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π/2-Angle Yao Graphs are Spanners

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Abstract

We show that the Yao graph $Y_4$ in the $L_2$ metric is a spanner with stretch factor $8(29+23\sqrt{2})$. Enroute to this, we also show that the Yao graph $Y_4^\infty$ in the $L_\infty$ metric is a planar spanner with stretch factor 8.

1 Introduction

Let $V$ be a finite set of points in the plane and let $G = (V, E)$ be the complete Euclidean graph on $V$. We will refer to the points in $V$ as nodes, to distinguish them from other points in the plane. The Yao graph [6] with an integer parameter $k > 0$, denoted $Y_k$, is defined as follows. At each node $u \in V$, any $k$ equally-separated rays originating at $u$ define $k$ cones. In each cone, pick a shortest edge $uv$, if there is one, and add to $Y_k$ the directed edge $\vec{uv}$. Ties are broken arbitrarily. Most of the time we ignore the direction of an edge $uv$; we refer to the directed version $\vec{uv}$ of $uv$ only when its origin ($u$) is important and unclear from the context. We will distinguish between $Y_k$, the Yao graph in the Euclidean $L_2$ metric, and $Y_k^\infty$, the Yao graph in the $L_\infty$ metric. Unlike $Y_k$ however, in constructing $Y_k^\infty$ ties are broken by always selecting the most counterclockwise edge; the reason for this choice will become clear in Section 2.

For a given subgraph $H \subseteq G$ and a fixed $t \geq 1$, $H$ is called a $t$-spanner for $G$ if, for any two nodes $u, v \in V$, the shortest path in $H$ from $u$ to $v$ is no longer than $t$ times the length of $uv$. The value $t$ is called the dilation or the stretch factor of $H$. If $t$ is constant, then $H$ is called a length spanner, or simply a spanner.

The class of graphs $Y_k$ has been much studied. Bose et al. [1] showed that, for $k \geq 9$, $Y_k$ is a spanner with stretch factor $\cos \frac{\pi}{k} - \sin \frac{\pi}{k}$. In the appendix, we improve the stretch factor and show that, in fact, $Y_k$ is a spanner for any $k \geq 7$. Recently, Molla [4] showed that $Y_2$ and $Y_3$ are not spanners.

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spanners, and that $Y_4$ is a spanner with stretch factor $4(2 + \sqrt{2})$, for the special case when the nodes in $V$ are in convex position (see also [2]). The authors conjectured that $Y_4$ is a spanner for arbitrary point sets. In this paper, we settle their conjecture and prove that $Y_4$ is a spanner with stretch factor $8(29 + 23\sqrt{2})$.

The paper is organized as follows. In Section 2, we prove that the graph $Y_\infty^4$ is a spanner with stretch factor 8. In Section 3, we prove, in a sequence of Lemmas, several properties for the graph $Y_4$. Finally, in Section 4, we use the properties of Section 3 to prove that for every edge $ab$ in $Y_\infty^4$, there exists a path between $a$ and $b$ in $Y_4$, whose length is not much more than the Euclidean distance between $a$ and $b$. By combining this with the result of Section 2, it follows that $Y_4$ is a spanner.

## 2 $Y_\infty^4$: in the $L_\infty$ Metric

In this section we focus on $Y_\infty^4$, which has a nicer structure compared to $Y_4$. First we prove that $Y_\infty^4$ is planar. Then we use this property to show that $Y_\infty^4$ is an 8-spanner. To be more precise, we prove that for any two nodes $a$ and $b$, the graph $Y_\infty^4$ contains a path between $a$ and $b$ whose length (in the $L_\infty$-metric) is at most $8|ab|_\infty$.

We need a few definitions. We say that two edges $ab$ and $cd$ properly cross (or cross, for short) if they share a point other than an endpoint ($a$, $b$, $c$ or $d$); we say that $ab$ and $cd$ intersect if they share a point (either an interior point or an endpoint). Let $Q_1(a)$, $Q_2(a)$, $Q_3(a)$ and $Q_4(a)$ be the four quadrants at $a$, as in Figure 1a. Let $P_i(a)$ be the path that starts at point $a$ and follows the directed Yao edges in quadrant $Q_i$. Let $P_i(a, b)$ be the subpath of $P_i(a)$ that starts at $a$ and ends at $b$. Let $|ab|_\infty$ be the $L_\infty$ distance between $a$ and $b$. Let $sp(a, b)$ denote a shortest path in $Y_\infty^4$ between $a$ and $b$. Let $S(a, b)$ denote the open square with corner $a$ whose boundary contains $b$, and let $\partial S(a, b)$ denote the boundary of $S(a, b)$. These definitions are illustrated in Figure 1a. For a node $a \in V$, let $x(a)$ denote the $x$-coordinate of $a$ and $y(a)$ denote the $y$-coordinate of $a$.

**Lemma 1** $Y_\infty^4$ is planar.

**Proof.** The proof is by contradiction. Assume the opposite. Then there are two edges $\overrightarrow{ab}, \overrightarrow{cd} \in Y_\infty^4$ that cross each other. Since $\overrightarrow{ab} \in Y_\infty^4$, $S(a, b)$ must be empty of nodes in $V$, and similarly for $S(c, d)$. Let $j$ be the intersection point between $ab$ and $cd$. Then $j \in S(a, b) \cap S(c, d)$, meaning that $S(a, b)$

---

Figure 1: (a) Definitions: $Q_i(a)$, $P_i(a)$ and $S(a, b)$. (b) Lemma 1 $ab$ and $cd$ cannot cross.
and \( S(c, d) \) must overlap. However, neither square may contain \( a, b, c \) or \( d \). It follows that \( S(a, b) \) and \( S(c, d) \) coincide, meaning that \( c \) and \( d \) lie on \( \partial S(a, b) \) (see Figure 1). Since \( cd \) intersects \( ab \), \( c \) and \( d \) must lie on opposite sides of \( ab \). Thus either \( ac \) or \( ad \) lies counterclockwise from \( ab \). Assume without loss of generality that \( ac \) lies counterclockwise from \( ab \); the other case is identical. Because \( S(a, c) \) coincides with \( S(a, b) \), we have that \( |ac|_{\infty} = |ab|_{\infty} \). In this case however, \( Y_{4}^{\infty} \) would break the tie between \( ac \) and \( ab \) by selecting the most counterclockwise edge, which is \( ac \). This contradicts the fact that \( \overrightarrow{ab} \in Y_{4}^{\infty} \).

It can be easily shown that each face of \( Y_{4}^{\infty} \) is either a triangle or a quadrilateral (except for the outer face). We skip this proof however, since we do not make use of this property in this paper.

**Theorem 1** \( Y_{4}^{\infty} \) is an 8-spanner.

**Proof.** We show that, for any pair of points \( a, b \in V \), \( |sp(a, b)|_{\infty} < 8|ab|_{\infty} \). The proof is by induction on the pairwise distance between the points in \( V \). Assume without loss of generality that \( b \in Q_{1}(a) \), and \( |ab|_{\infty} = |x(b) - x(a)| \). Consider the case in which \( ab \) is a closest pair of points in \( V \) (the base case for our induction). If \( ab \in Y_{4}^{\infty} \), then \( |sp(a, b)|_{\infty} = |ab|_{\infty} \). Otherwise, there must be \( ac \in Y_{4}^{\infty} \), with \( |ac|_{\infty} = |ab|_{\infty} \). But then \( |bc|_{\infty} < |ab|_{\infty} \) (see Figure 2a), a contradiction.

Assume now that the inductive hypothesis holds for all pairs of points closer than \( |ab|_{\infty} \). If \( ab \in Y_{4}^{\infty} \), then \( |sp(a, b)|_{\infty} = |ab|_{\infty} \) and the proof is finished. If \( ab \notin Y_{4}^{\infty} \), then the square \( S(a, b) \) must be nonempty.

