The Atiyah Class of a dg-Vector Bundle

Rajan Amit Mehta  
*Smith College*, rmehta@smith.edu

Mathieu Stiénon  
*The Pennsylvania State University*

Ping Xu  
*The Pennsylvania State University*

Follow this and additional works at: https://scholarworks.smith.edu/mth_facpubs

Part of the Mathematics Commons

Recommended Citation

https://scholarworks.smith.edu/mth_facpubs/16

This Article has been accepted for inclusion in Mathematics and Statistics: Faculty Publications by an authorized administrator of Smith ScholarWorks. For more information, please contact scholarworks@smith.edu
THE ATIYAH CLASS OF A DG-VECTOR BUNDLE

RAJAN AMIT MEHTA, MATHIEU STIÉNON, AND PING XU

En hommage à Charles-Michel Marle à l’occasion de son quatre-vingtième anniversaire

Abstract. We introduce the notions of Atiyah class and Todd class of a differential graded vector bundle with respect to a differential graded Lie algebroid. We prove that the space of vector fields $\mathfrak{X}(M)$ on a dg-manifold $M$ with homological vector field $Q$ admits a structure of $L_{\infty}[1]$-algebra with the Lie derivative $L_Q$ as unary bracket $\lambda_1$, and the Atiyah cocycle $\text{At}_M$ corresponding to a torsion-free affine connection as binary bracket $\lambda_2$.

1. DG-MANIFOLDS AND DG-VECTOR BUNDLES

A $\mathbb{Z}$-graded manifold $\mathcal{M}$ with base manifold $M$ is a sheaf of $\mathbb{Z}$-graded, graded-commutative algebras $\{\mathcal{R}_U | U \subset M \text{ open} \}$ over $M$, locally isomorphic to $C^\infty(U) \otimes \hat{S}(V^\vee)$, where $U \subset M$ is an open submanifold, $V$ is a $\mathbb{Z}$-graded vector space, and $\hat{S}(V^\vee)$ denotes the graded algebra of formal polynomials on $V$. By $C^\infty(M)$, we denote the $\mathbb{Z}$-graded, graded-commutative algebra of global sections. By a dg-manifold, we mean a $\mathbb{Z}$-graded manifold endowed with a homological vector field, i.e. a vector field $Q$ of degree $+1$ satisfying $[Q,Q] = 0$.

Example 1.1. Let $A \to M$ be a Lie algebroid over $\mathbb{C}$. Then $A[1]$ is a dg-manifold with the Chevalley–Eilenberg differential $d_{CE}$ as homological vector field. In fact, according to Vaintrob [12], there is a bijection between the Lie algebroid structures on the vector bundle $A \to M$ and the homological vector fields on the $\mathbb{Z}$-graded manifold $A[1]$.

Example 1.2. Let $s$ be a smooth section of a vector bundle $E \to M$. Then $E[-1]$ is a dg-manifold with the contraction operator $i_s$ as homological vector field.

Example 1.3. Let $\mathfrak{g} = \sum_{i \in \mathbb{Z}} \mathfrak{g}_i$ be a $\mathbb{Z}$-graded vector space of finite type, i.e. each $\mathfrak{g}_i$ is a finite-dimensional vector space. Then $\mathfrak{g}[1]$ is a dg-manifold if and only if $\mathfrak{g}$ is an $L_{\infty}$-algebra.

A dg-vector bundle is a vector bundle in the category of dg-manifolds. We refer the reader to [10] for details on dg-vector bundles. The following example is essentially due to Kotov–Strobl [4].


Research partially supported by NSF grants DMS1406668, and NSA grants H98230-06-1-0047 and H98230-14-1-0153.

1
2 RAJAN AMIT MEHTA, MATHIEU STIÉNON, AND PING XU

The example above is a special case of a general fact [10], that LA-vector bundles [6, 7, 8] (also known as VB-algebroids [2]) give rise to dg-vector bundles.

