd (left) with a square from some other colorings of $ML_k$ (right).

Figure 9: Colorings of squares from $ML_k$ that are edge-Kempe-equivalent to a removable colored square of $ML_k$.

of $ML_{k-2}$. If there is no such square in the coloring, then every square must be colored as one of the options shown in Figure 9. In either case, we can do a single edge-Kempe switch to produce an edge-Kempe-equivalent coloring that contains a removable square. Therefore $K'(ML_k, 3) = K'(ML_{k-2}, 3)$. To complete the proof, it suffices to show (which direct computation does) that $K'(ML_4, 3) = 1$ and $K'(ML_3, 3) = 2$.

Next consider any 3-edge coloring of $Pr_k$. The same argument as for $ML_k$ applies, so by removing a square we see that $K'(Pr_k, 3) = K'(Pr_{k-2}, 3)$. Because $K'(Pr_3, 3) = K'(Pr_4, 3) = 1$ by direct computation, it then follows that $K'(Pr_k, 3) = 1$.

Finally, consider any 3-edge coloring of $CPr_k$. Any crossed square must have one of the local colorings shown in Figure 10. For the leftmost two

Figure 10: The possible colorings of a crossed square of $CPr_k$. 

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colorings of Figure 10, the crossed square may be removed (and the remaining half-edges glued together) to produce a 3-edge coloring of $CPr_{k-1}$. If there are only crossed squares with coloring type of the rightmost coloring in Figure 10 we can do a single edge-Kempe switch to produce an edge-Kempe-equivalent coloring that contains a removable crossed square. (A parity argument shows that there must be at least two edge-Kempe chains in a relevant color pair.) Because $K'(CPr_2, 3) = 1$ by direct computation, it then follows that $K'(CPr_k, 3) = 1$.

6 Areas for future work

Two major questions remain about $K'(G)$ for cubic, nonplanar, bipartite graphs. First, while we have shown that there are nonplanar cubic bipartite graphs with $K'(G, 3) = 1$ and also some with $K'(G, 3) > 1$, there is as yet no characterization for when each is true. Second, using Mathematica we have found bipartite cubic graphs where $K'(G, 3) = 1, 2, 3, 4, 6, 8, 9, 15, 17, 35, 131$. Which natural numbers, and in particular which primes, $k$ are achievable as $K'(G, 3) = k$ for $G$ a cubic nonplanar bipartite 3-connected graph, with no nontrivial edge cuts? These same questions can be asked for cubic 3-colorable graphs more generally: which have $K'(G, 3) = 1$, and what possible $K'(G, 3)$ values can occur?

Beyond just examining the number of edge-Kempe connected components, what is the structure of the edge-Kempe-equivalence Graph of $G$, whose vertices represent colorings of $G$ and whose edges represent single edge-Kempe switches? This is the topic of [2].

References