Let \( A \) be the rectangle \( ab'ba' \) as in Figure 2b, where \( ba' \) and \( bb' \) are parallel to the diagonals of \( S \). If \( A \) is nonempty, then we can use induction to prove that \( |sp(a, b)|_{\infty} <= 8|ab|_{\infty} \) as follows. Pick \( c \in A \) arbitrary. Then \( |ac|_{\infty} + |cb|_{\infty} = |x(c) - x(a)| + |x(b) - x(c)| = |ab|_{\infty} \), and by the inductive hypothesis \( sp(a, c) \oplus sp(c, b) \) is a path in \( Y_{4}^{\infty} \) no longer than \( 8|ac|_{\infty} + 8|cb|_{\infty} = 8|ab|_{\infty} \); here \( \oplus \) represents the concatenation operator. Assume now that \( A \) is empty. Let \( c \) be at the intersection between the line supporting \( ba' \) and the vertical line through \( a \) (see Figure 2b). We discuss two cases, depending on whether \( \triangle abc \) is empty of points or not.

**Case 1:** \( \triangle abc \) is empty of points. Let \( ad \in P_{1}(a) \). We show that \( P_{4}(d) \) cannot contain an edge crossing \( ab \). Assume the opposite, and let \( st \in P_{4}(d) \) cross \( ab \). Since \( \triangle abc \) is empty, \( s \) must lie

![Figure 2: (a) Base case. (b) \( \triangle abc \) empty (c) \( \triangle abc \) non-empty, \( P_{ar} \cap P_{2}(b) = \{ e \} \) (d) \( \triangle abc \) non-empty, \( P_{ar} \cap P_{2}(b) = \emptyset \), \( e \) above \( r \) (e) \( \triangle abc \) non-empty, \( P_{ar} \cap P_{2}(b) = \emptyset \), \( e \) below \( r \).](image-url)
Then in this case, it is unclear whether the path $Y$ is a path in Figure 2c). Let $P$ be the clockwise neighbor of $ab$ of length $d$ intersecting in a point $j$. If $e$ belongs to $P$ and $d$ lies outside $\triangle abc$ (see Figure 2d), then $P_{ab} = P_{1}(a, i) \oplus P_{2}(r, i) \oplus P_{3}(d, j) \oplus P_{2}(b, j)$ extending from $a$ to $b$ in $Y_{4}^{\infty}$. If $|rd|_{\infty} = |x(d) - x(r)|$, then $|rd|_{\infty} < |x(b) - x(a)| = |ab|_{\infty}$, and the path $P_{ab}$ has length

$$|P_{ab}|_{\infty} \leq 2|y(d) - y(a)| + |ab|_{\infty} < 7|ab|_{\infty}. \quad (3)$$

In the above, we used the fact that $|y(d) - y(a)| = |y(d) - y(r)| + |y(r) - y(a)| < |ab|_{\infty} + 2|ab|_{\infty}$. Suppose now that

$$|rd|_{\infty} = |y(d) - y(r)|. \quad (4)$$

In this case, it is unclear whether the path $P_{ab}$ defined by (2) is short, since $rd$ can be arbitrarily long compared to $ab$. Let $e$ be the clockwise neighbor of $d$ along the path $P_{ab}$ ($e$ and $b$ may coincide). Then $e$ lies below $d$, and either $de \in P_{3}(d)$, or $ed \in P_{2}(e)$ (or both).

1. If $e$ lies above $r$, or at the same level as $r$ (i.e., $e \in Q_{1}(r)$, as in Figure 2d), then

$$|y(e) - y(r)| < |y(d) - y(r)| \quad (5)$$
We have established that nodes $a$ and $b$ whose boundary contains $D$ denote the boundary of Fischer et al. [3] show that $\oplus$ Recall that a few definitions. In this section we establish basic properties of $Y$ if two edges in $a(b)$, then
\[ |ed|_\infty \geq |y(d) - y(e)| \geq |y(d) - y(r)| = |rd|_\infty. \] (6)
Assume first that $ed \in P_2(e)$, or $|ed|_\infty = |x(e) - x(d)|$. In either case,
\[ |ed|_\infty \leq |er|_\infty < 2|ab|_\infty. \]
This along with inequality (6) shows that $|rd|_\infty < 2|ab|_\infty$. Substituting this upper bound in (2), we get
\[ |P_{ab}|_\infty \leq 2|y(d) - y(a)| + 2|ab|_\infty < 8|ab|_\infty. \]
Assume now that $ed \notin P_2(e)$, and $|ed|_\infty = |y(e) - y(d)|$. Then $ee' \in P_2(e)$ cannot go above $d$ (otherwise $|ed|_\infty < |ee'|_\infty$, contradicting $ee' \in P_2(e)$). This along with the fact $de \in P_4(d)$ implies that $P_2(e)$ intersects $P_{ar}$ in a point $k$. Redefine
\[ P_{ab} = P_{ar}(a,k) \oplus P_2(e,k) \oplus P_4(e,j) \oplus P_2(b,j). \]
Then $P_{ab}$ is a path in $Y_4^\infty$ from $a$ to $b$ of length
\[ |P_{ab}| \leq 2|y(r) - y(a)| + |ab|_\infty \leq 5|ab|_\infty. \]
We have established that $|sp(a,b)|_\infty \leq |P_{ab}|_\infty < 8|ab|_\infty$. This concludes the proof.

This theorem will be employed in Section 4.

3 $Y_4$: in the $L_2$ Metric

In this section we establish basic properties of $Y_4$. The ultimate goal of this section is to show that, if two edges in $Y_4$ cross, there is a short path between their endpoints (Lemma 8). We begin with a few definitions.

Let $Q(a,b)$ denote the infinite quadrant with origin at $a$ that contains $b$. For a pair of nodes $a,b \in V$, define recursively a directed path $P(a \rightarrow b)$ from $a$ to $b$ in $Y_4$ as follows. If $a = b$, then $P(a \rightarrow b) = \text{null}$. If $a \neq b$, there must exist $\overrightarrow{ac} \in Y_4$ that lies in $Q(a,b)$. In this case, define
\[ P(a \rightarrow b) = \overrightarrow{ac} \oplus P(c \rightarrow b). \]
Recall that $\oplus$ represents the concatenation operator. This definition is illustrated in Figure 3a. Fischer et al. [3] show that $P(a \rightarrow b)$ is well defined and lies entirely inside the square centered at $b$ whose boundary contains $a$.

For any node $a \in V$, let $D(a,r)$ denote the open disk centered at $a$ of radius $r$, and let $\partial D(a,r)$ denote the boundary of $D(a,r)$. Let $D[a,r] = D(a,r) \cup \partial D(a,r)$. For any path $P$ and any pair of nodes $a$ and $b$ on $P$, let $P[a,b]$ denote the subpath of $P$ that starts at $a$ and ends at $b$. Let $R(a,b)$ denote the closed rectangle with diagonal $ab$. 

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2. If $e$ lies below $r$ (as in Figure 2a), then
\[ |ed|_\infty \geq |y(d) - y(e)| \geq |y(d) - y(r)| = |rd|_\infty. \] (6)
Assume first that $ed \in P_2(e)$, or $|ed|_\infty = |x(e) - x(d)|$. In either case,
\[ |ed|_\infty \leq |er|_\infty < 2|ab|_\infty. \]
This along with inequality (6) shows that $|rd|_\infty < 2|ab|_\infty$. Substituting this upper bound in (2), we get
\[ |P_{ab}|_\infty \leq 2|y(d) - y(a)| + 2|ab|_\infty < 8|ab|_\infty. \]
Assume now that $ed \notin P_2(e)$, and $|ed|_\infty = |y(e) - y(d)|$. Then $ee' \in P_2(e)$ cannot go above $d$ (otherwise $|ed|_\infty < |ee'|_\infty$, contradicting $ee' \in P_2(e)$). This along with the fact $de \in P_4(d)$ implies that $P_2(e)$ intersects $P_{ar}$ in a point $k$. Redefine
\[ P_{ab} = P_{ar}(a,k) \oplus P_2(e,k) \oplus P_4(e,j) \oplus P_2(b,j). \]
Then $P_{ab}$ is a path in $Y_4^\infty$ from $a$ to $b$ of length
\[ |P_{ab}| \leq 2|y(r) - y(a)| + |ab|_\infty \leq 5|ab|_\infty. \]
We have established that $|sp(a,b)|_\infty \leq |P_{ab}|_\infty < 8|ab|_\infty$. This concludes the proof. 