Given a vector bundle \( E \xrightarrow{\pi} M \) of graded manifolds, its space of sections, denoted \( \Gamma(E) \), is defined to be \( \bigoplus_{j \in \mathbb{Z}} \Gamma_j(E) \), where \( \Gamma_j(E) \) consists of degree preserving maps \( s \in \text{Hom}(M, E[-j]) \) such that \( (\pi[-j]) \circ s = \text{id}_M \), where \( \pi[-j] : E[-j] \to M \) is the natural map induced from \( \pi \); see [10] for more details. When \( E \to M \) is a dg-vector bundle, the homological vector fields on \( E \) and \( M \) naturally induce a degree 1 operator \( Q \) on \( \Gamma(E) \), making \( \Gamma(E) \) a dg-module over \( C\infty(M) \). Since the space \( \Gamma(T^\ast M) \) of linear functions on \( E \) generates \( C\infty(E) \), the converse is also true.

**Lemma 1.5.** Let \( E \to M \) be a vector bundle object in the category of graded manifolds and suppose \( M \) is a dg-manifold. If \( \Gamma(E) \) is a dg-module over \( C\infty(M) \), then \( E \) admits a natural dg-manifold structure such that \( E \to M \) is a dg-vector bundle. In fact, the categories of dg-vector bundles and of locally free dg-modules are equivalent.

In this case, the degree +1 operator \( Q \) on \( \Gamma(E) \) gives rise to a cochain complex

\[
\cdots \to \Gamma_i(E) \xrightarrow{Q} \Gamma_{i+1}(E) \to \cdots,
\]

whose cohomology group will be denoted by \( H^\ast(\Gamma(E), Q) \).

In particular, the space \( \mathfrak{x}(M) \) of vector fields on a dg-manifold \( (M, Q) \) (i.e. graded derivations of \( C\infty(M) \)), which can be regarded as the space of sections \( \Gamma(T^\ast M) \), is naturally a dg-module over \( C\infty(M) \) with the Lie derivative \( \mathcal{L}_Q : \mathfrak{x}(M) \to \mathfrak{x}(M) \) playing the role of the degree +1 operator \( Q \).

Thus we have the following

**Corollary 1.6.** For every dg-manifold \( (M, Q) \), the Lie derivative \( \mathcal{L}_Q \) makes \( \Gamma(T^\ast M) \) into a dg-module over \( C\infty(M) \) and therefore \( T^\ast M \to M \) is naturally a dg-vector bundle.

Following the classical case, the corresponding homological vector field on \( T^\ast M \) is called the tangent lift of \( Q \).

Differential graded Lie algebroids are another useful notion. Roughly, a dg-Lie algebroid can be thought of as a Lie algebroid object in the category of dg-manifolds.

**Lemma 1.7.** Let \( \mathcal{D} \subset T^\ast M \) be an integrable distribution on a graded manifold \( M \). Suppose there exists a homological vector field \( Q \) on \( M \) such that \( \Gamma(\mathcal{D}) \) is stable under \( \mathcal{L}_Q \). Then \( \mathcal{D} \to M \) is a dg-Lie algebroid with the inclusion \( \rho : \mathcal{D} \to T^\ast M \) as its anchor map.

2. Atiyah class and Todd class of a dg-vector bundle

Let \( \mathcal{E} \to M \) be a dg-vector bundle and let \( \mathcal{A} \to M \) be a dg-Lie algebroid with anchor \( \rho : \mathcal{A} \to T^\ast M \). An \( \mathcal{A} \)-connection on \( \mathcal{E} \to M \) is a degree 0 map \( \nabla : \Gamma(\mathcal{A}) \otimes \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E}) \) such that

