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Computational Geometry Column 45

Joseph O'Rourke
Smith College, jorourke@smith.edu

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Computational Geometry Column 45

Joseph O’Rourke*

Abstract

The algorithm of Edelsbrunner for surface reconstruction by “wrapping” a set of points in $\mathbb{R}^3$ is described.

Curve reconstruction [O’R00] seeks to find a “best” curve passing through a given finite set of points, usually in $\mathbb{R}^2$. Surface reconstruction seeks to find a best surface passing through a set of points in $\mathbb{R}^3$. Both problems have numerous applications, usually deriving from the need to reconstruct the curve or surface from a sample. Both problems are highly underconstrained, for there are usually many curves/surfaces through the points. Surface reconstruction in particular is notoriously difficult to control. Although significant advances have been made in recent years [Dey04]—especially in the direction of performance guarantees based on sample density—we turn here to a beautiful and now relatively old “wrapping” algorithm due to Edelsbrunner, which, although implemented in 1996 at Raindrop Geomagic, has been published only recently [Ede03] after issuance of a patent in 2002.

Sample results of the algorithm are illustrated in Figs. 1 and 2. Although both of these examples reconstruct surfaces of genus one, we concentrate on the genus-zero case (a topological sphere) and only mention extensions for higher genus reconstructions.

An attractive aspect of the algorithm is that it reconstructs a unique surface without assumptions on sample density and without adjustment of heuristic parameters. Although the algorithm uses discrete methods, underneath it relies on continuous Morse functions. The discrete scaffolding on

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*Dept. of Computer Science, Smith College, Northampton, MA 01063, USA. orourke@cs.smith.edu. Supported by NSF Distinguished Teaching Scholar Grant DUE-0123154.

1. stl (stereolithography) files for shapes from http://www.cs.duke.edu/~edels/Tubes/
which the algorithm depends is the Delaunay complex, which we now informally describe. A *simplex* is a point, segment, triangle, or tetrahedron. A *simplicial complex* $\mathcal{K}$ is a “proper” gluing together of simplicies, in that (1) if a simplex $\sigma$ is in $\mathcal{K}$, then so are all its faces, and (2) if two simplices $\sigma$ and $\sigma'$ are in $\mathcal{K}$, then either $\sigma \cap \sigma'$ is empty or a face of each. Let $S$ be the finite set of points whose surface is to be reconstructed. The *Delaunay complex* $\text{Del} \ S$ is the dual of the Voronoi diagram of $S$. Under a general-position assumption, $\text{Del} \ S$ contains a simplex that is the convex hull of the sites $T \subset S$ iff there is an empty sphere that passes through the points of $T$. The outer boundary of $\text{Del} \ S$ is the convex hull of $S$. Augmenting $\text{Del} \ S$ with a dummy “simplex” $\emptyset$ for the space exterior to the hull, covers $\mathbb{R}^3$.

The algorithm seeks to find a “wrapping” surface $\mathcal{W}$, a connected simplicial subcomplex in $\text{Del} \ S$. It accomplishes this by finding a simplicial subcomplex $\mathcal{X}$ of $\text{Del} \ S$ whose boundary is $\mathcal{W}$. The vertices of $\mathcal{X}$ will be precisely the input points $S$, and the vertices of $\mathcal{W}$ will be a subset of $S$.

The algorithm uncovers $\mathcal{X}$ in $\text{Del} \ S$ by “sculpting” away simplices from $\text{Del} \ S$ one-by-one, starting from $\emptyset$, until $\mathcal{X}$ remains. The simplices are removed according to an acyclic partial ordering. It is the definition of this ordering that involves continuous mathematics.

A function $g(x)$ assigns to every point $x \in \mathbb{R}^3$ a number dependent on the closest Voronoi vertex. In particular, if $x$ is in a tetrahedron $T$ of $\text{Del} \ S$ whose empty circumsphere has center $z$ and radius $r$, then $g(x) = r^2 - ||z - x||^2$. Thus $g(x)$ is zero at the corners of $T$ and rises to $r^2$ at $z$, the closest Voronoi vertex.
vertex. Points outside the hull are assigned an effectively infinite value. \( g(x) \)
is continuous but not smooth enough to qualify as a Morse function, needed for the subsequent development. It will suffice here to claim that \( g \) can be smoothed sufficiently to define the vector field \( \nabla g \), and from this, by a limiting process, \textit{flow curves} through every point \( x \in \mathbb{R}^3 \) aiming toward higher values.

These flow curves are in turn used to define an acyclic relation on all the simplices of \( \text{Del} S \) and \( \phi \). Let \( \tau \) and \( \sigma \) be two simplices (of any dimension) and \( v \) a face shared between them. For example, if \( \tau \) and \( \sigma \) are both tetrahedra, \( v \) could be a triangle, or a segment, or a vertex. Define the \textit{flow relation} \( \rightarrow \) so that \( \tau \rightarrow v \rightarrow \sigma \) if there is a flow curve passing from \( \text{int } \tau \) to \( \text{int } v \) to \( \text{int } \sigma \).

A sink of the relation is a simplex that has no flow successor. \( \phi \) is always a sink (recall \( g(x) \) is large outside the hull), with the hull faces of \( \text{Del} S \) its immediate predecessors. Sinks are like critical points of the flow, with the simplices that gravitate toward a sink corresponding to a stable manifold in Morse terminology.

A key theorem is that the flow relation on simplices is acyclic, which reflects the increase of \( g(x) \) along every flow curve. The algorithms starts with \( \phi \) and methodically “collapses” its flow predecessors until no more collapses are possible, yielding the complex \( \mathcal{X} \).

Let \( v \) be a face of \( \tau \); then \( \tau \) is called a \textit{coface} of \( v \). Assume \( \tau \rightarrow v \); for example, \( \tau \) might be a tetrahedron and \( v \) one of its edges, with the flow from \( \tau \) through \( v \). We give some indication of when the pair \((v, \tau)\) is collapsible, without defining it precisely. First, \( \tau \) must be the highest dimension coface of \( v \), and \( v \) should not have any cofaces not part of \( \tau \). Thus, \( v \) is in a sense “exposed.” Second, the flow curves should pass right through every point of \( v \) (as opposed to running along or in \( v \)). Collapse of the pair removes all the cofaces of \( v \), thus eating away the parts of \( \tau \) sharing \( v \).

A second key theorem is that any sequence of collapses from \( \phi \) leads to the same simplicial complex \( \mathcal{X} \). Collapses also maintain the homotopy type, which, because \( \text{Del} S \) is a topological ball, result in \( \mathcal{X} \) a ball and \( \mathcal{W} \) a topological sphere.

To produce surfaces of higher genus, the contraction is pushed through

\[ \text{int } v \text{ is the interior of } v; \text{ for a } v \text{ a vertex, } \text{int } v = v. \]

\[ \text{One can think of this as a containing face, although its origins are more in complementary topological terminology.} \]
holes: the most “significant” sink (in terms of $g(x)$) is deleted (changing the homotopy type), and then the collapses resume as before. This is how the shapes shown in Figs. 1 and 2 were produced. Repeating this process on the sorted sinks results in a series of nested complexes $\mathcal{X} = \mathcal{X}_0, \mathcal{X}_1, \ldots, \emptyset$.

Finally, the algorithm works in any dimension, although most applications are in $\mathbb{R}^3$.

References