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\[ P(a \rightarrow b) = \overrightarrow{ac} \oplus P(c \rightarrow b). \]
Recall that $\oplus$ represents the concatenation operator. This definition is illustrated in Figure 3a. Fischer et al. [3] show that $P(a \rightarrow b)$ is well defined and lies entirely inside the square centered at $b$ whose boundary contains $a$.

For any node $a \in V$, let $D(a,r)$ denote the open disk centered at $a$ of radius $r$, and let $\partial D(a,r)$ denote the boundary of $D(a,r)$. Let $D[a,r] = D(a,r) \cup \partial D(a,r)$. For any path $P$ and any pair of nodes $a$ and $b$ on $P$, let $P[a,b]$ denote the subpath of $P$ that starts at $a$ and ends at $b$. Let $R(a,b)$ denote the closed rectangle with diagonal $ab$. 

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Figure 3: Definitions. (a) \( Q(a,b) \) and \( P(a \rightarrow b) \). (b) \( R(a \rightarrow b) \).

For a fixed pair of nodes \( a,b \in V \), define a path \( R(a \rightarrow b) \) as follows. Let \( e \in V \) be the first node along \( P(a \rightarrow b) \) that is not strictly interior to \( R(a,b) \). Then \( R(a \rightarrow b) \) is the subpath of \( P(a \rightarrow b) \) that extends between \( a \) and \( e \). In other words, \( R(a \rightarrow b) \) is the path that follows the \( Y_4 \) edges pointing towards \( b \), truncated as soon as it leaves the rectangle with diagonal \( ab \), or as it reaches \( b \). Formally,

\[
R(a \rightarrow b) = P(a \rightarrow b)[a,e]
\]

This definition is illustrated in Figure 3b.

Our proofs will make use of the following two propositions.

**Proposition 1** The sum of the lengths of crossing diagonals of a nondegenerate (necessarily convex) quadrilateral \( abcd \) is strictly greater than the sum of the lengths of either pair of opposite sides:

\[
|ac| + |bd| > |ab| + |cd|
\]

\[
|ac| + |bd| > |bc| + |da|
\]

This can be proved by partitioning the diagonals into two pieces each at their intersection point, and then applying the triangle inequality twice.

**Proposition 2** For any triangle \( \triangle abc \), the following inequalities hold:

\[
|ac|^2 \begin{cases} 
< |ab|^2 + |bc|^2, & \text{if } \angle abc < \pi/2 \\
= |ab|^2 + |bc|^2, & \text{if } \angle abc = \pi/2 \\
> |ab|^2 + |bc|^2, & \text{if } \angle abc > \pi/2 
\end{cases}
\]

This proposition follows immediately from the Law of Cosines applied to triangle \( \triangle abc \).

**Lemma 2** For each pair of nodes \( a,b \in V \),

\[
|P_R(a \rightarrow b)| \leq |ab|\sqrt{2}
\]  

(7)

Furthermore, each edge of \( P_R(a \rightarrow b) \) is no longer than \( |ab| \).
Proof. Let c be one of the two corners of \( R(a, b) \), other than a and b. Let \( \overrightarrow{de} \in \mathcal{P}_R(a \to b) \) be the last edge on \( \mathcal{P}_R(a \to b) \), which necessarily intersects \( \partial R(a, b) \) (note that it is possible that \( e = b \)). Refer to Figure 3b. Then \( |de| \leq |db| \), otherwise \( \overrightarrow{de} \) could not be in \( Y_4 \). Since \( db \) lies in the rectangle with diagonal \( ab \), we have that \( |db| \leq |ab| \), and similarly for each edge on \( \mathcal{P}_R(a \to b) \). This establishes the latter claim of the lemma. For the first claim of the lemma, let

\[
p = \mathcal{P}_R(a \to b)[a, d] \oplus db
\]

Since \( |de| \leq |db| \), we have that \( |\mathcal{P}_R(a \to b)| \leq |p| \). Since p lies entirely inside \( R(a, b) \) and consists of edges pointing towards b, we have that p is an \( xy \)-monotone path. It follows that \( |p| \leq |ac| + |cb| \).

We now show that \( |ac| + |cb| \leq |ab| \sqrt{2} \), thus establishing the first claim of the lemma.

Let \( x = |ac| \) and \( y = |cb| \). Then the inequality \( |ac| + |cb| \leq |ab| \sqrt{2} \) can be written as \( x + y \leq \sqrt{2x^2 + 2y^2} \), which is equivalent to \((x - y)^2 \geq 0\). This latter inequality obviously holds, completing the proof of the lemma.

Lemma 3 Let \( a, b, c, d \in V \) be four disjoint nodes such that \( \overrightarrow{ab}, \overrightarrow{cd} \in Y_4 \), \( b \in Q_i(a) \) and \( d \in Q_i(c) \), for some \( i \in \{1, 2, 3, 4\} \). Then \( ab \) and \( cd \) cannot cross each other.

Proof. We may assume without loss of generality that \( i = 1 \) and \( c \) is to the left of \( a \). The proof is by contradiction. Assume that \( ab \) and \( cd \) cross each other. Let \( j \) be the intersection point between \( ab \) and \( cd \) (see Figure 4a). Since \( j \in Q_1(a) \cap Q_1(c) \), it follows that \( d \in Q_2(a) \) and \( b \in Q_1(c) \). Thus \( |ab| \leq |ad| \), because otherwise, \( ab \) cannot be in \( Y_4 \). By Proposition applied to the quadrilateral \( abcd \),

\[
|ad| + |cb| < |ab| + |cd|
\]

This along with the fact that \( |ab| \leq |ad| \) implies that \( |cb| < |cd| \), contradicting the fact that \( \overrightarrow{cd} \in Y_4 \).

The next four lemmas (4–8) each concern a pair of crossing \( Y_4 \) edges, culminating (in Lemma 8) in the conclusion that there is a short path in \( Y_4 \) between a pair of endpoints of those edges.

Lemma 4 Let \( a, b, c \) and \( d \) be four disjoint nodes in \( V \) such that \( \overrightarrow{ab}, \overrightarrow{cd} \in Y_4 \), and \( ab \) crosses \( cd \). Then the following are true: (i) the ratio between the shortest side and the longer diagonal of the quadrilateral \( acbd \) is no greater than \( 1/\sqrt{2} \), and (ii) the shortest side of the quadrilateral \( acbd \) is strictly shorter than either diagonal.

Proof. The first part of the lemma is a well-known fact that holds for any quadrilateral (see [5], for instance). For the second part of the lemma, let \( ab \) be the shorter of the diagonals of \( acbd \), and assume without loss of generality that \( \overrightarrow{bd} \in Q_1(a) \). Imagine two disks \( D_a = D(a, |ab|) \) and \( D_b = D(b, |ab|) \), as in Figure 4b. If either \( c \) or \( d \) belongs to \( D_a \cup D_b \), then the lemma follows: a shortest quadrilateral edge is shorter than \( |ab| \).

So suppose that neither \( c \) nor \( d \) lies in \( D_a \cup D_b \). In this case, we use the fact that \( cd \) crosses \( ab \) to show that \( \overrightarrow{cd} \) cannot be an edge in \( Y_4 \). Define the following regions (see Figure 4b):

\[
\begin{align*}
R_1 &= (Q_1(a) \cap Q_2(b)) \backslash (D_a \cup D_b) \\
R_2 &= (Q_2(a) \cap Q_3(b)) \backslash (D_a \cup D_b) \\
R_3 &= (Q_4(a) \cap Q_3(b)) \backslash (D_a \cup D_b) \\
R_4 &= (Q_1(a) \cap Q_4(b)) \backslash (D_a \cup D_b)
\end{align*}
\]
If the node $c$ is not inside any of the regions $R_i$, for $i \in \{1, 2, 3, 4\}$, then the nodes $a$ and $b$ are in the same quadrant of $c$ as $d$. In this case, note that either $\angle cad > \pi/2$ or $\angle cbd > \pi/2$, which implies that either $|ca|$ or $|cb|$ is strictly smaller than $|cd|$. These together show that $\overrightarrow{cd} \notin Y_4$.