\[
\nabla_{f \mathcal{X}} = f \nabla \mathcal{X}
\]
and
\[ \nabla_X(f s) = \rho(X)(f)s + (-1)^{|X||f|}f \nabla_X s \]
for all \( f \in C^\infty(\mathcal{M}) \), \( X \in \Gamma(\mathcal{A}) \), and \( s \in \Gamma(\mathcal{E}) \). Here we use the ‘absolute value’ notation to denote the degree of the argument. When we say that \( \nabla \) is of degree 0, we actually mean that \( |\nabla_X s| = |X| + |s| \) for every pair of homogeneous elements \( X \) and \( s \). Such connections always exist since the standard partition of unity argument holds in the context of graded manifolds. Given a dg-vector bundle \( \mathcal{E} \to \mathcal{M} \) and an \( \mathcal{A} \)-connection \( \nabla \) on it, we can consider the bundle map \( \text{At}_\mathcal{E} : \mathcal{A} \otimes \mathcal{E} \to \mathcal{E} \) defined by

\[ \text{At}_\mathcal{E}(X, s) := Q(\nabla_X s) - \nabla_Q(X)s - (-1)^{|X|} \nabla_X (Q(s)), \quad \forall X \in \Gamma(\mathcal{A}), s \in \Gamma(\mathcal{E}). \]

**Proposition 2.1.**

1. \( \text{At}_\mathcal{E} : \mathcal{A} \otimes \mathcal{E} \to \mathcal{E} \) is a degree +1 bundle map and therefore can also be regarded as a degree +1 section of \( \mathcal{A}^\vee \otimes \text{End} \mathcal{E} \).

2. \( \text{At}_\mathcal{E} \) is a cocycle: \( Q(\text{At}_\mathcal{E}) = 0 \).

3. The cohomology class of \( \text{At}_\mathcal{E} \) is independent of the choice of the connection \( \nabla \).

Thus there is a natural cohomology class \( \alpha_\mathcal{E} := [\text{At}_\mathcal{E}] \) in \( H^1(\Gamma(\mathcal{A}^\vee \otimes \text{End} \mathcal{E}), Q) \). The class \( \alpha_\mathcal{E} \) is called the **Atiyah class** of the dg-vector bundle \( \mathcal{E} \to \mathcal{M} \) relative to the dg-Lie algebroid \( \mathcal{A} \to \mathcal{M} \).

The Atiyah class of a dg-manifold, which is the obstruction to the existence of connections compatible with the differential, was first investigated by Shoikhet [11] in relation with Kontsevich’s formality theorem and Duflo formula. More recently, the Atiyah class of a dg-manifold appeared in Costello’s work [1].

We define the **Todd class** \( \text{Td}_\mathcal{E} \) of a dg-vector bundle \( \mathcal{E} \to \mathcal{M} \) relative to a dg-Lie algebroid \( \mathcal{A} \to \mathcal{M} \) by

\[ \text{Td}_\mathcal{E} := \text{Ber} \left( \frac{1 - e^{-\alpha_\mathcal{E}}}{\alpha_\mathcal{E}} \right) \in \prod_{k \geq 0} H^k(\Gamma(\Lambda^k \mathcal{A}^\vee), Q), \]

where Ber denotes the Berezinian [9] and \( \Lambda^k \mathcal{A}^\vee \) denotes the dg vector bundle \( S^k(\mathcal{A}^\vee[-1])[k] \to \mathcal{M} \). One checks that \( \text{Td}_\mathcal{E} \) can be expressed in terms of scalar Atiyah classes \( \omega_k = \frac{1}{k!} (\omega)^k \text{str} \alpha^k \in H^k(\Gamma(\Lambda^k \mathcal{A}^\vee), Q) \). Here \( \text{str} : \text{End} \mathcal{E} \to C^\infty(\mathcal{M}) \) denotes the supertrace. Note that \( \text{str} \alpha^k \in \Gamma(\Lambda^k \mathcal{A}^\vee) \) since \( \alpha^k \in \Gamma(\Lambda^k \mathcal{A}^\vee) \otimes C^\infty(\mathcal{M}) \) End \( \mathcal{E} \). If \( \mathcal{A} = T\mathcal{M} \), we write \( \Omega^k(\mathcal{M}) \) instead of \( \Gamma(\Lambda^k T^\vee \mathcal{M}) \).

