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Computational Geometry Column 45

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Computational Geometry Column 45

Joseph O’Rourke*

Abstract

The algorithm of Edelsbrunner for surface reconstruction by “wrapping” a set of points in $\mathbb{R}^3$ is described.

Curve reconstruction [O’R00] seeks to find a “best” curve passing through a given finite set of points, usually in $\mathbb{R}^2$. Surface reconstruction seeks to find a best surface passing through a set of points in $\mathbb{R}^3$. Both problems have numerous applications, usually deriving from the need to reconstruct the curve or surface from a sample. Both problems are highly underconstrained, for there are usually many curves/surfaces through the points. Surface reconstruction in particular is notoriously difficult to control. Although significant advances have been made in recent years [Dey04]—especially in the direction of performance guarantees based on sample density—we turn here to a beautiful and now relatively old “wrapping” algorithm due to Edelsbrunner, which, although implemented in 1996 at Raindrop Geomagic, has been published only recently [Ede03] after issuance of a patent in 2002.

Sample results of the algorithm are illustrated in Figs. 1 and 2. Although both of these examples reconstruct surfaces of genus one, we concentrate on the genus-zero case (a topological sphere) and only mention extensions for higher genus reconstructions.

An attractive aspect of the algorithm is that it reconstructs a unique surface without assumptions on sample density and without adjustment of heuristic parameters. Although the algorithm uses discrete methods, underneath it relies on continuous Morse functions. The discrete scaffolding on...

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1. $\text{.stl}$ (stereolithography) files for shapes from http://www.cs.duke.edu/~edels/Tubes/
which the algorithm depends is the Delaunay complex, which we now informally describe. A simplex is a point, segment, triangle, or tetrahedron. A simplicial complex $K$ is a “proper” gluing together of simplicies, in that (1) if a simplex $\sigma$ is in $K$, then so are all its faces, and (2) if two simplices $\sigma$ and $\sigma'$ are in $K$, then either $\sigma \cap \sigma'$ is empty or a face of each. Let $S$ be the finite set of points whose surface is to be reconstructed. The Delaunay complex $\text{Del} S$ is the dual of the Voronoi diagram of $S$. Under a general-position assumption, $\text{Del} S$ contains a simplex that is the convex hull of the sites $T \subset S$ iff there is an empty sphere that passes through the points of $T$. The outer boundary of $\text{Del} S$ is the convex hull of $S$. Augmenting $\text{Del} S$ with a dummy “simplex” $\emptyset$ for the space exterior to the hull, covers $\mathbb{R}^3$.

The algorithm seeks to find a “wrapping” surface $W$, a connected simplicial subcomplex in $\text{Del} S$. It accomplishes this by finding a simplicial subcomplex $\mathcal{X}$ of $\text{Del} S$ whose boundary is $W$. The vertices of $\mathcal{X}$ will be precisely the input points $S$, and the vertices of $W$ will be a subset of $S$.

The algorithm uncovers $\mathcal{X}$ in $\text{Del} S$ by “sculpting” away simplices from $\text{Del} S$ one-by-one, starting from $\emptyset$, until $\mathcal{X}$ remains. The simplices are removed according to an acyclic partial ordering. It is the definition of this ordering that involves continuous mathematics.

A function $g(x)$ assigns to every point $x \in \mathbb{R}^3$ a number dependent on the closest Voronoi vertex. In particular, if $x$ is in a tetrahedron $T$ of $\text{Del} S$ whose empty circumsphere has center $z$ and radius $r$, then $g(x) = r^2 - ||z - x||^2$. Thus $g(x)$ is zero at the corners of $T$ and rises to $r^2$ at $z$, the closest Voronoi.
vertex. Points outside the hull are assigned an effectively infinite value. $g(x)$ is continuous but not smooth enough to qualify as a Morse function, needed for the subsequent development. It will suffice here to claim that $g$ can be smoothed sufficiently to define the vector field $\nabla g$, and from this, by a limiting process, \textit{flow curves} through every point $x \in \mathbb{R}^3$ aiming toward higher values.

These flow curves are in turn used to define an acyclic relation on all the simplices of $\text{Del} S$ and $\varnothing$. Let $\tau$ and $\sigma$ be two simplices (of any dimension) and $v$ a face shared between them. For example, if $\tau$ and $\sigma$ are both tetrahedra, $v$ could be a triangle, or a segment, or a vertex. Define the \textit{flow relation} “$\rightarrow$” so that $\tau \rightarrow v \rightarrow \sigma$ if there is a flow curve passing from $\text{int } \tau$ to $\text{int } v$ to $\text{int } \sigma$.\footnote{\text{int } v \text{ is the interior of } v; \text{ for a } v \text{ a vertex, int } v = v.}

A sink of the relation is a simplex that has no flow successor. $\varnothing$ is always a sink (recall $g(x)$ is large outside the hull), with the hull faces of $\text{Del} S$ its immediate predecessors. Sinks are like critical points of the flow, with the simplices that gravitate toward a sink corresponding to a stable manifold in Morse terminology.

A key theorem is that the flow relation on simplices is acyclic, which reflects the increase of $g(x)$ along every flow curve. The algorithms starts with $\varnothing$ and methodically “collapses” its flow predecessors until no more collapses are possible, yielding the complex $\mathcal{X}$.

Let $v$ be a face of $\tau$; then $\tau$ is called a \textit{coface} of $v$.\footnote{\text{One can think of this is a } containing \textit{face}, although its origins are more in complementary topological terminology.} Assume $\tau \rightarrow v$; for example, $\tau$ might be a tetrahedron and $v$ one of its edges, with the flow from $\tau$ through $v$. We give some indication of when the pair $(v, \tau)$ is collapsible, without defining it precisely. First, $\tau$ must be the highest dimension coface of $v$, and $v$ should not have any cofaces not part of $\tau$. Thus, $v$ is in a sense “exposed.” Second, the flow curves should pass right through every point of $v$ (as opposed to running along or in $v$). Collapse of the pair removes all the cofaces of $v$, thus eating away the parts of $\tau$ sharing $v$.

A second key theorem is that any sequence of collapses from $\varnothing$ leads to the same simplicial complex $\mathcal{X}$. Collapses also maintain the homotopy type, which, because $\text{Del} S$ is a topological ball, result in $\mathcal{X}$ a ball and $\mathcal{W}$ a topological sphere.

To produce surfaces of higher genus, the contraction is pushed through
holes: the most “significant” sink (in terms of $g(x)$) is deleted (changing the homotopy type), and then the collapses resume as before. This is how the shapes shown in Figs. 1 and 2 were produced. Repeating this process on the sorted sinks results in a series of nested complexes $\mathcal{X} = \mathcal{X}_0, \mathcal{X}_1, \ldots, \emptyset$.

Finally, the algorithm works in any dimension, although most applications are in $\mathbb{R}^3$.

References