So assume that $c$ is in $R_i$ for some $i \in \{1, 2, 3, 4\}$. In this situation, the node $d$ must lie in the region $R_j$, with $j = (i + 2) \mod 4$ (with the understanding that $R_0 = R_4$), because otherwise, (i) $a$ and $d$ are in the same quadrant of $c$ and $|ca| < |cd|$ or (ii) $b$ and $d$ are in the same quadrant of $c$ and $|cb| < |cd|$. Either case contradicts the fact $\overrightarrow{cd} \in Y_4$. Consider now the case $c \in R_1$ and $d \in R_3$; the other cases are treated similarly. Let $i$ and $j$ be the intersection points between $D_a$ and the vertical line through $a$. Similarly, let $k$ and $\ell$ be the intersection points between $D_b$ and the vertical line through $b$ (see Figure 4b). Since $ij$ is a diameter of $D_a$, we have that $\angle ibj = \pi/2$ and similarly $\angle kal = \pi/2$. Also note that $\angle cbd \geq \angle ibj = \pi/2$, meaning that $|cd| > |cb|$. Similarly, $\angle cad \geq \angle kal = \pi/2$, meaning that $|cd| > |ca|$. These along with the fact that at least one of $a$ and $b$ is in the same quadrant for $c$ as $d$, imply that $\overrightarrow{cd} \notin Y_4$. This completes the proof.

**Lemma 5** Let $a, b, c, d$ be four distinct nodes in $V$, with $c \in Q_1(a)$, such that

(a) $\overrightarrow{ab} \in Q_1(a)$ and $\overrightarrow{cd} \in Q_2(c)$ are two edges in $Y_4$ that cross each other.

(b) $ad$ is a shortest side of the quadrilateral $abcd$.

Then $\mathcal{P}_R(a \to d)$ and $\mathcal{P}_R(d \to a)$ have a nonempty intersection.

**Proof.** The proof consists of two parts showing that the following claims hold: (i) $d \in Q_2(a)$ and (ii) $\mathcal{P}_R(d \to a)$ does not cross $ab$.

Before we prove these two claims, let us argue that they are sufficient to prove the lemma. Lemma 3 and (i) imply that $\mathcal{P}_R(a \to d)$ cannot cross $cd$. As a result, $\mathcal{P}_R(a \to d)$ intersects the left side of the rectangle $R(d, a)$. Consider the last edge $\overrightarrow{x'y'}$ of the path $\mathcal{P}_R(d \to a)$. If this edge crosses the right side of $R(a, d)$, then (ii) implies that $y'$ is in the wedge bounded by $ab$ and the upwards vertical ray starting at $a$; this implies that $|ay'| < |ab|$, contradicting the fact that $\overrightarrow{ab}$ is an edge in $Y_4$. Therefore, $\overrightarrow{x'y'}$ intersects the bottom side of $R(d, a)$, and the lemma follows (see Figure 5b).

To prove the first claim (i), we observe that the assumptions in the lemma imply that $d \in Q_1(a) \cup Q_2(a)$. Therefore, it suffices to prove that $d$ is not in $Q_1(a)$. Assume to the contrary that
d ∈ Q_1(a). Since c ∈ Q_1(a), it must be that b ∈ Q_2(c); otherwise, ∠acb ≥ π/2, which implies |ab| > |ac|, contradicting the fact that \( \overrightarrow{ab} \in Y_4 \). Let i and j be the intersection points between cd and \( \partial D(a,|ab|) \), where i is to the left of j. Since ∠dbc ≥ ∠ibj > π/2, we have |cb| < |cd|. This, together with the fact that b and d are in the same quadrant Q_2(c), contradicts the assumption that \( \overrightarrow{cd} \) is an edge in Y_4. This completes the proof of claim (i).

Next we prove claim (ii) by contradiction. Thus, we assume that there is an edge \( \overrightarrow{xy} \) on the path \( P_R(d \rightarrow a) \) that crosses ab. Then necessarily \( x \in R(a,d) \) and \( y \in Q_1(a) \cup Q_4(a) \). If \( y \in Q_4(a) \), then \( ∠xay > π/2 \), meaning that \( |xy| > |xa| \), a contradiction to the fact that \( \overrightarrow{xy} \in Y_4 \). Thus, it must be that \( y \in Q_1(a) \), as in Figure 5a. This implies that \(|ab| ≤ |ay|\), because \( \overrightarrow{ab} \in Y_4 \).

![Figure 5](image)

Figure 5: (a) Lemma 5: \( xy \in P_R(d \rightarrow a) \) cannot cross ab.

The contradiction to our assumption that \( \overrightarrow{xy} \) crosses ab will be obtained by proving that \( |xy| > |xa| \). Indeed, this inequality contradicts the fact that \( \overrightarrow{xy} \in Y_4 \).

Let \( δ \) be the distance from \( x \) to the horizontal line through \( a \). Our intermediate goal is to show that

\[ δ ≤ |ab|/\sqrt{2}. \tag{8} \]

We claim that ∠acb < π/2. Indeed, if this is not the case, then \(|ac| < |ab|\), contradicting the fact that \( \overrightarrow{ab} \) is an edge in Y_4. By a similar argument, and using the fact that \( \overrightarrow{cd} \) is an edge in Y_4, we obtain the inequality ∠cbd < π/2. We now consider two cases, depending on the relative lengths of ac and cb.

1. Assume first that \(|ac| > |cb|\). If ∠cad ≥ π/2, then \(|cd| ≥ |ac| > |cb|\), contradicting the fact that \( \overrightarrow{cd} \) is an edge in Y_4 (recall that b and d are in the same quadrant of c). Therefore, we have ∠cad < π/2. Thus far we have established that three angles of the convex quadrilateral acbd are acute. It follows that the fourth one (∠adb) is obtuse. Proposition 2 applied to \( \triangle adb \) tells us that

\[ |ab|^2 > |ad|^2 + |db|^2 ≥ 2|ad|^2, \]

where the latter inequality follows from the assumption that \( \overrightarrow{ad} \) is a shortest side of acbd (and, therefore, \(|db| ≥ |ad|\)). Thus, we have that \(|ad| ≤ |ab|/\sqrt{2}\). This along with the fact that \( x \in R(a,d) \) implies inequality (8).

2. Assume now that \(|ac| ≤ |cb|\). Let i be the intersection point between ab and the horizontal line through c (refer to Figure 5a). Note that ∠aic ≥ π/2 and ∠bic ≤ π/2 (these two angles
sum to \( \pi \)). This along with Proposition \( \text{II} \) applied to triangle \( \triangle aic \) shows that

\[
|ac|^2 \geq |ai|^2 + |ic|^2.
\]

Similarly, Proposition \( \text{II} \) applied to triangle \( \triangle bic \) shows that

\[
|bc|^2 \leq |bi|^2 + |ic|^2.
\]

The two inequalities above along with our assumption that \( |ac| \leq |cb| \) imply that \( |ai| \leq |bi| \), which in turn implies that \( |ai| \leq |ab|/2 \), because \( |ai| + |ib| = |ab| \). Since \( x \) is below \( i \) (otherwise, \( |cx| < |cd| \), contradicting the fact that \( \overrightarrow{cd} \) is an edge in \( Y_4 \)), we have \( \delta \leq |ai| \). It follows that \( \delta \leq |ab|/2 \).

Finally we derive a contradiction using the now established inequality (8). Let \( j \) be the orthogonal projection of \( x \) onto the vertical line through \( a \) (thus \( |aj| = \delta \)). Note that \( \angle ajy < \pi/2 \), because \( y \in Q_4(x) \). By Proposition \( \text{II} \) applied to \( \triangle ajy \), we have

\[
|ay|^2 < |aj|^2 + |jy|^2 = \delta^2 + |jy|^2.
\]

Since \( y \) and \( b \) are in the same quadrant of \( a \), and since \( \overrightarrow{ab} \in Y_4 \), we have that \( |ab| \leq |ay| \). This along with the inequality above and (5) implies that \( |jy| \geq |ab|/\sqrt{2} \geq \delta \). By Proposition \( \text{II} \) applied to \( \triangle xjy \), we have \( |xy|^2 > |xj|^2 + |jy|^2 \geq |xj|^2 + \delta^2 = |xj|^2 + |ja|^2 = |xa|^2 \). It follows that \( |xy| > |xa| \), contradicting our assumption that \( \overrightarrow{xy} \in Y_4 \).