3. **Atiyah class and Todd class of a dg-manifold**

Consider a dg-manifold \( (\mathcal{M}, Q) \). According to Lemma [17], its tangent bundle \( T\mathcal{M} \) is indeed a dg-Lie algebroid. By the Atiyah class of a dg-manifold \( (\mathcal{M}, Q) \), denoted \( \alpha_\mathcal{M} \), we mean the Atiyah class of the tangent dg-vector bundle \( T\mathcal{M} \to \mathcal{M} \) with respect to the dg-Lie algebroid \( T\mathcal{M} \). Similarly, the Atiyah 1-cocycle of a dg manifold \( \mathcal{M} \) corresponding to an affine connection on \( \mathcal{M} \) (i.e. a \( T\mathcal{M} \)-connection on \( T\mathcal{M} \to \mathcal{M} \)) is the 1-cocycle defined as in Eq. (1).

**Lemma 3.1.** Suppose \( V \) is a vector space. The only connection on the graded manifold \( V[1] \) is the trivial connection.

**Proof.** Since the graded algebra of functions on \( V[1] \) is \( \Lambda(\Lambda V^\vee) \), every vector \( v \in V \) determines a degree \(-1\) vector field \( \iota_v \) on \( V[1] \), which acts as a contraction operator on \( \Lambda(\Lambda V^\vee) \). The \( C^\infty(V[1]) \)-module of all vector fields on \( V[1] \) is generated by its subset \( \{ \iota_v \}_{v \in V} \). It follows that a connection \( \nabla \) on \( V[1] \) is completely determined
by the knowledge of \( \nabla_{v,w} \) for all \( v, w \in V \). Since the degree of every vector field \( \nabla_{v,w} \) must be \(-2\) and there are no nonzero vector fields of degree \(-2\), it follows that \( \nabla_{v,w} = 0 \).

Given a finite-dimensional Lie algebra \( g \), consider the dg-manifold \((\mathcal{M}, Q)\), where \( \mathcal{M} = g[1] \) and \( Q \) is the Chevalley-Eilenberg differential \( d_{CE} \). The following result can be easily verified using the canonical trivialization \( T\mathcal{M} \cong g[1] \times g[1] \).

**Lemma 3.2.** Let \((\mathcal{M}, Q) = (g[1], d_{CE})\) be the canonical dg-manifold corresponding to a finite-dimensional Lie algebra \( g \). Then,

\[
H^k(\Gamma(T^\vee \mathcal{M} \otimes \text{End}\, T\mathcal{M}), Q) \cong H^{k-1}_{CE}(g, g^\vee \otimes g^\vee \otimes g),
\]

and

\[
H^k(\Omega^k(\mathcal{M}), Q) \cong (\mathcal{S}^k g^\vee)^g.
\]

**Proposition 3.3.** Let \((\mathcal{M}, Q) = (g[1], d_{CE})\) be the canonical dg-manifold corresponding to a finite-dimensional Lie algebra \( g \). Then the Atiyah class \( \alpha_{g[1]} \) is precisely the Lie bracket of \( g \) regarded as an element of \((g^\vee \otimes g^\vee \otimes g)^g \cong H^1(\Gamma(T^\vee \mathcal{M} \otimes \text{End}\, T\mathcal{M}), Q)\). Consequently, the isomorphism

\[
\prod_k H^k(\Omega^k(\mathcal{M}), Q) \xrightarrow{\sim} (\mathcal{S}(g^\vee))^g
\]

maps the Todd class \( Td_{g[1]} \) onto the Duflo element of \( g \).

**Example 3.4.** Consider a dg-manifold of the form \( \mathcal{M} = ([R^{m|n}], Q) \). Let \((x_1, \ldots, x_m; x_{m+1}, \ldots, x_{m+n})\) be coordinate functions on \( R^{m|n} \), and write \( Q = \sum_k Q_k(x) \frac{\partial}{\partial x_k} \). Then the Atiyah 1-cocycle associated to the trivial connection \( \nabla = \frac{\partial}{\partial x_k} \) is given by

\[
\text{At}_{\mathcal{M}} \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = (-1)^{|x_i|+|x_j|} \sum_k \frac{\partial^2 Q_k}{\partial x_i \partial x_j} \frac{\partial}{\partial x_k}.
\]

As we can see from (3), the Atiyah 1-cocycle \( \text{At}_{\mathcal{M}} \) includes the information about the homological vector field of second-order and higher.

4. **Atiyah class and homotopy Lie algebras**

Let \( \mathcal{M} \) be a graded manifold. A \((1,2)\)-tensor of degree \( k \) on \( \mathcal{M} \) is a \( C \)-linear map \( \alpha : \mathfrak{X}(\mathcal{M}) \otimes \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M}) \) such that \( |\alpha(X,Y)| = |X| + |Y| + k \) and

\[
\alpha(fX,Y) = (-1)^{|f|+|X|} f\alpha(X,Y) = (-1)^{|f|+|X|} \alpha(X,fY).
\]

We denote the space of \((1,2)\)-tensors of degree \( k \) by \( \mathcal{T}^{1,2}_k(\mathcal{M}) \), and the space of all \((1,2)\)-tensors by \( \mathcal{T}^{1,2}(\mathcal{M}) = \bigoplus_k \mathcal{T}^{1,2}_k(\mathcal{M}) \).

The torsion of an affine connection \( \nabla \) is given by

\[
T(X,Y) = \nabla_X Y - (-1)^{|X||Y|} \nabla_Y X - [X,Y].
\]

The torsion is an element in \( \mathcal{T}^{1,2}_0(\mathcal{M}) \). Given any affine connection, one can define its opposite affine connection \( \nabla^{op} \), given by

\[
\nabla^{op}_X Y = \nabla_X Y - T(X,Y) = [X,Y] + (-1)^{|X||Y|} \nabla_Y X.
\]

The average \( \frac{1}{2}(\nabla + \nabla^{op}) \) is a torsion-free affine connection. This shows that torsion-free affine connections always exist on graded manifolds.
In this section, we show that, as in the classical situation considered by Kapranov in [3, 8], the Atiyah 1-cocycle of a dg-manifold gives rise to an interesting homotopy Lie algebra. As in the last section, let $(\mathcal{M}, Q)$ be a dg-manifold and let $\nabla$ be an affine connection on $\mathcal{M}$. The following can be easily verified by direct computation.

1. The anti-symmetrization of the Atiyah 1-cocycle $At_{\mathcal{M}}$ is equal to $L_Q T$, so $At_{\mathcal{M}}$ is graded antisymmetric up to an exact term. In particular, if $\nabla$ is torsion-free, we have

$$At_{\mathcal{M}}(X, Y) = (-1)^{|X||Y|} At_{\mathcal{M}}(Y, X).$$

2. The degree 1 + $|X|$ operator $At_{\mathcal{M}}(X, -)$ need not be a derivation of the degree +1 ‘product’ $\mathcal{X}(\mathcal{M}) \otimes_{\mathcal{C}} \mathcal{X}(\mathcal{M}) \xrightarrow{At_{\mathcal{M}}} \mathcal{X}(\mathcal{M})$. However, the Jacobiator

$$(X, Y, Z) \mapsto At_{\mathcal{M}}(X, At_{\mathcal{M}}(Y, Z)) - \{(-1)^{|X|+1} At_{\mathcal{M}}( At_{\mathcal{M}}(X, Y), Z) + (-1)^{(|X|+1)(|Y|+1)} At_{\mathcal{M}}( Y, At_{\mathcal{M}}(X, Z))\},$$

of $At_{\mathcal{M}}$, which vanishes precisely when $At_{\mathcal{M}}(X, -)$ is a derivation of $At_{\mathcal{M}}$, is equal to $\pm L_Q(\nabla At_{\mathcal{M}})$. Hence $At_{\mathcal{M}}$ satisfies the graded Jacobi identity up to the exact term $L_Q(\nabla At_{\mathcal{M}})$.