**Lemma 6** Let \( a, b, c, d \) be four distinct nodes in \( V \), with \( c \in Q_1(a) \), such that

(a) \( \overrightarrow{ab} \in Q_1(a) \) and \( \overrightarrow{cd} \in Q_3(c) \) are two edges in \( Y_4 \) that cross each other.

(b) \( ad \) is a shortest side of the quadrilateral \( acbd \).

Then \( P_R(d \rightarrow a) \) does not cross \( ab \).

**Proof.** We first show that \( d \notin Q_3(a) \). Assume the opposite. Since \( c \in Q_1(a) \) and \( d \in Q_3(a) \), we have that \( \angle cad > \pi/2 \). This implies that \( |ca| < |cd| \), which along with the fact that \( a, d \in Q_3(c) \) contradict the fact that \( \overrightarrow{cd} \in Y_4 \). Also note that \( d \notin Q_1(a) \), since in that case \( ab \) and \( cd \) could not intersect. In the following we discuss the case \( d \in Q_2(a) \); the case \( d \in Q_4(a) \) is symmetric.

A first observation is that \( c \) must lie below \( b \); otherwise \( |cb| < |cd| \) (since \( \angle cbd > \pi/2 \)), which would contradict the fact that \( \overrightarrow{cd} \in Y_4 \). We now prove by contradiction that there is no edge in \( P_R(d \rightarrow a) \) crossing \( ab \). Assume the contrary, and let \( \overrightarrow{xy} \in P_R(d \rightarrow a) \) be such an edge. Then necessarily \( x \in R(a,d) \) and \( \overrightarrow{xy} \in Q_4(x) \). Note that \( y \) cannot lie below \( a \); otherwise \( |xa| < |xy| \) (since \( \angle xay > \pi/2 \)), which would contradict the fact that \( \overrightarrow{xy} \in Y_4 \). Also \( y \) must lie outside \( D(c, |cd|) \cap Q(c,d) \), otherwise \( \overrightarrow{cd} \) could not be in \( Y_4 \). These together show that \( y \) sits to the right of \( c \). See Figure (6a). Then the following inequalities regarding the quadrilateral \( xayb \) must hold:

(i) \( |by| > |bc| \), due to the fact that \( \angle bcy > \pi/2 \).

(ii) \( |bx| \geq |bd| \) (\( |bx| = |bd| \) if \( x \) and \( d \) coincide). If \( x \) and \( d \) are distinct, the inequality \( |bx| > |bd| \) follows from the fact that \( |cx| \geq |cd| \) (since \( x \) is outside \( D(c, |cd|) \)), and Proposition \( \text{I} \) applied to the quadrilateral \( xcbd \):

\[
|bd| + |cx| < |bx| + |cd|
\]
Figure 6: Lemma 6: (a) $P_{R}(d \rightarrow a)$ does not cross $ab$. (b) If $ad$ is not the shortest side of $acbd$, the lemma conclusion might not hold.

Inequalities (i) and (ii) show that $by$ and $bx$ are longer than sides of the quadrilateral $acbd$, and so they must be longer than the shortest side of $acbd$, which by assumption (b) of the lemma is $ad$: \[ \min\{|bx|, |by|\} \geq |ad| \geq |ax| \] (this latter inequality follows from the fact that $x \in R(d,a)$). Also note that $|ab| \leq |ay|$, since $ab \in Y_4$ and $y$ lies in the same quadrant of $a$ as $b$. The fact that both diagonals of $xayb$ are in $Y_4$ enables us to apply Lemma 4(ii) to conclude that $ay$ is not a shortest side of the quadrilateral $xayb$. Thus $xa$ is a shortest side of the quadrilateral $xayb$, and we can use Lemma 4(ii) to claim that \[ |xa| < \min\{|xy|, |ab|\} \leq |xy|. \] This contradicts our assumption that $xy \in Y_4$.

Figure 6(b) shows that the claim of the lemma might be false without assumption (b). The next lemma relies on all of Lemmas 2–6.

**Lemma 7** Let $a, b, c, d \in V$ be four distinct nodes such that $\overrightarrow{ab} \in Y_4$ crosses $\overrightarrow{cd} \in Y_4$, and let $xy$ be a shortest side of the quadrilateral $abcd$. Then there exist two paths $P_x$ and $P_y$ in $Y_4$, where $P_x$ has $x$ as an endpoint and $P_y$ has $y$ as an endpoint, with the following properties:

(a) $P_x$ and $P_y$ have a nonempty intersection.

(b) $|P_x| + |P_y| \leq 3\sqrt{2}|xy|$.

(c) Each edge on $P_x \cup P_y$ is no longer than $|xy|$.

**Proof.** Assume without loss of generality that $b \in Q_1(a)$. We discuss the following exhaustive cases:

1. $c \in Q_1(a)$, and $d \in Q_1(c)$. In this case, $ab$ and $cd$ cannot cross each other (by Lemma 3), so this case is finished.

2. $c \in Q_1(a)$, and $d \in Q_2(c)$, as in Figure 7b. Since $ab$ crosses $cd$, $b \in Q_2(c)$. Since $\overrightarrow{ab} \in Y_4$, $|ab| \leq |ac|$. Since $\overrightarrow{cd} \in Y_4$, $|cd| \leq |cb|$. These along with Lemma 4 imply that $ad$ and $db$ are the only candidates for a shortest edge of $acbd$. 

---

Note: The text continues with further details and proofs, which are not fully transcribed here due to the length and complexity of the content.
Figure 7: Lemma 7: (a) $c \in Q_1(a)$, and $d \in Q_2(c)$ (b) $c \in Q_1(a)$, and $d \in Q_3(c)$ (c) $c \in Q_2(a)$ (d) $c \in Q_4(a)$.

Assume first that $ad$ is a shortest edge of $acbd$. By Lemma $3$, $P_a = P_R(a \rightarrow d)$ does not cross $cd$. It follows from Lemma $2$ that $P_a$ and $P_d = P_R(d \rightarrow a)$ have a nonempty intersection. Furthermore, by Lemma $3$ $|P_a| \leq |ad|\sqrt{2}$ and $|P_d| \leq |ad|\sqrt{2}$, and no edge on these paths is longer than $|ad|$, proving the lemma true for this case.

Consider now the case when $db$ is a shortest edge of $acbd$ (see Figure 7a). Note that $d$ is below $b$ (otherwise, $d \in Q_2(c)$ and $|cd| > |cb|$) and, therefore, $b \in Q_1(d)$). By Lemma $3$ $P_d = P_R(d \rightarrow b)$ does not cross $ab$. If $P_b = P_R(b \rightarrow d)$ does not cross $cd$, then $P_b$ and $P_d$ have a nonempty intersection, proving the lemma true for this case. Otherwise, there exists $\overrightarrow{xy} \in P_R(b \rightarrow d)$ that crosses $cd$ (see Figure 7a). Define

$$P_b = P_R(b \rightarrow d) \oplus P_R(y \rightarrow d)$$
$$P_d = P_R(d \rightarrow y)$$

By Lemma $3$ $P_R(y \rightarrow d)$ does not cross $cd$. Then $P_b$ and $P_d$ must have a nonempty intersection. We now show that $P_b$ and $P_d$ satisfy conditions (b) and (c) of the lemma. Proposition $1$ applied on the quadrilateral $xdyc$ tells us that

$$|xc| + |yd| < |xy| + |cd|$$

We also have that $|cx| \geq |cd|$, since $\overrightarrow{cd} \in Y_4$ and $x$ is in the same quadrant of $c$ as $d$. This along with the inequality above implies $|yd| < |xy|$. Because $xy \in P_R(b \rightarrow d)$, by Lemma $2$ we have that $|xy| \leq |bd|$, which along with the previous inequality shows that $|yd| < |bd|$. This along with Lemma $2$ shows that condition (c) of the lemma is satisfied. Furthermore, $|P_R(y \rightarrow d)| \leq |yd|\sqrt{2}$ and $|P_R(d \rightarrow y)| \leq |yd|\sqrt{2}$. It follows that $|P_b| + |P_d| \leq 3\sqrt{2}|bd|$. 