Armed with this motivation, we can now state the main result of this note.

**Theorem 4.1.** Let $(\mathcal{M}, Q)$ be a dg-manifold and let $\nabla$ be a torsion-free affine connection on $\mathcal{M}$. There exists a sequence $(\lambda_k)_{k \geq 2}$ of maps $\lambda_k \in \text{Hom}(S^k(T\mathcal{M}), T\mathcal{M}[-1])$ starting with $\lambda_2 := At_{\mathcal{M}} \in \text{Hom}(S^2(T\mathcal{M}), T\mathcal{M}[-1])$ which, together with $\lambda_1 := L_Q : \mathcal{X}(\mathcal{M}) \rightarrow \mathcal{X}(\mathcal{M})$, satisfy the $L_{\infty}[1]$-algebra axioms. As a consequence, the space of vector fields $\mathcal{X}(\mathcal{M})$ on a dg-manifold $(\mathcal{M}, Q)$ admits an $L_{\infty}[1]$-algebra structure with the Lie derivative $L_Q$ as unary bracket $\lambda_1$ and the Atiyah cocycle $At_{\mathcal{M}}$ as binary bracket $\lambda_2$.

To prove Theorem 4.1, we introduce a Poincaré–Birkhoff–Witt map for graded manifolds.

It was shown in [5] that every torsion-free affine connection $\nabla$ on a smooth manifold $M$ determines an isomorphism of coalgebras (over $C^\infty(M)$)

$$\text{pbw}^\nabla : \Gamma(S(T\mathcal{M})) \xrightarrow{\cong} D(M),$$

(6)
called the Poincaré–Birkhoff–Witt (PBW) map. Here $D(M)$ denotes the space of differential operators on $M$.

Geometrically, an affine connection $\nabla$ induces an exponential map $TM \rightarrow M \times M$, which is a well-defined diffeomorphism from a neighborhood of the zero section of $TM$ to a neighborhood of the diagonal $\Delta(M)$ of $M \times M$. Sections of $S(T\mathcal{M})$ can be viewed as fiberwise distributions on $TM$ supported on the zero section, while $D(M)$ can be viewed as the space of source-fiberwise distributions on $M \times M$ supported on the diagonal $\Delta(M)$. The map $\text{pbw}^\nabla : \Gamma(S(T\mathcal{M})) \rightarrow D(M)$ is simply the push-forward of fiberwise distributions through the exponential map $\exp^\nabla : TM \rightarrow M \times M$ and is clearly an isomorphism of coalgebras over $C^\infty(M)$.

Even though, for a graded manifold $\mathcal{M}$ endowed with a torsion-free affine connection $\nabla$, we lack an exponential map $\exp^\nabla : TM \rightarrow M \times M$, a PBW map can still be defined purely algebraically thanks to the iteration formula introduced in [5].
Lemma 4.2. Let $\mathcal{M}$ be a $\mathbb{Z}$-graded manifold and let $\nabla$ be a torsion-free affine connection on $\mathcal{M}$. The Poincaré-Birkhoff-Witt map inductively defined by the relations

\[
\begin{align*}
pbw^{\nabla}(f) &= f, \quad \forall f \in C^\infty(\mathcal{M}); \\
pbw^{\nabla}(X) &= X, \quad \forall X \in \mathfrak{X}(\mathcal{M});
\end{align*}
\]