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3. \(c \in Q_1(a), \text{ and } d \in Q_3(c), \) as in Figure 7b. Then \(|ac| \geq \max\{ab, cd\}\), and by Lemma 4 \(ac\) is not a shortest edge of \(abcd\). The case when \(bd\) is a shortest edge of \(abcd\) is settled by Lemmas 3 and 2. Lemma 2 tells us that \(P = P_R(d \to b)\) does not cross \(ab\), and \(P_b = P_R(b \to d)\) does not cross \(cd\). It follows that \(P_d\) and \(P_b\) have a nonempty intersection. Furthermore, Lemma 2 guarantees that \(P_d\) and \(P_b\) satisfy conditions (b) and (c) of the lemma. Consider now the case when \(ad\) is a shortest edge of \(abcd\); the case when \(bc\) is shortest is symmetric. By Lemma 6, \(P_R(d \to a)\) does not cross \(ab\). If \(P_R(a \to d)\) does not cross \(cd\), then this case is settled: \(P_d = P_R(d \to a)\) and \(P_a = P_R(a \to d)\) satisfy the three conditions of the lemma. Otherwise, let \(\overline{xy} \in P_R(a \to d)\) be the edge crossing \(cd\). Arguments similar to the ones used in case 1 above show that

\[
\begin{align*}
\mathcal{P}_a &= \mathcal{P}_R(a \to d) \oplus \mathcal{P}_R(y \to d) \\
\mathcal{P}_d &= \mathcal{P}_R(d \to y)
\end{align*}
\]

are two paths that satisfy the conditions of the lemma.

4. \(c \in Q_1(a), \text{ and } d \in Q_4(c), \) as in Figure 7f. Note that a horizontal reflection of Figure 7f, followed by a rotation of \(\pi/2\), depicts a case identical to case 1, which has already been settled.

5. \(c \in Q_2(a), \) as in Figure 7j. Note that Figure 7d rotated by \(\pi/2\) depicts a case identical to case 1, which has already been settled.

6. \(c \in Q_3(a). \) Then it must be that \(d \in Q_1(c), \) otherwise \(cd\) cannot cross \(ab\). By Lemma 4 however, \(ab\) and \(cd\) may not cross, unless one of them is not in \(Y_4\).

7. \(c \in Q_4(a), \) as in Figure 7h. Note that a vertical reflection of Figure 7b depicts a case identical to case 1, so this case is settled as well.

Having exhausted all cases, we conclude that the lemma holds.

We are now ready to establish the main lemma of this section, showing that there is a short path between the endpoints of two intersecting edges in \(Y_4\).

**Lemma 8** Let \(a, b, c, d \in V\) be four distinct nodes such that \(\overrightarrow{ab} \in Y_4\) crosses \(\overrightarrow{cd} \in Y_4\), and let \(xy\) be a shortest side of the quadrilateral \(abcd\). Then \(Y_4\) contains a path \(p(x, y)\) connecting \(x\) and \(y\), of length

\[|p(x, y)| \leq \frac{6}{\sqrt{2} - 1} \cdot |xy|.
\]

Furthermore, no edge on \(p(x, y)\) is longer than \(|xy|\).

**Proof.** Let \(\mathcal{P}_x\) and \(\mathcal{P}_y\) be the two paths whose existence in \(Y_4\) is guaranteed by Lemma 7. By condition (c) of Lemma 7, no edge on \(\mathcal{P}_x\) and \(\mathcal{P}_y\) is longer than \(|xy|\). By condition (a) of Lemma 7, \(\mathcal{P}_x\) and \(\mathcal{P}_y\) have a nonempty intersection. If \(\mathcal{P}_x\) and \(\mathcal{P}_y\) share a node \(u \in V\), then the path

\[p(x, y) = \mathcal{P}_x[x, u] \oplus \mathcal{P}_y[y, u]\]

is a path from \(x\) to \(y\) in \(Y_4\) no longer than \(3\sqrt{2}|xy|\); the length restriction follows from guarantee (b) of Lemma 7. Otherwise, let \(\overrightarrow{ab} \in \mathcal{P}_x\) and \(\overrightarrow{cd} \in \mathcal{P}_y\) be two edges crossing each other. Let \(x'y'\)
be a shortest side of the quadrilateral $a'c'b'd'$, with $x' \in P_x$ and $y' \in P_y$. Lemma 7 tells us that $|a'b'| \leq |xy|$ and $|c'd'| \leq |xy|$. These along with Lemma 4 imply that

$$|x'y'| \leq |xy|/\sqrt{2}. \quad (9)$$

This enables us to derive a recursive formula for computing a path $p(x, y) \in Y_4$ as follows:

$$p(x, y) = \begin{cases} 
 x, & \text{if } x = y \\
 P_x[x, x'] \oplus P_y[y, y'] \oplus p(x', y'), & \text{if } x \neq y 
\end{cases} \quad (10)$$

Next we use induction on the length of $xy$ to prove the claim of the lemma. The base case corresponds to $x = y$, case in which $p(x, y)$ degenerates to a point and $|p(x, y)| = 0$. To prove the inductive step, pick a shortest side $xy$ of a quadrilateral $abcd$, with $\overrightarrow{ab}, \overrightarrow{cd} \in Y_4$ crossing each other, and assume that the lemma holds for all such sides shorter than $xy$. Let $p(x, y)$ be the path determined recursively as in (10). By the inductive hypothesis, we have that $p(x', y')$ contains no edges longer than $|x'y'| \leq |xy|$, and

$$|p(x', y')| \leq \frac{6}{\sqrt{2} - 1} |x'y| \leq \frac{6}{2 - \sqrt{2}} |xy|.$$  

This latter inequality follows from (9). This along with Lemma 7 and formula (10) implies

$$|p(x, y)| \leq (3\sqrt{2} + \frac{6}{2 - \sqrt{2}}) \cdot |xy| = \frac{6}{\sqrt{2} - 1} \cdot |xy|.$$  

This completes the proof.

\section*{4 $Y_4^\infty$ and $Y_4$}

We prove that every individual edge of $Y_4^\infty$ is spanned by a short path in $Y_4$. This, along with the result of Theorem 1, establishes that $Y_4$ is a spanner.

Fix an edge $ab \in Y_4^\infty$. Define an edge or a path as short if its length is within a constant factor of $|ab|$. In our proof that $ab$ is spanned by a short path in $Y_4$, we will make use of the following three statements (which will be proved in the appendix).

\begin{itemize}
  \item \textbf{S1} If $ab$ is short, then $P_R(a \rightarrow b)$, and therefore its reverse, $P_R^{-1}(a \rightarrow b)$, are short by Lemma 2.
  \item \textbf{S2} If $ab \in Y_4$ and $cd \in Y_4$ are short, and if $ab$ intersects $cd$, Lemma 8 shows that then there is a short path between any two of the endpoints of these edges.
  \item \textbf{S3} If $p(a, b)$ and $p(c, d)$ are short paths that intersect, then there is a short path $P$ between any two of the endpoints of these paths, by \textbf{S2}.
\end{itemize}

\textbf{Lemma 9} \textit{For any edge $ab \in Y_4^\infty$, there is a short path $p(a, b) \in Y_4$ of length}

$$|p(a, b)| \leq (29 + 23\sqrt{2})|ab|.$$  

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Proof. For the sake of clarity, we only prove here that there is a short path \( p(a, b) \), and defer the calculations of the actual stretch factor of \( p(a, b) \) to the appendix. Assume without loss of generality that \( \overrightarrow{ab} \in Y_4^∞ \), and \( \overrightarrow{ab} \in Q_1(a) \). If \( \overrightarrow{ab} \in Y_4 \), then \( p(a, b) = ab \) and the proof is finished. So assume the opposite, and let \( \overrightarrow{ac} \in Q_1(a) \) be the edge in \( Y_4 \); since \( Q_1(a) \) is nonempty, \( \overrightarrow{ac} \) exists. Because \( \overrightarrow{ac} \in Y_4 \) and \( b \) is in the same quadrant of \( a \) as \( c \), we have that

\[
\begin{align*}
|ac| & \leq |ab| & \text{(i)} \\
|bc| & \leq |ac|\sqrt{2} & \text{(ii)}
\end{align*}
\]

Thus both \( ac \) and \( bc \) are short. And this in turn implies that \( P_R(b \to c) \) is short by S1. We next focus on \( P_R(b \to c) \). Let \( b' \notin R(b, c) \) be the other endpoint of \( P_R(b \to c) \). We distinguish three cases.

**Case 1:** \( P_R(b \to c) \) and \( ac \) intersect. Then by S3 there is a short path \( p(a, b) \) between \( a \) and \( b \).

**Case 2:** \( P_R(b \to c) \) and \( ac \) do not intersect, and \( P_R(b' \to a) \) and \( ab \) do not intersect (see Figure 8b). Note that because \( b' \) is the endpoint of the short path \( P_R(b \to c) \), the triangle inequality on \( \triangle abb' \) implies that \( ab' \) is short, and therefore \( P_R(b' \to a) \) is short. We consider two cases:

(i) \( P_R(b' \to a) \) intersects \( ac \). Then by S3 there is a short path \( p(a, b') \). So

\[
p(a, b) = p(a, b') \oplus P_R^{-1}(b \to c)
\]

is short.

(ii) \( P_R(b' \to a) \) does not intersect \( ac \). Then \( P_R(c \to b') \) must intersect \( P_R(b \to c) \oplus P_R(b' \to a) \). Next we establish that \( b'c \) is short. Let \( e \) be the last edge of \( P_R(b \to c) \), and so incident to \( b' \) (note that \( e \) and \( b \) may coincide). Because \( P_R(b \to c) \) does not intersect \( ac, b' \) and \( c \) are in the same quadrant for \( e \). It follows that \( |b'e| \leq |ec| \) and \( \angle b'ec < \pi/2 \). These along with Proposition 2 for \( \triangle b'ec \) imply that

\[
|b'c|^2 < |b'e|^2 + |ec|^2 \leq 2|ec|^2 < 2|bc|^2
\]

(latter inequality uses the fact that \( \angle b'ec > \pi/2 \), which implies that \( |ec| < |bc| \)). It follows that

\[
|b'c| \leq |bc|\sqrt{2} \leq 2|ac| & \text{(by (1)ii)}
\]

\[
|b'c| \leq 2|ac|
\]

Figure 8: Lemma 9. (a) Case 1: \( P_R(b \to c) \) and \( ac \) have a nonempty intersection. (b) Case 2: \( P_R(b' \to a) \) and \( ab \) have an empty intersection. (c) Case 3: \( P_R(b' \to a) \) and \( ab \) have a non-empty intersection.

**Case 1:** \( P_R(b \to c) \) and \( ac \) intersect. Then by S3 there is a short path \( p(a, b) \) between \( a \) and \( b \).
Thus $b'c$ is short, and by $S1$ we have that $P_R(c \to b')$ is short. Since $P_R(c \to b')$ intersects the short path $P_R(b \to c) \oplus P_R(b' \to a)$, there is by $S3$ a short path $p(c, b)$, and so
\[ p(a, b) = ac \oplus p(c, b) \]
is short.

**Case 3:** $P_R(b \to c)$ and $ac$ do not intersect, and $P_R(b' \to a)$ intersects $ab$ (see Figure 8c). If $P_R(b' \to a)$ intersects $ab$ at $a$, then $p(a, b) = P_R(b \to c) \oplus P_R(b' \to a)$ is short. So assume otherwise, in which case there is an edge $de \in P_R(b' \to a)$ that crosses $ab$. Then $d \in Q_4(a)$, $e \in Q_3(a) \cup Q_4(a)$, and $e$ and $a$ are in the same quadrant for $d$. Note however that $e$ cannot lie in $Q_3(a)$, since in that case $\angle dae > \pi/2$, which would imply $|de| > |da|$, which in turn would imply $de \notin Y_4$. So it must be that $e \in Q_4(a)$.

Next we show that $P_R(e \to a)$ does not cross $ab$. Assume the opposite, and let $rs \in P_R(e \to a)$ cross $ab$. Then $r \in Q_4(a)$, $s \in Q_4(a) \cup Q_2(a)$, and $s$ and $a$ are in the same quadrant for $r$. Arguments similar to the ones above show that $s \notin Q_2(a)$, so $s$ must lie in $Q_1(a)$. Let $d$ be the $L_\infty$ distance from $a$ to $b$. Let $x$ be the projection of $r$ on the horizontal line through $a$. Then
\[ |rs| \geq |rx| + d \geq |rx| + |xa| > |ra| \quad \text{(by the triangle inequality)} \]
Because $a$ and $s$ are in the same quadrant for $r$, the inequality above contradicts $rs \notin Y_4$.

We have established that $P_R(e \to a)$ does not cross $ab$. Then $P_R(a \to e)$ must intersect $P_R(e \to a) \oplus de$. Note that $de$ is short because it is in the short path $P_R(b' \to a)$. Thus $ae$ is short, and so $P_R(a \to e)$ and $P_R(e \to a)$ are short. Thus we have two intersecting short paths, and so by $S3$ there is a short path $p(a, e)$. Then
\[ p(a, b) = p(a, e) \oplus P_R^{-1}(b' \to a) \oplus P_R^{-1}(b \to c) \]
is short. Calculations deferred to the appendix show that, in each of these cases, the stretch factor for $p(a, b)$ does not exceed $29 + 23\sqrt{2}$.

Our main result follows immediately from Theorem 1 and Lemma 9.

**Theorem 2** $Y_4$ is a $t$-spanner, for $t \geq 8(29 + 23\sqrt{2})$.

**5 Conclusion**

Our results settle a long-standing open problem, asking whether $Y_4$ is a spanner or not. We answer this question positively, and establish a loose stretch factor of $8(29 + 23\sqrt{2})$. Experimental results, however, indicate a stretch factor of the order $1 + \sqrt{2}$, a factor of 200 smaller. Finding tighter stretch factors for both $Y_4^\infty$ and $Y_4$ remain interesting open problems. Establishing whether $Y_5$ and $Y_6$ are spanners or not is also open.

**References**

6 Appendix

6.1 Calculations for the stretch factor of $p(a,b)$ in Lemma 9

We start by computing the stretch factor of the short paths claimed by statements S2 and S3.

**S2** If $ab \in Y_4$ and $cd \in Y_4$ are short, and if $ab$ intersects $cd$, then there is a short path $P$ between any two of the endpoints of these edges, of length

$$|P| \leq |ab| + |cd| + 3(2 + \sqrt{2}) \max\{|ab|, |cd|\}$$

This upper bound can be derived as follows. Let $xy$ be a shortest side of the quadrilateral $acbd$. By Lemma 8, $Y_4$ contains a path $p(x,y)$ no longer than $6(\sqrt{2} + 1)|xy|$. By Lemma 4, $|xy| \leq \max\{|ab|, |cd|\}/\sqrt{2}$. These together with the fact that $|P| \leq |ab| + |cd| + |p(x,y)|$ yield inequality (13).

**S3** If $p(a,b)$ and $p(c,d)$ are short paths that intersect, then there is a short path $P$ between any two of the endpoints of these paths, of length

$$|P| \leq |p(a,b)| + |p(c,d)| + 3(2 + \sqrt{2}) \max\{|ab|, |cd|\}$$

This follows immediately from S2 and the fact that no edge on $p(a,b) \cup p(c,d)$ is longer than $\max\{|ab|, |cd|\}$ (by Lemma 8).