and

\[
pbw^{\nabla}(X_0 \circ \cdots \circ X_n) = \frac{1}{n+1} \sum_{k=0}^n (-1)^{|X_k|} \sum_{|X_k|+\cdots+|X_{k-1}|} \{ X_k \cdot pbw^{\nabla}(X_0 \circ \cdots \circ \hat{X}_k \circ \cdots \circ X_n) \\
&\quad - pbw^{\nabla}(\nabla X_k(X_0 \circ \cdots \circ \hat{X}_k \circ \cdots \circ X_n)) \},
\]

for all $n \in \mathbb{N}$ and $X_0, \ldots, X_n \in \mathfrak{X}(\mathcal{M})$, establishes an isomorphism

\[
(pbw^{\nabla}: \Gamma(S(T\mathcal{M})) \xrightarrow{\cong} D(\mathcal{M}).)
\]

of coalgebras over $C^\infty(\mathcal{M})$.

Now assume that $(\mathcal{M}, Q)$ is a dg-manifold. The homological vector field $Q$ induces a degree +1 coderivation of $D(\mathcal{M})$ defined by the Lie derivative:

\[
L_Q(X_1 \cdots X_n) = \sum_{k=1}^n (-1)^{|X_1|+\cdots+|X_{k-1}|} X_1 \cdots X_{k-1} [Q, X_k] X_{k+1} \cdots X_n.
\]

Now using the isomorphism of coalgebras $pbw^{\nabla}$ as in Eq. (7) to transfer $L_Q$ from $D(\mathcal{M})$ to $\Gamma(S(T\mathcal{M}))$, we obtain $\delta := (pbw^{\nabla})^{-1} \circ L_Q \circ pbw^{\nabla}$, a degree 1 coderivation of $\Gamma(S(T\mathcal{M}))$. Finally, dualizing $\delta$, we obtain an operator

\[
D : \Gamma(\hat{S}(T^\vee \mathcal{M})) \to \Gamma(\hat{S}(T^\vee \mathcal{M}))
\]

as

\[
\Gamma(\hat{S}(T^\vee \mathcal{M})) \cong \text{Hom}_{C^\infty(\mathcal{M})}(\Gamma(S(T\mathcal{M})), C^\infty(\mathcal{M})).
\]

Theorem 4.3. Let $(\mathcal{M}, Q)$ be a dg-manifold and let $\nabla$ be a torsion-free affine connection on $\mathcal{M}$.

1. The operator $D$, dual to $(pbw^{\nabla})^{-1} \circ L_Q \circ pbw^{\nabla}$, is a degree +1 derivation of the graded algebra $\Gamma(\hat{S}(T^\vee \mathcal{M}))$ satisfying $D^2 = 0$.

2. There exists a sequence $\{R_k\}_{k \geq 2}$ of homomorphisms $R_k \in \text{Hom}(S^k T^\vee \mathcal{M}, T^\vee \mathcal{M})[-1]$, whose first term $R_2$ is precisely the Atiyah 1-cocycle $\text{At}_\mathcal{M}$, such that $D = L_Q + \sum_{k=2}^{\infty} R_k$, where $R_k$ denotes the $C^\infty(\mathcal{M})$-linear operator on $\Gamma(\hat{S}(T^\vee \mathcal{M}))$ corresponding to $R_k$.

Finally we note that Theorem 4.1 is a consequence of Theorem 4.3.

Acknowledgements

We would like to thank several institutions for their hospitality while work on this project was being done: Penn State University (Mehta), and Université Paris Diderot (Xu). We also wish to thank Hsuan-Yi Liao, Dmitry Roytenberg and Boris Shoikhet for inspiring discussions.

\[\footnote{We would like to thank Hsuan-Yi Liao for correcting a sign error in the inductive formula defining the map $pbw^{\nabla}$.}\]
REFERENCES


DEPARTMENT OF MATHEMATICS & STATISTICS, SMITH COLLEGE, 44 COLLEGE LANE, NORTHAMPTON, MA 01063

E-mail address: rmehta@smith.edu

DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PA 16802

E-mail address: stienon@psu.edu

E-mail address: ping@math.psu.edu