**Case 1:** $P_R(b \rightarrow c)$ and $ac$ intersect. Then by S3 we have

$$|p(a,b)| \leq |P_R(b,c)| + |ac| + 3(2 + \sqrt{2}) \max\{|bc|, |ac|\}$$

$$\leq \sqrt{2}|bc| + |ac| + 3(2 + \sqrt{2})\sqrt{2}|ac|$$

(by 7), (11)ii

$$= 3(3 + 2\sqrt{2})|ac| \leq 3(3 + 2\sqrt{2})|ab|$$

(by 11)i
Case 2(i): \( P_R(b \to c) \) and \( ac \) do not intersect; \( P_R(b' \to a) \) and \( ab \) do not intersect; and \( P_R(b' \to a) \) intersects \( ac \). By S3, there is a short path \( p(a, b') \) of length
\[
|p(a, b')| \leq |P_R(b', a)| + |ac| + 3(2 + \sqrt{2}) \max\{|b'a|, |ac|\}
\leq |b'a|\sqrt{2} + |ac| + 3(2 + \sqrt{2}) \max\{|b'a|, |ac|\} \quad \text{(by (7))}
\]
Next we establish an upper bound on \( |b'a| \). By the triangle inequality,
\[
|ab| < |ac| + |cb'| \leq 3|ac| \quad \text{(by (12))}
\]
Substituting this inequality in (15) yields
\[
|p(a, b')| \leq (19 + 12\sqrt{2})|ac| \quad \text{(17)}
\]
Thus \( p(a, b) = p(a, b') \oplus P_R^{-1}(b \to c) \) is a path in \( Y_4 \) of length
\[
|p(a, b)| \leq |p(a, b')| + |bc|\sqrt{2} \quad \text{(by (7))}
\leq |p(a, b')| + 2|ac| \quad \text{(by (11)ii)}
\leq (21 + 12\sqrt{2})|ac| \quad \text{(by (17))}
\leq (21 + 12\sqrt{2})|ab| \quad \text{(by (11)i)}
\]
Case 2(ii): \( P_R(b \to c) \) and \( ac \) do not intersect; \( P_R(b' \to a) \) and \( ab \) do not intersect; and \( P_R(b' \to a) \) does not intersect \( ac \). Then \( P_R(c \to b') \) must intersect \( P_R(b \to c) \oplus P_R(b' \to a) \). By S3 there is a short path \( p(c, b) \) of length
\[
|p(c, b)| \leq |P_R(c \to b')| + |P_R(b \to c)| + |P_R(b' \to a)| + 3(2 + \sqrt{2}) \max\{|cb'|, |bc|, |b'a|\}
\leq (|cb'| + |bc| + |b'a|)\sqrt{2} + 3(2 + \sqrt{2}) \max\{|cb'|, |bc|, |b'a|\} \quad \text{(by (7))}
\]
Inequalities (11)ii, (12) and (16) imply that \( \max\{|cb'|, |bc|, |b'a|\} \leq 3|ac| \). Substituting in the above, we get
\[
|p(c, b)| \leq (2 + \sqrt{2} + 3)\sqrt{2}|ac| + 9(2 + \sqrt{2})|ac|
\leq (20 + 14\sqrt{2})|ac| \quad \text{(by (11)i)}
\]
Thus \( p(a, b) = ac \oplus p(c, b) \) is a path in \( Y_4 \) from \( a \) to \( b \) of length
\[
|p(a, b)| \leq (21 + 14\sqrt{2})|ac| \leq (21 + 14\sqrt{2})|ab| \quad \text{(by (11)i)}
\]
Case 3: \( P_R(b \to c) \) and \( ac \) do not intersect, and \( P_R(b' \to a) \) intersects \( ab \). If \( P_R(b' \to a) \) intersects \( ab \) at \( a \), then \( p(a, b) = P_R(b \to c) \oplus P_R(b' \to a) \) is clearly short and does not exceed the spanning ratio of the lemma. Otherwise, there is an edge \( \overline{de} \in P_R(b' \to a) \) that crosses \( ab \), and \( P_R(a \to e) \) intersects \( P_R(c \to a) \oplus \overline{de} \) (as established in the proof of Lemma 9). By S3 there is a short path \( p(a, e) \) of length
\[
|p(a, e)| \leq |P_R(a \to e)| + |P_R(e \to a)| + |\overline{de}| + 3(2 + \sqrt{2}) \max\{|ae|, |de|\}
\leq 2|ae|\sqrt{2} + |\overline{de}| + 3(2 + \sqrt{2}) \max\{|ae|, |de|\} \quad \text{(by (7))}
\]
\[18\]
A loose upper bound on $|ae|$ can be obtained by employing Proposition 1 to the quadrilateral $aebd$: $|ac| + |bd| < |ab| + |de| < |ab| + |ab'|$. Substituting the upper bound for $|ab'|$ from (16) yields

$$|ae| < |ab| + 3|ac| \leq 4|ab| \quad (19)$$

By Lemma 2, $|de| \leq |ab'|$ (since $de \in \mathcal{P}_R(b' \rightarrow a)$), which along with (16) implies

$$|de| \leq 3|ab| \quad (20)$$

Substituting (19) and (20) in (18) yields

$$|p(a,e)| \leq (27 + 20\sqrt{2})|ab|$$

Then

$$p(a,b) = p(a,e) \oplus \mathcal{P}_R^{-1}(b' \rightarrow a) \oplus \mathcal{P}_R^{-1}(b \rightarrow c)$$

is a path from $a$ to $b$ of length

$$|p(a,b)| \leq |p(a,e)| + |b'a|\sqrt{2} + |bc|\sqrt{2} \quad (by \ (7))$$

$$\leq (27 + 20\sqrt{2})|ab| + 3\sqrt{2}|ab| + 2|ab| \quad (by \ (16), \ (11))$$

$$= (29 + 23\sqrt{2})|ab|$$

6.2 $Y_k$ is a Spanner, for $k \geq 7$

**Lemma 10** Let $\theta$ be a real number with $0 < \theta < \pi/3$, and let

$$t = \frac{1 + \sqrt{2} - 2\cos \theta}{2\cos \theta - 1}.$$ 

Let $a, b,$ and $c$ be three distinct points in the plane such that $|ac| \leq |ab|$, let $\alpha = \angle bac$, and assume that $0 \leq \alpha \leq \theta$. Then

$$|bc| \leq |ab| - |ac|/t. \quad (21)$$

**Proof.** Refer to Figure 9. By the Law of Cosines, we have

$$|bc|^2 = |ac|^2 + |ab|^2 - 2|ac| \cdot |ab| \cdot \cos \alpha.$$ 

Since $t > 1$ and $|ac| \leq |ab|$, the right-hand side in (21) is positive, so (21) is equivalent to

Figure 9: Lemma 1: If $\alpha < 60$ and $|ac| \leq |ab|$, then $|bc| \leq |ab| - |ac|/t.$
\[ |bc|^2 \leq (|ab| - |ac|/t)^2. \]

Thus, we have to show that
\[ |ac|^2 + |ab|^2 - 2|ac| \cdot |ab| \cos \alpha \leq (|ab| - |ac|/t)^2, \]
which simplifies to
\[ (1 - 1/t^2) |ac| \leq 2(\cos \alpha - 1/t)|ab|. \tag{22} \]

Since \(|ac| \leq |ab|\) and \(\cos \theta \leq \cos \alpha\), (22) holds if
\[ 1 - 1/t^2 \leq 2(\cos \theta - 1/t), \]
which can be rewritten as
\[ (2 \cos \theta - 1)t^2 - 2t + 1 \geq 0. \tag{23} \]

By our choice of \(t\), equality holds in (23). \(\Box\)

An immediate consequence of Lemma 10 is the following result.

**Theorem 3** For any \(\theta\) with \(0 < \theta < \pi/3\), the Yao-graph with cones of angle \(\theta\), is a \(t\)-spanner for
\[ t = \frac{1 + \sqrt{2 - 2 \cos \theta}}{2 \cos \theta - 1}. \]

**Proof.** The proof of this claim is by induction on the distances defined by the \(\binom{n}{2}\) pairs of nodes. Since \(\theta < \pi/3\), any closest pair is connected by an edge in the Yao-graph; this proves the basis of the induction. The induction step follows from Lemma 10. \(\Box\)

What happens to the value of \(t\) from Lemma 10 if \(\theta\) gets close to \(\pi/3\)? Let \(\varepsilon = \cos \theta - 1/2\), so that \(\varepsilon\) is close to zero. Then
\[
t = \frac{1}{2\varepsilon} + \sqrt{\frac{1 - 2\varepsilon}{4\varepsilon^2}} \\
= \frac{1}{2\varepsilon} + \frac{\sqrt{1 - 2\varepsilon}}{2\varepsilon} \\
\sim \frac{1}{2\varepsilon} + \frac{1 - \varepsilon}{2\varepsilon} \\
= \frac{1}{2} + \frac{1}{\varepsilon} \\
= \frac{1}{2} + \frac{1}{\cos \theta - 1/2}. \]